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THE PREDICTION OF PLASTIC PROPERTIES OF POLYCRYSTALLINE AGGREGATES OF BCC METALS DEFORMING BY <111> PENCIL GLIDE

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Abstract: The Bishop and Hill-type isostrain analysis for the deformation of BCC crystals by pencil glide has been re-examined. Expressions have been derived for slip-plane orientations and shears for simultaneous slip along four <111> directions. The expressions for shears, in conjunction with expressions for the stress states previously calculated by Piehler and Backofen, permit comparison of external and internal work, which must be equal for the active stress state. Additional relations are introduced which must be satisfied for simultaneous operation of three <111> slip systems. These relations permit a straightforward computational procedure for determining possible stress states, and insure that external and internal work are equal. If for a possible stress state involving activation of three slip directions, the shear stress on the supposedly inactive system is less than the yield stress, that stress state is active.

INTRODUCTION

As Piehler and Backofen have noted,[1] at least three of the four <111> slip systems must be activated to provide the 5 degrees of freedom necessary to accommodate an arbitrary imposed strain in a BCC crystal deforming by <111> pencil glide. They obtained closed-form expressions for all stress states which simultaneously operate four <111> slip systems, and derived conditions for simultaneous operation of three

<111> slip systems. In the present paper we shall introduce additional relations which must be satisfied for simultaneous operation of three <111> slip systems. These relations permit a straightforward computational procedure for determining slip planes, shears, rotations, and Bishop and Hill stresses.[2] In addition, the shears associated with the activation of all four slip directions are tabulated. Hence, for these stress states, the internal and external work can be compared, and it can be easily determined whether one of these stress states is activated. In order to be brief, and to avoid duplication of prior work, it will be necessary to make frequent reference to the paper[1] by Piehler and Backofen (PB).

A. Four <111> Slip Directions Activated

Piehler and Backofen analyzed cases involving activation of four slip systems by examining slip along arbitrary directions in {111} planes, the results of which are equivalent to those from analysis of slip along <111> directions in arbitrary planes containing these directions. The polar form of the yield criterion for pencil glide is[1]

$$\tau_{np} \cos \theta + \tau_{nq} \sin \theta = +K. \qquad (1)$$

It is always possible to choose θ so that K is positive. The slip plane is defined by the rotation θ about the <111> direction from a <110> direction toward a <112> direction. τ_{np} and τ_{nq} are the shear stresses acting on a {111} plane in a <110> and <112> direction, respectively. For specific <111>, <110>, <112> groupings, τ_{np} and τ_{nq} are given in terms of the Bishop and Hill stresses in PB Table I. Equation (1) is satisfied if $\tau_{np} = K \cos \theta$ and $\tau_{nq} = K \sin \theta$. For PB groups I, IIIa, IIIb, IIIc the slip-plane orientations can be found from PB Tables I and IV, since the stress states in PB Table IV assure that $\tau_{np} = K \cos \theta$ and $\tau_{nq} = K \sin \theta$ for each slip system. These slip plane orientations are listed in Table I. D_{I}, D_{IIIa}, D_{IIIb}, D_{IIIc} are defined in PB Table IV. The stress states in PB Table IV are also those which are most favorable from a maximum work viewpoint.[2] Furthermore, it can be noted that PB Groups IIa, IIb, and IIc are special cases of PB stress states in Groups IIIa, IIIb, IIIc, respectively, and need not be considered separately.

With these slip plane orientations, the strains, $d\varepsilon_{ij}$, and rotations, $d\omega_{ij}$, referred to the cubic axes, may be expressed in terms of the shears, $s_p = d\gamma_{13}^{(p)}$, on the four slip systems

$$d\varepsilon_{ij} = \sum_p (a_{i3}a_{j1} + a_{i1}a_{j3}) \frac{s_p}{2}, \text{ and} \qquad (2a)$$

$$d\omega_{ij} = \sum_p (a_{i3}a_{j1} - a_{i1}a_{j3}) \frac{s_p}{2}, \qquad (2b)$$

where a_{ij} are the direction cosines of the angles between the cubic axes and directions in the system defined by the slip plane normal (3), the slip direction (1), and the direction

