Texture and Lamellae Distribution Functions in Lamellar Eutectics

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The crystallographic orientation distribution and the geometrical lamellae orientation distribution in lamellar eutectics are, in general, not independent of each other. The combined orientation-lamellae distribution function depends on five angular parameters. X-ray diffraction in such eutectics may exhibit an anisotropic macroscopic absorption factor if the penetration depth of the X-rays is large compared with their planar size. As a consequence, the reflected X-ray intensity may depend on a third angle γ , i.e. a rotation of the sample about the diffraction vector s additionally to the usual pole figure angles α , β which describe the orientation of the diffraction vector s with respect to the sample coordinate system. It is thus necessary to measure three-dimensional generalized pole figures instead of conventional two-dimensional ones.

GENERAL DEFINITIONS

In order to describe the orientation distribution in binary lamellar eutectics the lamellae orientation distribution has to be considered along with the crystal orientation distributions in both phases. In order to describe these orientation distributions we fix a coordinate system K_A in the sample and a system K_B^α in each crystal of the phase α . The orientation of the crystal is then described by the rotation g^α which transforms K_A into K_B^α , Figure 1.

$$K_B^\alpha = g^\alpha \cdot K_A \quad (1)$$

FIGURE 1 The sample coordinate system K_A , the crystal coordinate system K_B^α , the crystal orientation g^α and the lamellae orientation n .

A crystal direction k^α is then parallel to the sample direction x so that

$$x = g^\alpha \cdot k^\alpha, \quad x_i = g_{ij}^\alpha \cdot k_j^\alpha \quad (2)$$

The orientation distribution in the lamellar eutectic is defined by the volume fractions of crystals having the orientation g^α and the normal direction n to the lamellae plane. The normal direction can be referred to the sample coordinate system or to the crystal coordinate system, Figure 1. Hence, the orientation distribution in the phase α can be described by the two equivalent but complementary functions

$$\begin{aligned} \frac{dV^\alpha}{V^\alpha} &= F^\alpha(g, x) dg dx \\ \frac{dV^\alpha}{V^\alpha} &= \phi^\alpha(g, k) dg dk \end{aligned} \quad (3)$$

where V^α is the volume fraction of the phase α such that

$$V^1 + V^2 = 1 \quad (4)$$

The two functions F^α and ϕ^α are related to each other according to Eq. (2)

$$\begin{aligned} F^\alpha(g, g \cdot k) &= \phi^\alpha(g, k) \\ \phi^\alpha(g, g^{-1} \cdot x) &= F^\alpha(g, x) \end{aligned} \quad (5)$$

The textures in the usual sense are obtained by integration over x

or k

$$\begin{aligned} f^\alpha(g) &= \int F^\alpha(g, x) dx \\ f^\alpha(g) &= \int \phi^\alpha(g, k) dk \end{aligned} \quad (6)$$

The lamellae distribution functions are obtained by integration over g

$$\begin{aligned} L^\alpha(x) &= \int F^\alpha(g, x) dg \\ \Lambda^\alpha(k) &= \int \phi^\alpha(g, k) dg \end{aligned} \quad (7)$$

The lamellae orientation distribution functions $L^\alpha(x)$ in both phases are assumed to be identical

$$L^1(x) = L^2(x) \quad (8)$$

This relationship is not strictly necessary but will hold, for example, if one assumes a constant and orientation independent lamellae thickness in both phases. A similar relationship is, of course, not required for the crystallographic lamellae distribution function $\Lambda(k)$.

In the following we shall restrict our considerations mostly to one of the phases. The upper index α will then be omitted. Particular axis distribution functions of the functions F and ϕ can be defined which describe the volume fraction of crystals, the crystal direction h of which is parallel to the sample direction y while the lamellae normal direction is either x or k .

$$\begin{aligned} D(h, y, x) &= \frac{1}{2\pi} \int_{h \parallel y} F(g, x) d\psi \\ \Delta(h, y, k) &= \frac{1}{2\pi} \int_{h \parallel y} \phi(g, k) d\psi \end{aligned} \quad (9)$$

They can be interpreted as pole figures or inverse pole figures depending on whether h or y is kept constant. The total axis distribution function is then obtained by

$$A(h, y) = \frac{1}{2\pi} \int_{h \parallel y} f(g) d\psi = \begin{aligned} &= P_h(y) \quad (\text{pole figure}) \\ &= R_y(h) \quad (\text{inverse pole figure}) \end{aligned} \quad (10)$$