TABLE I

SLIP PLANE ORIENTATIONS CORRESPONDING TO STRESS STATES WHICH ACTIVATE FOUR SLIP DIRECTIONS

Group	$\cos\theta_a/\sin\theta_a$	$\cos\theta_b/\sin\theta_b$	$\cos\theta_c/\sin\theta_c$	$\cos\theta_d/\sin\theta_d$
I	$\dfrac{(d\varepsilon_{22} - d\varepsilon_{11})/D_I}{-\sqrt{3}(d\varepsilon_{11} + d\varepsilon_{22})/D_I}$	$-\cos\theta_a/\sin\theta_a$	$\cos\theta_a/\sin\theta_a$	$-\cos\theta_a/\sin\theta_a$
IIIa	$\dfrac{(d\varepsilon_{22} - d\varepsilon_{33} + 6d\varepsilon_{23})/D_{IIIa}}{\sqrt{3}(-d\varepsilon_{22} + d\varepsilon_{33} + 2d\varepsilon_{23})/D_{IIIa}}$	$\dfrac{(-d\varepsilon_{22} + d\varepsilon_{33} + 6d\varepsilon_{23})/D_{IIIa}}{\sqrt{3}(-d\varepsilon_{22} + d\varepsilon_{33} - 2d\varepsilon_{23})/D_{IIIa}}$	$-\cos\theta_b/\sin\theta_b$	$-\cos\theta_a/\sin\theta_a$
IIIb	$\dfrac{(d\varepsilon_{33} - d\varepsilon_{11} - 6d\varepsilon_{31})/D_{IIIb}}{\sqrt{3}(d\varepsilon_{33} - d\varepsilon_{11} + 2d\varepsilon_{31})/D_{IIIb}}$	$-\cos\theta_a/\sin\theta_a$	$\dfrac{(d\varepsilon_{33} - d\varepsilon_{11} + 6d\varepsilon_{31})/D_{IIIb}}{\sqrt{3}(d\varepsilon_{33} - d\varepsilon_{11} - 2d\varepsilon_{31})/D_{IIIb}}$	$-\cos\theta_c/\sin\theta_c$
IIIc	$\dfrac{2(d\varepsilon_{22} - d\varepsilon_{11})/D_{IIIc}}{-4\sqrt{3}\, d\varepsilon_{12}/D_{IIIc}}$	$-\cos\theta_a/-\sin\theta_a$	$\cos\theta_a/\sin\theta_a$	$-\cos\theta_a/-\sin\theta_a$

(2) perpendicular to (1) and (3) such that (1), (2), (3) form a right-handed set. From Eqs. (2a) and (2b) we obtain

$$\sqrt{6}\, d\varepsilon_{11} = -(\cos\theta_a + \frac{1}{\sqrt{3}}\sin\theta_a)s_a + (\cos\theta_b - \frac{1}{\sqrt{3}}\sin\theta_b)s_b$$
$$-(\cos\theta_c + \frac{1}{\sqrt{3}}\sin\theta_c)s_c + (\cos\theta_d - \frac{1}{\sqrt{3}}\sin\theta_d)s_d, \quad (3a)$$

$$\sqrt{6}\, d\varepsilon_{22} = (\cos\theta_a - \frac{1}{\sqrt{3}}\sin\theta_a)s_a - (\cos\theta_b + \frac{1}{\sqrt{3}}\sin\theta_b)s_b$$
$$+(\cos\theta_c - \frac{1}{\sqrt{3}}\sin\theta_c)s_c - (\cos\theta_d + \frac{1}{\sqrt{3}}\sin\theta_d)s_d, \quad (3b)$$

$$2\sqrt{6}\, d\varepsilon_{23} = (\cos\theta_a + \frac{1}{\sqrt{3}}\sin\theta_a)s_a + (\cos\theta_b - \frac{1}{\sqrt{3}}\sin\theta_b)s_b$$
$$-(\cos\theta_c + \frac{1}{\sqrt{3}}\sin\theta_c)s_c - (\cos\theta_d - \frac{1}{\sqrt{3}}\sin\theta_d)s_d, \quad (3c)$$

$$2\sqrt{6}\, d\varepsilon_{31} = -(\cos\theta_a - \frac{1}{\sqrt{3}}\sin\theta_a)s_a + (\cos\theta_b + \frac{1}{\sqrt{3}}\sin\theta_b)s_b$$
$$+(\cos\theta_c - \frac{1}{\sqrt{3}}\sin\theta_c)s_c - (\cos\theta_d + \frac{1}{\sqrt{3}}\sin\theta_d)s_d, \quad (3d)$$

$$3\sqrt{2}\, d\varepsilon_{12} = -s_a\sin\theta_a + s_b\sin\theta_b - s_c\sin\theta_c + s_d\sin\theta_d, \quad (3e)$$

$$2\sqrt{6}\, d\omega_{23} = -(\cos\theta_a - \sqrt{3}\sin\theta_a)s_a - (\cos\theta_b + \sqrt{3}\sin\theta_b)s_b$$
$$+(\cos\theta_c - \sqrt{3}\,\theta_c)s_c + (\cos\theta_d + \sqrt{3}\sin\theta_d)s_d, \quad (3f)$$

$$2\sqrt{6}\, d\omega_{31} = -(\cos\theta_a + \sqrt{3}\sin\theta_a)s_a + (\cos\theta_b - \sqrt{3}\sin\theta_b)s_b$$
$$+(\cos\theta_c + \sqrt{3}\sin\theta_c)s_c - (\cos\theta_d - \sqrt{3}\sin\theta_d)s_d, \quad (3g)$$

$$\sqrt{6}\, d\omega_{12} = s_a\cos\theta_a + s_b\cos\theta_b + s_c\cos\theta_c + s_d\cos\theta_d\ . \quad (3h)$$

By substituting $\cos\theta_a$, $\sin\theta_a$, etc., for Group I (from Table I) into Eqs. (3b) through (3e), we obtain

$$s_a + s_b + s_c + s_d = \sqrt{6}\, D_I/2, \quad (4a)$$

$$-s_a + s_b + s_c - s_d = \sqrt{6}\, D_I d\varepsilon_{23}/d\varepsilon_{11}, \quad (4b)$$

$$-s_a - s_b + s_c + s_d = \sqrt{6}\, D_I d\varepsilon_{31}/d\varepsilon_{22}, \text{ and} \quad (4c)$$

$$s_a - s_b + s_c - s_d = \sqrt{6}\, D\, d\varepsilon_{12}/(d\varepsilon_{11} + d\varepsilon_{22}), \qquad (4d)$$

Equations (3a) and (3b) are in this case linearly dependent, each leading to Eq. (4a). Expressions similar to Eqs. (4a) through (4d) may also be obtained for Groups IIIa, IIIb and IIIc. Equations (4a) through (4d) and similar relations for Groups IIIa, IIIb and IIIc may be solved for the shears s_a, s_b, s_c, and s_d in terms of the imposed strains, $d\varepsilon_{ij}$. The results are summarized in Table II.

PB Table IV and Tables I and II list the stress states, slip plane orientations, and shears which allow an arbitrary imposed strain to be accommodated by simultaneous slip in four <111> directions, and satisfy the yield condition for each slip system. We can now compare the external work, $\sum_{i,j} \sigma_{ij}\, d\varepsilon_{ij}$, to the internal work, $K \sum_p |s_p|$. Using Bishop and Hill notation, we may write

TABLE II

SHEARS PRODUCED BY ACTIVATION OF FOUR SLIP DIRECTIONS

Group I:

$$s_a = \frac{\sqrt{6}}{4} D_I\left(\frac{1}{2} - \frac{d\varepsilon_{23}}{d\varepsilon_{11}} - \frac{d\varepsilon_{31}}{d\varepsilon_{22}} - \frac{d\varepsilon_{12}}{d\varepsilon_{33}}\right)$$

$$s_b = \frac{\sqrt{6}}{4} D_I\left(\frac{1}{2} + \frac{d\varepsilon_{23}}{d\varepsilon_{11}} - \frac{d\varepsilon_{31}}{d\varepsilon_{22}} + \frac{d\varepsilon_{12}}{d\varepsilon_{33}}\right)$$

$$s_c = \frac{\sqrt{6}}{4} D_I\left(\frac{1}{2} + \frac{d\varepsilon_{23}}{d\varepsilon_{11}} + \frac{d\varepsilon_{31}}{d\varepsilon_{22}} - \frac{d\varepsilon_{12}}{d\varepsilon_{33}}\right)$$

$$s_d = \frac{\sqrt{6}}{4} D_I\left(\frac{1}{2} - \frac{d\varepsilon_{23}}{d\varepsilon_{11}} + \frac{d\varepsilon_{31}}{d\varepsilon_{22}} + \frac{d\varepsilon_{12}}{d\varepsilon_{33}}\right)$$

Group IIIa:

$$s_a = \frac{\sqrt{6}}{32} D_{IIIa}\left[2 - \frac{d\varepsilon_{11}}{d\varepsilon_{23}} - \frac{4(1+\alpha)d\varepsilon_{31} + 4(1-\alpha)d\varepsilon_{12}}{(1+\alpha^2)d\varepsilon_{23}}\right]$$

$$s_b = \frac{\sqrt{6}}{32} D_{IIIa}\left[2 + \frac{d\varepsilon_{11}}{d\varepsilon_{23}} + \frac{4(1-\alpha)d\varepsilon_{31} - 4(1+\alpha)d\varepsilon_{12}}{(1+\alpha^2)d\varepsilon_{23}}\right]$$

$$s_c = \frac{\sqrt{6}}{32} D_{IIIa}\left[2 + \frac{d\varepsilon_{11}}{d\varepsilon_{23}} - \frac{4(1-\alpha)d\varepsilon_{31} - 4(1+\alpha)d\varepsilon_{12}}{(1+\alpha^2)d\varepsilon_{23}}\right]$$

$$s_d = \frac{\sqrt{6}}{32} D_{IIIa}\left[2 - \frac{d\varepsilon_{11}}{d\varepsilon_{23}} + \frac{4(1+\alpha)d\varepsilon_{31} + 4(1-\alpha)d\varepsilon_{12}}{(1+\alpha^2)d\varepsilon_{23}}\right]$$

$$\alpha = \frac{d\varepsilon_{22} - d\varepsilon_{33}}{2d\varepsilon_{23}}$$

TABLE II (cont.)

Group IIIb:

$$s_a = \frac{\sqrt{6}}{32} D_{IIIb}\left[2 - \frac{d\varepsilon_{22}}{d\varepsilon_{31}} - \frac{4(1-\beta)d\varepsilon_{23} + 4(1+\beta)d\varepsilon_{12}}{(1+\beta^2)d\varepsilon_{31}}\right]$$

$$s_b = \frac{\sqrt{6}}{32} D_{IIIb}\left[2 - \frac{d\varepsilon_{22}}{d\varepsilon_{31}} + \frac{4(1-\beta)d\varepsilon_{23} + 4(1+\beta)d\varepsilon_{12}}{(1+\beta^2)d\varepsilon_{31}}\right]$$

$$s_c = \frac{\sqrt{6}}{32} D_{IIIb}\left[2 + \frac{d\varepsilon_{22}}{d\varepsilon_{31}} - \frac{4(1+\beta)d\varepsilon_{23} - 4(1-\beta)d\varepsilon_{12}}{(1+\beta^2)d\varepsilon_{31}}\right]$$

$$s_d = \frac{\sqrt{6}}{32} D_{IIIb}\left[2 + \frac{d\varepsilon_{22}}{d\varepsilon_{31}} + \frac{4(1+\beta)d\varepsilon_{23} - 4(1-\beta)d\varepsilon_{12}}{(1+\beta^2)d\varepsilon_{31}}\right]$$