It is the integral over the particular axis distribution functions

$$A(h, y) = \int D(h, y, x) dx = \int \Delta(h, y, k) dk \quad (11)$$

The functions are assumed to be normalized in multiples of the random density (mrd-units). This means

$$F_r = \phi_r = f_r = L_r = \Lambda_r = D_r = \Delta_r = A_r = 1 \quad (12)$$

where r indicates the random distribution. The orientation element dg and the solid angular elements dx and dk are defined such that

$$\int dg = \int dx = \int dk = 1 \quad (13)$$

The first part of this definition agrees with the definition used formerly (Bunge, 1982) whereas the second and third part deviate from the formerly used definitions by the factor $1/4\pi$. It seems, however, more convenient to use the analogous definition in the space of orientations g and in the space of directions x or k . Since the functions defined above are distribution functions their integrals over the whole orientation space are always identical with the corresponding integrals over the random functions. Hence, it is

$$\begin{aligned} \int f(g) dg &= 1 \\ \int L(x) dx &= \int \Lambda(k) dk = \int P_h(y) dy = \int R_y(h) dh = 1 \\ \int F(g, x) dg dx &= \int \phi(g, k) dg dk = 1 \end{aligned} \quad (14)$$

The sample is assumed to have the symmetry G^s with the symmetry operations g^s and u^s (symmetry rotations and inversions). This symmetry is symbolized by one dot on top of the functional symbol

$$G^s = \{g^s, u^s\}, \quad (\cdot) \quad (15)$$

The crystals have the symmetry G^c with the operations g^c and u^c . This symmetry is symbolized by two dots

$$G^c = \{g^c, u^c\}, \quad (:) \quad (16)$$

FIGURE 2 Influence of crystal symmetry g^c and sample symmetry g^s on the crystal orientation g and the lamellae normal direction x in the sample coordinate system and k in the crystal coordinate system.

Finally, the shape symmetry of the lamellae is to be considered. It is an axial symmetry $D_{\infty h}$. When the lamellae are considered in the crystal coordinate system, a reduced crystal symmetry can be defined which contains the operations common to G^c and $D_{\infty h}$. This symmetry may be characterized by

$$G'^c = \{g'^c, u'^c\}, \quad (\square) \quad (17)$$

In the most general cases when the lamellae are not in special orientation with respect to the crystal lattice then this symmetry is triclinic (1 or $\bar{1}$ respectively). In other cases it may be higher.

The defined distribution functions must obey the crystal and sample symmetry in the following ways, Figure 2.

$$\dot{F}(g^c \cdot g \cdot g^s, g^s \cdot x) = \dot{F}(g, x) \quad (18)$$

$$\dot{\phi}(g^c \cdot g \cdot g^s, g^{c^{-1}} \cdot k) = \dot{\phi}(g, k)$$

$$\dot{f}(g^c \cdot g \cdot g^s) = \dot{f}(g) \quad (19)$$

$$\dot{L}(g^s \cdot x) = \dot{L}(x) \quad (20)$$

$$\dot{A}(g^c \cdot k) = \dot{A}(k)$$

$$\dot{D}(g^c \cdot h, g^s \cdot y, g^s \cdot x) = \dot{D}(h, y, x) \quad (21)$$

$$\dot{\Delta}(g^c \cdot h, g^s \cdot y, g^c \cdot k) = \dot{\Delta}(h, y, k)$$

$$\dot{A}(g^c \cdot h, g^s \cdot y) = \dot{A}(h, y) \quad (22)$$

Furthermore, positive and negative lamellae normal direction

cannot be distinguished. Hence, it is also necessary

$$\begin{aligned} F(g, -x) &= F(g, x) \\ \phi(g, -k) &= \phi(g, k) \end{aligned} \quad (23)$$

Besides the general case, defined by Eq. (3) we consider two specific cases of texture and lamellae orientation distribution:

In the *first specific case* it is assumed that texture and lamellae orientation are independent of each other. Then the functions F and ϕ take on the specific form

$$\begin{aligned} F(g, x) &= f(g) \cdot L(x) \\ \phi(g, k) &= f(g) \cdot \Lambda(k) \end{aligned} \quad (24)$$

Because of the relationship Eq. (5) between the two functions F and ϕ we obtain with eq. (7)

$$\begin{aligned} L(x) &= \int \phi(g, g^{-1} \cdot x) dg \\ \Lambda(k) &= \int F(g, g \cdot k) dg \end{aligned} \quad (25)$$

from which it follows with Eq. (24)

$$\begin{aligned} L(x) &= \int f(g) \cdot \Lambda(g^{-1} \cdot x) dg \\ \Lambda(k) &= \int f(g) \cdot L(g \cdot k) dg \end{aligned} \quad (26)$$