$$\beta = \frac{d\varepsilon_{33} - d\varepsilon_{11}}{2d\varepsilon_{31}}$$

Group IIIc:

$$s_a = \frac{\sqrt{6}}{32} D_{IIIc}\left[2 - \frac{d\varepsilon_{33}}{d\varepsilon_{12}} - \frac{4(1+\gamma)d\varepsilon_{23} + 4(1-\gamma)d\varepsilon_{31}}{(1+\gamma^2)d\varepsilon_{12}}\right]$$

$$s_b = \frac{\sqrt{6}}{32} D_{IIIc}\left[2 + \frac{d\varepsilon_{33}}{d\varepsilon_{12}} - \frac{4(1-\gamma)d\varepsilon_{23} - 4(1+\gamma)d\varepsilon_{31}}{(1+\gamma^2)d\varepsilon_{12}}\right]$$

$$s_c = \frac{\sqrt{6}}{32} D_{IIIc}\left[2 - \frac{d\varepsilon_{33}}{d\varepsilon_{12}} + \frac{4(1+\gamma)d\varepsilon_{23} + 4(1-\gamma)d\varepsilon_{31}}{(1+\gamma^2)d\varepsilon_{12}}\right]$$

$$s_d = \frac{\sqrt{6}}{32} D_{IIIc}\left[2 + \frac{d\varepsilon_{33}}{d\varepsilon_{12}} + \frac{4(1-\gamma)d\varepsilon_{23} - 4(1+\gamma)d\varepsilon_{31}}{(1+\gamma^2)d\varepsilon_{12}}\right]$$

$$\gamma = \frac{d\varepsilon_{11} - d\varepsilon_{22}}{2d\varepsilon_{12}}$$

$$-Bd\varepsilon_{11} + Ad\varepsilon_{22} + 2Fd\varepsilon_{23} + 2Gd\varepsilon_{31} + 2Hd\varepsilon_{12} \leqq K(|s_a| + |s_b| + |s_c| + |s_d|). \quad (5)$$

The equality will apply only for the operative set of slip-planes and shears.[2] Since Eq. (5) involves absolute values of the shears, explicit conditions for operation of Group I, IIIa, IIIb, or IIIc stress state cannot be given. The calculation of the terms in Eq. (5) is, however, straightforward and will be discussed in the section on computational procedure. It is possible that none of the stress states involving activation of four slip directions will satisfy expression (5) with the equality sign. In such situations, stress states which activate three <III> slip directions must be considered.

B. Three <111> Slip Directions Activated

There are four distinct cases for which one <111> slip direction is inactive (Groups IVa, IVb, IVc, and IVd in Piehler and Backofen's convention[1]). In contrast to the approach taken for the cases in which simultaneous slip occurred in four <111> directions, we shall first find those shears which are capable of accommodating an arbitrary imposed strain. We shall use the notation due to Katoh.[3] From Eqs. (3a) through (3h), we obtain

$$\sqrt{6}(d\varepsilon_{22} - d\varepsilon_{11})/2 = X_1 - X_2 + X_3 - X_4, \tag{6a}$$

$$3\sqrt{2}d\varepsilon_{33}/2 = Y_1 + Y_2 + Y_3 + Y_4, \tag{6b}$$

$$3\sqrt{2}d\varepsilon_{12} = -Y_1 + Y_2 - Y_3 + Y_4, \tag{6c}$$

$$2\sqrt{6}d\varepsilon_{31} = -X_1 + X_2 + X_3 - X_4 + (Y_1 + Y_2 - Y_3 - Y_4)/\sqrt{3}, \tag{6d}$$

$$2\sqrt{6}d\varepsilon_{23} = X_1 + X_2 - X_3 - X_4 + (Y_1 - Y_2 + Y_3 + Y_4)/\sqrt{3}, \tag{6e}$$

$$\sqrt{6}d\omega_{12} = X_1 + X_2 + X_3 + X_4, \tag{6f}$$

$$2\sqrt{6}d\omega_{31} = -X_1 + X_2 + X_3 - X_4 - \sqrt{3}(Y_1 + Y_2 - Y_3 - Y_4), \tag{6g}$$

and

$$2\sqrt{6}d\omega_{23} = -X_1 - X_2 + X_3 + X_4 + \sqrt{3}(Y_1 - Y_2 - Y_3 + Y_4), \tag{6h}$$

where $X_1 = s_a \cos\theta_a$, $X_2 = s_b \cos\theta_b$, $X_3 = s_c \cos\theta_c$, $X_4 = s_d \cos\theta_d$, $Y_1 = s_a \sin\theta_a$, $Y_2 = s_b \sin\theta_b$, $Y_3 = s_c \sin\theta_c$ and $Y_4 = s_d \sin\theta_d$. Conservation of volume has been assumed, i.e., $d\varepsilon_{11} + d\varepsilon_{22} + d\varepsilon_{33} = 0$. Equations (6a) through (6h) may be solved for X_i and Y_i to yield:

$$X_1 = (\sqrt{6}/8)(-d\varepsilon_{11} + d\varepsilon_{22} + 3d\varepsilon_{23} - 3d\varepsilon_{31} - d\omega_{23} - d\omega_{31} + 2d\omega_{12}), \tag{7a}$$

$$X_2 = (\sqrt{6}/8)(d\varepsilon_{11} - d\varepsilon_{22} + 3d\varepsilon_{23} + 3d\varepsilon_{31} - d\omega_{23} + d\omega_{31} + 2d\omega_{12}), \tag{7b}$$

$$X_3 = (\sqrt{6}/8)(-d\varepsilon_{11} + d\varepsilon_{22} - 3d\varepsilon_{23} + 3d\varepsilon_{31} + d\omega_{23} + d\omega_{31} + 2d\omega_{12}), \tag{7c}$$

$$X_4 = (\sqrt{6}/8)(d\varepsilon_{11} - d\varepsilon_{22} - 3d\varepsilon_{23} - 3d\varepsilon_{31} + d\omega_{23} - d\omega_{31} + 2d\omega_{12}), \tag{7d}$$

$$Y_1 = (3\sqrt{2}/8)(d\varepsilon_{33} + d\varepsilon_{23} + d\varepsilon_{31} - 2d\varepsilon_{12} + d\omega_{23} - d\omega_{31}), \tag{7e}$$

$$Y_2 = (3\sqrt{2}/8)(d\varepsilon_{33} - d\varepsilon_{23} + d\varepsilon_{31} + 2d\varepsilon_{12} - d\omega_{23} - d\omega_{31}), \tag{7f}$$

$$Y_3 = (3\sqrt{2}/8)(d\varepsilon_{33} - d\varepsilon_{23} - d\varepsilon_{31} - 2d\varepsilon_{12} - d\omega_{23} - d\omega_{31}), \tag{7g}$$

and

$$Y_4 = (3\sqrt{2}/8)(d\varepsilon_{33} + d\varepsilon_{23} - d\varepsilon_{31} + 2d\varepsilon_{12} + d\omega_{23} + d\omega_{31}). \tag{7h}$$

For Group IVa, $s_a = X_1 = Y_1 = 0$, so that Eqs. (7a) and (7e) may be solved for $d\omega_{23}$ and $d\omega_{31}$ to yield:

$$d\omega_{23} = d\varepsilon_{22} + d\varepsilon_{23} - 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12}, \text{ and} \tag{8a}$$

$$d\omega_{31} = -d\varepsilon_{11} + 2d\varepsilon_{23} - d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12}. \tag{8b}$$

If $d\omega_{23}$ and $d\omega_{31}$ from Eqs. (8a) and (8b) are substituted in Eqs. (7b) through (7d) and (7f) through (7h) we obtain expressions for X_2, X_3, X_4, Y_2, Y_3 and Y_4 in terms of the strains referred to the crystal axes and the as yet undetermined rotation $d\omega_{12}$. Similar expressions may be obtained for Groups IVb, IVc and IVd. For Group IVb,

$$d\omega_{23} = -d\varepsilon_{22} + d\varepsilon_{23} + 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12}, \text{ and} \tag{8c}$$

$$d\omega_{31} = -d\varepsilon_{11} - 2d\varepsilon_{23} - d\varepsilon_{31} + d\varepsilon_{12} - d\omega_{12}. \tag{8d}$$

For Group IVc,

$$d\omega_{23} = -d\varepsilon_{22} + d\varepsilon_{23} - 2d\varepsilon_{31} - d\varepsilon_{12} - d\omega_{12}, \text{ and} \tag{8e}$$

$$d\omega_{31} = d\varepsilon_{11} + 2d\varepsilon_{23} - d\varepsilon_{31} + d\varepsilon_{12} - d\omega_{12}. \tag{8f}$$

For Group IVd,

$$d\omega_{23} = d\varepsilon_{22} + d\varepsilon_{23} + 2d\varepsilon_{31} - d\varepsilon_{12} - d\omega_{12}, \text{ and} \tag{8g}$$

$$d\omega_{31} = d\varepsilon_{11} - 2d\varepsilon_{23} - d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12}. \tag{8h}$$

Expressions for the X_i and Y_i for PB Groups IVa through IVd are summarized in Table III.

TABLE III

GROUP IV: DEFORMATIONS CAPABLE OF ACCOMMODATING AN ARBITRARY IMPOSED STRAIN

GROUP IVa ([111] inactive): $X_1 = Y_1 = 0$

$$X_2 = \sqrt{6}(-d\varepsilon_{22} + 2d\varepsilon_{23} + 2d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12})/4$$

$$X_3 = \sqrt{6}(-d\varepsilon_{11} + d\varepsilon_{22} + 2d\omega_{12})/4$$

$$X_4 = \sqrt{6}(d\varepsilon_{11} - 2d\varepsilon_{23} - 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12})/4$$

$$Y_2 = 3\sqrt{2}(-d\varepsilon_{22} - 2d\varepsilon_{23} + 2d\varepsilon_{31} + d\varepsilon_{12} - d\omega_{12})/4$$

$$Y_3 = 3\sqrt{2}(d\varepsilon_{33} - 2d\varepsilon_{12})/4$$

$$Y_4 = 3\sqrt{2}(-d\varepsilon_{11} + 2d\varepsilon_{23} - 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12})/4$$

TABLE III (Cont.)