The particular axis distribution functions take on the form

$$\begin{aligned} D(h, y, x) &= A(h, y) \cdot L(x) \\ \Delta(h, y, k) &= A(h, y) \cdot \Lambda(k) \end{aligned} \quad (27)$$

This case of independent lamellae and orientation distribution may be *further specialized* assuming that all the lamellae are perpendicular to the sample direction x_0 . The lamellar distribution function $L(x)$ then takes on the form

$$L(x) = \delta_{xx_0} \quad (28)$$

and Eq. (24a) reads

$$F(g, x) = f(g) \cdot \delta_{xx_0} \quad (29)$$

Replacing $L(g \cdot k)$ in Eq. (26) by Eq. (28) one obtains for the crystallographic lamellae distribution function in this case:

$$\Lambda(k) = \int_{k \parallel x_0} f(g) d\psi = A(k, x_0) = R_{x_0}(k) \quad (30)$$

The lamellae distribution function $\Lambda(k)$ is then the inverse pole figure of the sample direction x_0 .

In the *second specific case* it is assumed that the lamellae are perpendicular to one specific crystal direction k_0 and its crystallographically symmetrical directions. Hence it is

$$\Lambda(k) = \sum_{g^c} \delta_{k, g^c \cdot k_0} \quad (31)$$

The crystals with lamellae perpendicular to k_0 (and not to any other symmetrically equivalent direction $g^c \cdot k_0$) may have the orientation distribution function $\phi^{k_0}(g)$. This function must have sample symmetry but it may possess only the reduced crystal symmetry defined in Eq. (17).

$$\phi^{k_0}(g'^c \cdot g \cdot g^s) = \phi^{k_0}(g) \quad (32)$$

The function $\phi(g, k)$ then takes on the form:

$$\phi(g, k) = \frac{1}{N^c} \sum_{g^c} \phi^{g^c \cdot k_0}(g) \cdot \delta_{k, g^c \cdot k_0} \quad (33)$$

which can also be written:

$$\phi(g, k) = \frac{1}{N^c} \sum_{g^c} \phi^{k_0}(g^{c^{-1}} \cdot g) \cdot \delta_{k, g^c \cdot k_0} \quad (34)$$

This function has the full crystal and sample symmetry according to Eq. (18b). The texture function $f(g)$ is obtained according to Eq. (6b) by:

$$f(g) = \frac{1}{N^c} \sum_{g^c} \phi^{k_0}(g^c \cdot g) \quad (35)$$

In the definition of the function ϕ^{k_0} it was assumed that the lamellae were perpendicular to the crystal direction k_0 . Hence, the lamellae distribution function $L(x)$ in the sample coordinate system is the distribution of crystal directions k_0 with respect to the sample coordinate system K_A corresponding to the crystals the orientation

distribution of which is $\phi^{k_0}(g)$. This is the k_0 -pole figure $P'_{k_0}(x)$ of the function $\phi^{k_0}(g)$

$$L(x) = \frac{1}{2\pi} \int_{k_0 \parallel x} \phi^{k_0}(g) d\psi = P'_{k_0}(x) \quad (36)$$

Since the function $\phi^{k_0}(g)$ does not have the full crystal symmetry the pole figures of the crystallographically equivalent directions $g^c \cdot k_0$ of this function are in general different from $P'_{k_0}(x)$

$$P'_{g^c \cdot k_0}(x) \neq P'_{k_0}(x) \quad (37)$$

The k_0 pole figure of the total texture $f(g)$ defined according to Eq. (10) corresponds to the full crystal symmetry, Eq. (22)

$$P_{g^c \cdot k_0}(x) = P_{k_0}(x) \quad (38)$$

Hence, in general, the lamellae distribution function $L(x)$ is not identical with the k_0 -pole figure of the total texture $f(g)$

$$P_{k_0}(x) = \frac{1}{N^c} \sum_{g^c} P'_{g^c \cdot k_0}(x) \neq L^{k_0}(x) \quad (39)$$

The function $F(g, x)$ is related to $\phi(g, k)$ according to Eq. (5b). Hence, in the specific case considered here, it is according to Eq. (32):

$$F(g, x) = \frac{1}{N^c} \sum_{g^c} \phi^{k_0}(g^{c^{-1}} \cdot g) \cdot \delta_{g^{-1} \cdot x, g^c \cdot k_0} \quad (40)$$

In order to obtain the general axis distribution function $D(h, y, x)$ the function $F(g, x)$ is to be integrated according to Eq. (7a) over all those orientations g for which the crystal direction h is parallel to the sample direction y . At the same time, according to Eq. (40) the function $F(g, x)$ is different from zero only if one of the crystallographically equivalent directions $g^c \cdot k_0$ is parallel to the sample direction x . The directions x and y are well defined sample directions. The angle between them be

$$\alpha = \angle(x, y) \quad (41)$$

Similarly the crystal directions h and $g^c \cdot k_0$ are well defined directions, too, the angles between which may be

$$\beta^c = \angle(h, g^c \cdot k_0) \quad (42)$$

The integral Eq. (7a) over the function Eq. (40) is thus different from zero only if one of the angles β^c is identical with the angle α . Hence, three cases may be distinguished according to whether none of the angles β^c , one β^c , or several ones are equal to α . In these cases the integral $D(h, y, x)$ over $F(g, x)$ takes on the forms

$$D(h, y, x) = \begin{cases} \phi(g_0) & (\beta^c = \alpha) \\ 0 & (\beta^c \neq \alpha) \end{cases} \quad (43)$$

where the orientation g_0 is defined by

$$g_0 = \{h \parallel y, k_0 \parallel x\} \quad (44)$$

and the function $\phi(g_0)$ has the form

$$\phi(g_0) = \frac{1}{N} \sum_{i=1}^N \phi^{k_0}(g^{c_i^{-1}} \cdot g_0) \quad (45)$$

The summation in Eq. (45) is to be extended over all crystal symmetry operations for which $\beta^c = \alpha$ is fulfilled.

SERIES EXPANSIONS

The considered functions can be developed into series of spherical harmonics (ref. Bunge, 1982):

$$F(g, x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{n=-1}^{+l} \sum_{\lambda=0}^{\infty} \sum_{p=-\lambda}^{+\lambda} C_{l\lambda}^{mnp} T_l^{mn}(g) K_{\lambda}^p(x) \quad (46)$$

$$\phi(g, k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{n=-l}^{+l} \sum_{\lambda=0}^{\infty} \sum_{p=-\lambda}^{+\lambda} Q_{l\lambda}^{mnp} T_l^{mn}(g) K_{\lambda}^p(k)$$

whereby the symmetries have not been taken into account. With Eq. (5) and expressing $K_{\lambda}^p(g \cdot h)$ and $K_{\lambda}^p(g^{-1} \cdot x)$ by the addition theorem

$$K_{\lambda}^p(g \cdot k) = \sum_{s=-\lambda}^{+\lambda} T_{\lambda}^{*ps}(g) \cdot K_{\lambda}^s(k) \quad (47)$$

$$K_{\lambda}^p(g^{-1} \cdot x) = \sum_{s=-\lambda}^{+\lambda} T_{\lambda}^{sp}(g) \cdot K_{\lambda}^s(x)$$

Eq. (46) can be written

$$F(g, g \cdot k) = \sum_{lmn\lambda ps} C_{l\lambda}^{mnp} T_l^{n*}(g) T_{\lambda}^{nps}(g) K_{\lambda}^s(k) \quad (48)$$

$$\phi(g, g^{-1} \cdot x) = \sum_{lmn\lambda ps} Q_{l\lambda}^{mnp} T_l^{mn}(g) T_{\lambda}^{sp}(g) K_{\lambda}^s(x)$$

If the products of the harmonics T are expressed by sums and the Clebsh-Gordan coefficients these relationships take on the form

$$F(g, g \cdot k) = \sum_{lmn\lambda ps} \sum_{l'=|l-\lambda|}^{|l+\lambda|} C_{l\lambda}^{mnp} (l\lambda mp | l'q)(l\lambda ns | l'r) T_l^{q*}(g) K_{\lambda}^s(k) \quad (49)$$

$$\phi(g, g^{-1} \cdot x) = \sum_{lmn\lambda ps} \sum_{l'=|l-\lambda|}^{|l+\lambda|} Q_{l\lambda}^{mnp} (l\lambda mp | l'q)(l\lambda ns | l'r) T_l^{q*}(g) K_{\lambda}^s(x)$$

Comparison with Eq. (46) gives the relationships between the coefficients

$$C_{l\lambda}^{qrs} = \sum_{lmnp} Q_{l\lambda}^{mnp} (l\lambda mp | l'q)(l\lambda ns | l'r) \quad (50)$$

$$Q_{l\lambda}^{qrs} = \sum_{lmnp} C_{l\lambda}^{mnp} (l\lambda mp | l'q)(l\lambda ns | l'r)$$