GROUP IVb ($[1\bar{1}1]$ inactive): $X_2 = Y_2 = 0$

$$X_1 = \sqrt{6}(d\varepsilon_{22} + 2d\varepsilon_{23} - 2d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12})/4$$

$$X_3 = \sqrt{6}(-d\varepsilon_{11} - 2d\varepsilon_{23} + 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12})/4$$

$$X_4 = \sqrt{6}(d\varepsilon_{11} - d\varepsilon_{22} + 2d\omega_{12})/4$$

$$Y_1 = 3\sqrt{2}(-d\varepsilon_{22} + 2d\varepsilon_{23} + 2d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12})/4$$

$$Y_3 = 3\sqrt{2}(-d\varepsilon_{11} - 2d\varepsilon_{23} - 2d\varepsilon_{31} - d\varepsilon_{12} - d\omega_{12})/4$$

$$Y_4 = 3\sqrt{2}(d\varepsilon_{33} + 2d\varepsilon_{12})/4$$

GROUP IVc ($[\bar{1}\bar{1}1]$ inactive): $X_3 = Y_3 = 0$

$$X_1 = \sqrt{6}(-d\varepsilon_{11} + d\varepsilon_{22} + 2d\omega_{12})/4$$

$$X_2 = \sqrt{6}(d\varepsilon_{11} + 2d\varepsilon_{23} + 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12})/4$$

$$X_4 = \sqrt{6}(-d\varepsilon_{22} - 2d\varepsilon_{23} - 2d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12})/4$$

$$Y_1 = 3\sqrt{2}(d\varepsilon_{33} - 2d\varepsilon_{12})/4$$

$$Y_2 = 3\sqrt{2}(-d\varepsilon_{11} - 2d\varepsilon_{23} + 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12})/4$$

$$Y_4 = 3\sqrt{2}(-d\varepsilon_{22} + 2d\varepsilon_{23} - 2d\varepsilon_{31} + d\varepsilon_{12} - d\omega_{12})/4$$

GROUP IVd ($[\bar{1}11]$ inactive): $X_4 = Y_4 = 0$

$$X_1 = \sqrt{6}(-d\varepsilon_{11} + 2d\varepsilon_{23} - 2d\varepsilon_{31} + d\varepsilon_{12} + d\omega_{12})/4$$

$$X_2 = \sqrt{6}(d\varepsilon_{11} - d\varepsilon_{22} + 2d\omega_{12})/4$$

$$X_3 = \sqrt{6}(d\varepsilon_{22} - 2d\varepsilon_{23} + 2d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12})/4$$

$$Y_1 = 3\sqrt{2}(-d\varepsilon_{11} + 2d\varepsilon_{23} + 2d\varepsilon_{31} - d\varepsilon_{12} - d\omega_{12})/4$$

$$Y_2 = 3\sqrt{2}(d\varepsilon_{33} + 2d\varepsilon_{12})/4$$

$$Y_3 = 3\sqrt{2}(-d\varepsilon_{22} - 2d\varepsilon_{23} - 2d\varepsilon_{31} - d\varepsilon_{12} + d\omega_{12})/4$$

The rotations $d\omega_{12}$, and thus the X_i's and Y_i's in Table III, can be determined by examination of the yield conditions. If in PB Table I, we substitute $\tau_{np}^{(a)} = K\cos\theta_a$, $\tau_{nq}^{(a)} = K\sin\theta_a$, etc., we obtain the yield conditions:

$$-C + F - G = \sqrt{6}\ K \cos \theta_a, \tag{9a}$$

$$-A + B + F + G - 2H = 3\sqrt{2}\ K \sin \theta_a, \tag{9b}$$

$$C + F + G = \sqrt{6}\ K \cos \theta_b, \tag{9c}$$

$$-A + B - F + G + 2H = 3\sqrt{2}\ K \sin \theta_b, \tag{9d}$$

$$-C - F + G = \sqrt{6}\ K \cos \theta_c, \tag{9e}$$

$$-A + B - F - G - 2H = 3\sqrt{2}\ K \sin \theta_c, \tag{9f}$$

$$C - F - G = \sqrt{6}\ K \cos \theta_d, \text{ and} \tag{9g}$$

$$-A + B + F - G + 2H = 3\sqrt{2}\ K \sin \theta_d. \tag{9h}$$

In the case of Group IVa, yielding must occur on slip systems b, c, and d, and thus we must satisfy six equations (Eqs. (9c) through (9h)) in five unknowns (since $A + B + C = 0$). Equations (9c) and (9e) through (9h) may be solved for A, B, C, F, G and H in terms of $\cos \theta_b$, $\sin \theta_b$, etc. Similar expressions may be obtained for Groups IVb, IVc and IVd. The results are summarized in Table IV. In obtaining A, B, C, F, G and H for Group IVa, we did not use Eq. (9d). Satisfaction of Eq. (9d) requires that

$$\cos \theta_b + 2 \cos \theta_c + \cos \theta_d - \sqrt{3}(\sin \theta_b - \sin \theta_d) = 0 \tag{10a}$$

TABLE IV

GROUP IV: BISHOP AND HILL STRESSES

GROUP IVa:

$$A = -\frac{3\sqrt{2}}{4} K(\sin \theta_c + \sin \theta_d) - \frac{\sqrt{6}}{4} K(2 \cos \theta_b + \cos \theta_c + \cos \theta_d)$$

$$B = \frac{3\sqrt{2}}{4} K(\sin \theta_c + \sin \theta_d) + \frac{\sqrt{6}}{4} K(\cos \theta_c - \cos \theta_d)$$

$$C = \frac{\sqrt{6}}{2} K(\cos \theta_b + \cos \theta_d)$$

$$F = -\frac{\sqrt{6}}{2} K(\cos \theta_c + \cos \theta_d)$$

$$G = \frac{\sqrt{6}}{2} K(\cos \theta_b + \cos \theta_c)$$

$$H = \frac{3\sqrt{2}}{4} K(-\sin \theta_c + \sin \theta_d) + \frac{\sqrt{6}}{4} K(\cos \theta_c + \cos \theta_d)$$

TABLE IV (Cont.)