According to Eq. (6) and Eq. (46) the texture takes on the form

$$f(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{n=-l}^{+l} C_l^{mn} T_l^{mn}(g) \quad (51)$$

with the coefficients

$$C_l^{mn} = \frac{1}{\sqrt{4\pi}} C_{l0}^{mn0} = \frac{1}{\sqrt{4\pi}} Q_{l0}^{mn0} \quad (52)$$

The lamellae distribution functions can be expressed

$$L(x) = \sum_{\lambda=0}^{\infty} \sum_{p=-\lambda}^{+\lambda} C_{0\lambda}^{00p} K_{\lambda}^p(x) \quad (53)$$

$$\Lambda(k) = \sum_{\lambda=0}^{\infty} \sum_{p=-\lambda}^{+\lambda} Q_{0\lambda}^{00p} K_{\lambda}^p(k)$$

The particular axis distribution functions are expanded in the form

$$D(h, y, x) = \sum_{lmn\lambda p} \frac{4\pi}{2l+1} C_{l\lambda}^{mnp} K_l^{*m}(h) K_l^n(y) K_\lambda^p(x) \quad (54)$$

$$\Delta(h, y, k) = \sum_{lmn\lambda p} \frac{4\pi}{2l+1} Q_{l\lambda}^{mnp} K_l^{*m}(h) K_l^n(y) K_\lambda^p(k)$$

The total axis distribution function has the form

$$A(h, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{n=-l}^{+l} \frac{4\pi}{2l+1} C_l^{mn} K_l^{*m}(h) K_l^n(y) \quad (55)$$

which may be written in terms of the total pole figure

$$P_h(y) = \sum_{l=0}^{\infty} \sum_{n=-l}^{+l} F_l^n(h) K_l^n(y) \quad (56)$$

with the coefficients

$$F_l^n(h) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} C_l^{mn} K_l^{*m}(h) \quad (57)$$

whereby the coefficients C_l^{mn} are further expressed by Eq. (52). In the specific case of texture and lamellae distribution being independent of each other the coefficients C and Q take on the specific form

$$C_{l\lambda}^{mnp} = C_{l0}^{mn0} \cdot C_{0\lambda}^{00p} \quad (58)$$

$$Q_{l\lambda}^{mnp} = Q_{l0}^{mn0} \cdot Q_{0\lambda}^{00p}$$

The specific function ϕ^{k_0} , Eq. (32); may be written in the form

$$\phi^{k_0}(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{n=-l}^{+l} C_l^{mn}(k_0) T_l^{mn}(g) \quad (59)$$

with

$$C_l^{mn}(k_0) = \sum_{\lambda=0}^{\infty} \sum_{p=-\lambda}^{+\lambda} Q_{l\lambda}^{mnp} K_\lambda^p(k_0) \quad (60)$$

THE ABSORPTION FACTOR IN A LAMELLAR SANDWICH STRUCTURE

In order to describe X-ray diffraction in a polycrystalline, textured, lamellar eutectic, anisotropic absorption, as an essential factor, has

to be taken into account. We consider X-ray reflection in a two-phase material. The intensity reflected at the lattice planes perpendicular to h in the phase α in the sample direction y is given by

$$I_h^\alpha(y) = I_0 \cdot V^\alpha \cdot P_h^\alpha(y) \cdot A^\alpha \quad (61)$$

where I_0 is the intensity of the incident beam, V^α is the volume fraction of the phase α , $P_h^\alpha(y)$ is its pole figure and A^α is an absorption factor which depends on the paths of the X-rays in the two phases (Bunge, 1985)

We consider two limiting cases:

In the case of *large particles*, compared with the penetration depths of the X-rays, the path of rays reflected in the phase α is nearly exclusively in this same phase α , Fig. 3a. This case is identical with single phase reflection. Hence, the absorption factor

FIGURE 3 Reflection of X-rays in one phase of a two-phase material. (a) Large particles, (b) Small particles.

FIGURE 4 X-ray diffraction with the diffraction plane parallel to the plane of the eutectic lamellae corresponds to the large particle case.

has the form

$$A^{\alpha} = \frac{1}{\mu^{\alpha}} \quad (62)$$

where μ^{α} is the absorption coefficient of the phase α .

In the case of *small particles* the path of rays reflected in the phase α is in both phases, Figure 3b. If the particles are small enough and do not have a preferred arrangement, we may assume in a good approximation

$$A^{\alpha} = \frac{1}{\bar{\mu}}, \quad \bar{\mu} = V^1 \mu^1 + V^2 \mu^2 \quad (63)$$

We now consider a lamellar eutectic, the lamellar diameter of which is large compared with the penetration depth, the lamellar thickness, however, is small. Then X-ray reflection corresponds to the large particle case only if the plane of the lamellae is parallel to the reflection plane, Figure 4, in all other cases it corresponds to the small particle case.

FIGURE 5 Definition of the experimental coordinate system K_E (a) and an experimental direction r (b) normal to the lamellae plane.

We assume an experimental coordinate system K_E , Figure 5a, made up of the diffraction direction s , the diffraction plane normal t and a third direction q perpendicular to both.

$$K_E = \{s, q, t\} \quad (64)$$

The normal direction n to the lamellae plane may be parallel to the experimental direction r expressed by the angles θ and ρ in K_E , Figure 5b.

$$r = \{\theta, \rho\} \quad (65)$$

In the most general case the absorption factor A^α is a function of the lamellar orientation r with respect to K_E . The shape of this function depends on the lamellae thicknesses D^α , the respective absorption coefficients μ^α and the Bragg angle ϑ

$$A^\alpha(r) = A_{D, \mu, \vartheta}^\alpha(\theta, \rho) \quad (66)$$

A general expression for the absorption coefficient $A^\alpha(r)$ and numerical values will be given elsewhere.

In the specific case of large but thin lamellae and sufficiently different absorption coefficients the absorption factor can be ap-

FIGURE 6 The absorption factor in the case of large but thin lamellae with sufficiently different absorption coefficients μ^1 and μ^2 .

proximated in the form

$$A^\alpha(r) = \frac{1}{\bar{\mu}} + \left(\frac{1}{\mu^\alpha} - \frac{1}{\bar{\mu}} \right) \delta_{r,t} \quad (67)$$

which is shown schematically in Figure 6.

X-RAY DIFFRACTION IN LAMELLAR EUTECTICS

In order to describe X-ray diffraction in polycrystalline, textured lamellar eutectics the sample orientation Γ with respect to the experimental coordinate system K_E has to be specified as is shown in Figure 7a

$$K_A = \Gamma \cdot K_E \quad (68)$$

It is the rotation Γ which transforms the experimental coordinate system K_E into the sample coordinate system K_A . An experimental direction r is then parallel to a sample direction x such that

$$r = \Gamma \cdot x, \quad r_i = \Gamma_{ij} \cdot x_j \quad (69)$$

The sample orientation Γ may be expressed by three Euler angles $\omega\chi\phi$ which are the rotation angles of the Eulerian cradle of a

FIGURE 7 The sample orientation Γ relative to the experimental coordinate system K_E . (a) The sample in the system K_E , (b) The directions y (diffraction directions s) and z (diffraction plane normal t) in the sample coordinate system, expressed by the three angles $\alpha\beta\gamma$.

texture goniometer (ref. Bunge and Puch, 1984)

$$\Gamma = \{\omega, \chi, \phi\} \quad (70)$$

Another description of the sample orientation Γ is obtained by specifying the sample directions y and z parallel to the experimental directions s and t respectively, i.e. the diffraction direction and the diffraction plane normal, Figure 7b, by

$$\begin{aligned} s &= \Gamma \cdot y \\ t &= \Gamma \cdot z \end{aligned} \quad (71)$$

In normal pole figure measurements only the sample direction y is relevant. It is expressed by the angles α, β in the sample coordinate system K_A . The direction z is perpendicular to y . Hence, it can be fixed by a third angle γ , i.e. a rotation about the diffraction direction $y \parallel s$ which is irrelevant in usual pole figure measurements. Hence, the sample orientation can also be expressed by the three angles $\alpha\beta\gamma$ or the two directions y, z .

$$\Gamma = \{y, z\} = \{\alpha, \beta, \gamma\} \quad (72)$$

as is shown in Figure 7b.