GROUP IVb:

$$A = -\frac{3\sqrt{2}}{4}K(\sin\theta_c + \sin\theta_d) + \frac{\sqrt{6}}{4}K(2\cos\theta_a + \cos\theta_c + \cos\theta_d)$$

$$B = \frac{3\sqrt{2}}{4}K(\sin\theta_c + \sin\theta_d) + \frac{\sqrt{6}}{4}K(\cos\theta_c - \cos\theta_d)$$

$$C = -\frac{\sqrt{6}}{2}K(\cos\theta_a + \cos\theta_c)$$

$$F = -\frac{\sqrt{6}}{2}K(\cos\theta_c + \cos\theta_d)$$

$$G = -\frac{\sqrt{6}}{2}K(\cos\theta_a + \cos\theta_d)$$

$$H = \frac{3\sqrt{2}}{4}K(\sin\theta_d - \sin\theta_c) + \frac{\sqrt{6}}{4}K(\cos\theta_c + \cos\theta_d)$$

GROUP IVc:

$$A = -\frac{3\sqrt{2}}{4}K(\sin\theta_a + \sin\theta_b) - \frac{\sqrt{6}}{4}K(\cos\theta_a + \cos\theta_b + 2\cos\theta_d)$$

$$B = \frac{3\sqrt{2}}{4}K(\sin\theta_a + \sin\theta_b) + \frac{\sqrt{6}}{4}K(\cos\theta_a - \cos\theta_b)$$

$$C = \frac{\sqrt{6}}{2}K(\cos\theta_b + \cos\theta_d)$$

$$F = \frac{\sqrt{6}}{2}K(\cos\theta_a + \cos\theta_b)$$

$$G = -\frac{\sqrt{6}}{2}K(\cos\theta_a + \cos\theta_d)$$

$$H = \frac{3\sqrt{2}}{4}K(\sin\theta_b - \sin\theta_a) + \frac{\sqrt{6}}{4}K(\cos\theta_a + \cos\theta_b)$$

GROUP IVd:

$$A = -\frac{3\sqrt{2}}{4}K(\sin\theta_a + \sin\theta_b) + \frac{\sqrt{6}}{4}K(\cos\theta_a + \cos\theta_b + 2\cos\theta_c)$$

$$B = \frac{3\sqrt{2}}{4}K(\sin\theta_a + \sin\theta_b) + \frac{\sqrt{6}}{4}K(\cos\theta_a - \cos\theta_b)$$

$$C = -\frac{\sqrt{6}}{2}K(\cos\theta_a + \cos\theta_c)$$

$$F = \frac{\sqrt{6}}{2}K(\cos\theta_a + \cos\theta_b)$$

$$G = \frac{\sqrt{6}}{2}K(\cos\theta_b + \cos\theta_c)$$

$$H = \frac{3\sqrt{2}}{4}K(\sin\theta_b - \sin\theta_a) + \frac{\sqrt{6}}{4}K(\cos\theta_a + \cos\theta_b)$$

For Group IVb, satisfaction of Eq. (9b) requires that

$$\cos \theta_a + \cos \theta_c + 2 \cos \theta_d + \sqrt{3}(\sin \theta_a - \sin \theta_c) = 0. \quad (10b)$$

For Group IVc, satisfaction of Eq. (9h) requires that

$$2 \cos \theta_a + \cos \theta_b + \cos \theta_d + \sqrt{3}(\sin \theta_b - \sin \theta_d) = 0. \quad (10c)$$

For Group IVd, satisfaction of Eq. (9f) requires that

$$\cos \theta_a + 2 \cos \theta_b + \cos \theta_c + \sqrt{3}(\sin \theta_c - \sin \theta_a) = 0. \quad (10d)$$

For Group IVa, the six equations in Table III together with Eq. (10a) form a system of seven equations in seven unknowns, s_b, s_c, s_d, θ_b, θ_c, θ_d and $d\omega_{12}$, where $X_2 = s_b \cos \theta_b$, $Y_2 = s_b \sin \theta_b$, etc. The system is nonlinear, and a solution in closed form has not yet been found. It may, however, be solved to any desired precision by an iterative trial-and-error process, in which an initial value is chosen for $d\omega_{12}$, e.g., $d\omega_{12}^{(1)} = 0$, and X_i and Y_i are determined from Table III. From X_2 and Y_2, we obtain $s_b = (X_2{}^2 + Y_2{}^2)^{1/2}$, $\cos \theta_b = X_2/s_b$ and $\sin \theta_b = Y_2/s_b$. We obtain s_c, s_d, $\cos \theta_c$, $\sin \theta_c$, $\cos \theta_d$, and $\sin \theta_d$ in similar fashion, and substitute these values in the left-hand side of Eq. (10a), which will in general not be exactly satisfied. A second trial value is chosen for $d\omega_{12}$, e.g., $d\omega_{12}^{(2)} = 0.01$, and the residual of Eq. (10a) is again calculated. Successive trial values of $d\omega_{12}$ are selected so as to reduce the residual to an acceptable value, e.g., 10^{-5}.