With an absorption factor according to Eq. (66) the reflected X-ray intensity, Eq. (61), can be written more precisely using the particular pole figure instead of the total one

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \cdot \int D^\alpha(h, y, x) \cdot A^\alpha(\Gamma \cdot x) dx \quad (73)$$

With the specific form of the absorption factor according to Eq. (67) this expression takes on the form

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \left[\frac{1}{\bar{\mu}} P_h^\alpha(y) + \left(\frac{1}{\mu^\alpha} - \frac{1}{\bar{\mu}} \right) D^\alpha(h, y, z) \right] \quad (74)$$

The reflected intensity depends on the full sample orientation Γ and not only on two orientation angles α, β as in the case of the usual pole figures. Hence, in the case of X-ray diffraction in polycrystalline, textured lamellar eutectics the usual two-dimensional pole figures are to be replaced by generalized three-dimensional pole figures such that

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \cdot Q_h^\alpha(\Gamma) \quad (75)$$

where the sample orientation Γ may be expressed according to Eq. (70) or to Eq. (72). This latter form shows that the three-dimensional pole figures depend on a rotation of the sample about the diffraction direction s which is not the case in the usual pole figures, especially those of single phase materials.

The experimental arrangement (Figure 5) exhibits a certain symmetry G^e with the operations g^e and u^e symbolized by three dots

$$G^e = \{g^e, u^e\}, \quad (\because) \quad (76)$$

This symmetry depends on the experimental conditions i.e. whether X-ray diffraction is based on normal or anomalous scattering. In these cases it is

$$\begin{aligned} G^e &= D_{2h} = mmm && \text{normal scattering} \\ &= C_{2v} = 2mm && \text{anomalous scattering} \end{aligned} \quad (77)$$

In the case of anomalous scattering the two-fold axis is parallel to the diffraction vector s and in the case of normal scattering an inversion centre is added, Figure 8.

Since the sample also has a symmetry described in Eq. (15), the

FIGURE 8 Experimental symmetry in the case of normal diffraction, mmm and anomalous diffraction, $2mm$.

three-dimensional pole figure $Q_h(\Gamma)$ Eq. (75) has the symmetries

$$\dot{Q}_h(g^s \cdot \Gamma \cdot g^e) = \dot{Q}_h(\Gamma) \quad (78)$$

The three-dimensional pole figure as a function of the sample orientation Γ with respect to the experimental coordinate system K_E must not be mixed up with the three-dimensional orientation distribution function (ODF) which is a function of the crystal orientation g with respect to the sample coordinate system K_A .

In the *specific case* of lamellae orientation independent of the texture, Eq. (73) is transformed by Eq. (27a) into

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \cdot P_h^\alpha(y) \cdot \int L(x) \cdot A^\alpha(\Gamma \cdot x) dx \quad (79)$$

With Eq. (67) this takes on the form

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \cdot P_h^\alpha(y) \cdot \left[\frac{1}{\bar{\mu}} + \left(\frac{1}{\mu^\alpha} - \frac{1}{\bar{\mu}} \right) \cdot L(z) \right] \quad (80)$$

This means, each pole figure is multiplied by the same lamellae distribution factor.

An even *more specific case* is obtained if the lamellae are perpendicular to one specific sample direction x_0 which may then be taken as the x_3 direction. This case is obtained, for instance, if textured foils of two different materials are stacked upon each other such as to form an “artificial eutectic”. In this case Eq. (79) takes on the form

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \cdot P_h^\alpha(y) \cdot A^\alpha(r) \quad (81)$$

where $r = \{\theta, \rho\}$ is the normal direction x_3 of the sample expressed in the experimental coordinate system K_E .

Equation (81) can be used to determine the absorption factor for lamellar structures experimentally. With the absorption factor Eq. (67) finally this takes on the form

$$I_h^\alpha(\Gamma) = I_0 \cdot V^\alpha \cdot P_h^\alpha(y) \cdot \left[\frac{1}{\bar{\mu}} + \left(\frac{1}{\mu^\alpha} - \frac{1}{\bar{\mu}} \right) \delta_{x_3, t} \right] \quad (82)$$

The absorption factor was measured experimentally in a “synthetic eutectic” prepared by stacking aluminium and copper foils upon each other. This fulfills the two conditions mentioned above that texture and lamellae orientation are independent of each other and that the lamellae are perpendicular to a specific sample direction. A cross section through the stack is shown in Figure 9. The lamellae

FIGURE 9 Cross-section through a “synthetic eutectic”, i.e. a stack of aluminium and copper foils.

TABLE I
The lamellae thicknesses, linear absorption coefficients and penetration depths for MoK α -radiation for the sandwich sample Figure 9

	D [μm]	μ^α [cm^{-1}]	Penetration depth $1/\mu^\alpha$ [μm]
Al	31	14.3	699
Cu	3.4	444	22.5

thicknesses, absorption coefficients and penetration depths are shown in Table I. X-ray diffraction measurements were carried out at the five-circle diffractometer in the HASYLAB at DESY in Hamburg because of the small angular divergence of the X-ray beam available with this equipment (Kupčik *et al.* 1983). The absorption coefficients correspond to MoK α -radiation. Measurements were carried out with the sample normal direction (which is perpendicular to the foil normal direction) parallel to the diffraction direction s . The angle γ is a rotation of the sample about this direction whereby in the orientation $\gamma = 0$ the direction q is parallel to the lamellae plane. The measured intensity of the Al(200) reflection is shown in Figure 10. Hence, the form of the absorption factor is, in this case, very near to that assumed in Eq. (67) and Figure 6.

In the specific case of lamellae perpendicular to one crystal direction k_0 and its symmetrically equivalent directions the axis distribution function $D(h, y, z)$ in Eq. (67) has the form Eq. (43). Hence the reflected intensity, in this case, can be written:

$$\begin{aligned}
 I_h^\alpha(\Gamma) &= I_0 \cdot V^\alpha \left[\frac{1}{\bar{\mu}} P_h^\alpha(y) + \left(\frac{1}{\mu^\alpha} - \frac{1}{\bar{\mu}} \right) \phi(g_0) \right] \\
 &= I_0 \cdot V^\alpha \frac{1}{\bar{\mu}} P_h^\alpha(y) \quad (83)
 \end{aligned}$$

where q and $\phi(g_0)$ are defined according to Eqs. (44) and (45) respectively whereby the sample direction x is now the specific direction z perpendicular to y , hence, it is $\alpha = 90^\circ$. The two cases are illustrated in Figure 11. In the first case the reflected intensity depends on all three orientation angles of the sample whereas in the second case it depends only on the usual pole figure angles α , β and

FIGURE 10 The absorption factor for MoK_α -radiation measured in the “synthetic eutectic”, Figure 9, as a function of the rotation angle γ about the sample normal direction.

FIGURE 11 The lamellae normal direction is parallel to the crystal direction k_0 . The normal direction h to the reflecting lattice, plane is: (a) perpendicular to k_0 . Definition of the orientation g_0 , (b) not perpendicular to k_0 .

not on the rotation γ about the diffraction vector s . If a three-dimensional dependence of the intensity according to Eq. (83a) is found for several pole figures h_i then the crystallographic orientation k_0 of the lamellae normal direction can easily be determined by the condition:

$$k_0 \perp h_i \quad (\text{all } i) \quad (84)$$

CONCLUSIONS

In order to describe the orientation distribution in lamellar eutectics or eutectoids it is necessary to consider the crystallographic orientation of the crystallites and the geometrical orientation of the lamellae in both phases. In the most general case, the distributions of these two orientations may be interdependent. It is then necessary to define two combined orientation-shape distribution functions which depend on five orientation angles. The five-dimensional orientation distribution functions may be written in two forms, with the lamellae orientation referred either to the crystal coordinate system K_B or to the sample coordinate system K_A . The textures, in the conventional definition, and the lamellae distribution functions are only three-dimensional or two-dimensional projections, respectively, of these five-dimensional distribution functions. Textures and lamellae distribution functions together are thus, in general, not sufficient to describe the orientation relations in lamellar eutectics completely.

The reflection of X-rays in eutectic alloys may be influenced by the crystallographic texture as well as by the lamellae orientation if the penetration depth of the X-rays is large compared with lamellae thickness and small compared with their planar dimensions. In this case the different absorption of the X-rays in the two phases may give rise to a macroscopically anisotropic absorption factor. The reflected X-ray intensity then depends on the lamellae distribution additionally to the usual dependence on the crystallographic orientation distribution, i.e. the texture. As a consequence of this anisotropic absorption the reflected intensity may depend on a rotation of the sample about the direction of the diffraction vector s . This is quite different from usual pole figure measurements where

the reflected intensity depends only on the orientation of the diffraction vector s (the bisectrix between incident and reflected beam) relative to the sample coordinate system which is usually described by the two pole figure angles α , β . In the case of eutectic alloys it is thus necessary to measure three-dimensional generalizations of the usual two-dimensional pole figures which depend also on a third angle γ i.e. a rotation about the diffraction vector s . These three-dimensional pole figures depend on a complete five-dimensional orientation-lamellae-distribution. This provides the possibility to obtain information about the lamellae orientation distribution function of which, by optical methods, only a planar section function can be obtained. It also provides information about the crystallographic orientations of lamellae planes and their respective distribution which is otherwise difficult to obtain.

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