When a value of $d\omega_{12}$ has been found such that the residual of Eq. (10a) is sufficiently small, it is necessary to assure that the stress on the system which has been assumed to be inactive is less than the yield stress. The quadratic forms of the yield conditions for pencil glide are given in PB Table II. A, B, C, F, G and H are calculated for Group IVa, using the expressions in Table IV, and substituted in the left side of the first equation in PB Table II. The result must be less than $9K^2$ if the [111] system is to be inactive. The solutions of Groups IVb, IVc and IVd are accomplished in similar fashion.

It may be shown that Eqs. (10) insure that the external work is equal to the internal work. Hence, the stress states and shears defined by Tables III and IV and Eqs. (10) define those deformations which can accommodate the imposed strain, satisfy the yield condition on three <111> slip directions, and give rise to internal work equal to the external work.

C. Computational Procedure

A general procedure to determine the operative stress states, slip plane orientations, and shears which accommodate an arbitrary strain by <111> pencil glide may now be outlined:

(1) The imposed strain state referred to the macroscopic

specimen axes is transformed to obtain the strain tensor relative to the cubic crystal axes, $d\varepsilon_{ij}$.

(2) The stress states and shears corresponding to Groups I, IIIa, IIIb and IIIc may be calculated from the relations in PB Table IV and Table II. If, for any of these groups, the external work is equal to the internal work [Eq. (5)], the operative stress state and shears have been found. If not, the Group IV stress states must be investigated.

(3) For each Group IV stress state, the X_i and Y_i are determined from the relations in Table III for trial values of $d\omega_{12}$. Values of s_a, $\cos\theta_a$, $\sin\theta_a$, etc. are then determined from $s_a = (X_1{}^2 + Y_1{}^2)^{\frac{1}{2}}$, $\cos\theta_a = X_1/s_a$, $\sin\theta_a = Y_1/s_a$, etc. Values of the sines and cosines are substituted in the left hand side of the appropriate Eq. (10), e.g., Eq. (10a) for Group IVa. Successive trial values of $d\omega_{12}$ are tried until the appropriate Eq. (10) is satisfied within acceptable limits, e.g., 10^{-5}. The Bishop and Hill Stresses, A, B, C, F, G and H are then calculated according to Table IV and substituted in the left hand side of the appropriate equation in PB Table II to assure that this stress state does not give rise to yielding on the supposedly inactive system. If such is the case for some group, it is operative.

The stress state so obtained may then be transformed back to the macroscopic specimen axes. If this procedure is repeated for a number of crystallites which represent a given texture, the stresses may be averaged,and one point on the upper-bound yield locus obtained.[4] An entire upper-bound (isostrain) yield locus can be obtained by following this procedure for a number of imposed strain states.

The use of this procedure for the analysis of the plastic deformation of textured low-carbon sheet steels will be the subject of a subsequent paper. Measurements of r-values and the ratio of plane-strain flow stress to uniaxial flow stress will be compared to isostrain and isostress predictions, and the range of application of these models will be discussed.

SUMMARY

1. Relations which facilitate analysis of the pencil-glide deformation of BCC crystals undergoing an arbitrary shape change have been derived. Stress states which activate all four slip directions or just three slip directions have been investigated.

2. Expressions have been derived for slip-plane orientations and shears for simultaneous slip along four <111> directions. The expressions for the shears, in conjunction with expressions for the stress states previously calculated by Piehler and Backofen, permit comparison of external and internal work. A given stress state is active if the external and internal work resulting from its activation are equal.

3. In the case of simultaneous slip along three <111> directions, expressions have been derived relating $X_1 = s_a \cos\theta_a$, $Y_1 = s_a \sin\theta_a$, etc., to the strain components and one rotation component related to the cubic axes, where s_i are the shears along <111>, and the θ_i are the angles between the slip plane normal and <$\bar{1}$10>. Additional relations among

sines and cosines of angles specifying the orientations of active slip planes have been derived. These relations permit a straightforward computational procedure for obtaining possible stress states, and insure that the external work is equal to the internal work. A given stress state involving just three slip directions is active provided the shear stress on the supposedly inactive system is less than the yield stress.

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REFERENCES

1. H. R. Piehler and W. A. Backofen, *Met. Trans.*, 2, 249 (1971).
2. J. F. W. Bishop and R. Hill, *Phil. Mag.*, 42, 1298 (1951).
3. I. L. Dillamore and H. Katoh, in *Quantitative Analysis of Textures*, Proc. of the International Seminar, Krakow. Sept. 1-6, 1971 (SITPH, Katowice, Poland), p. 315.
4. H. R. Piehler and W. A. Backofen, in *Textures in Research and Practice*, J. Grewen and G. Wassermann, eds. (Springer-Verlag, Berlin, 1969), p. 436.