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ON SOME INVENTORY CONTROL PROBLEMS WITH STATE-DEPENDENT PARAMETERS *

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ABSTRACT

This article analyzes discrete Markov processes that arise in some general control problems of a storage (or a dam) with an infinite capacity. Employing a special "step-wise" structure of the transition matrices of these processes, the author obtains conditions for their ergodicity and develops an implementable algorithm for finding the generating function of the corresponding distributions. This algorithm is applied to a dam with a multilevel state-dependent control strategy, according to which parameters of the input and output processes may depend on the amount of the water in the dam. The method is illustrated with a case of a single-level control strategy where explicit results are obtained.

Key words: Inventory, Dams, Markov Chains, Ergodicity, Control, State Dependent, Parameters
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1. INTRODUCTION.

In maintaining an inventory it is important to control its size to avoid both overstocking and insufficiency of the supply.

A natural way of achieving this goal is to implement a control strategy which regulates input and output processes depending on present inventory. In such cases the inventory systems can, for example, respond to an excessively large inventory by either decreasing the rate of the input or increasing the rate of the output. Similarly, the corresponding adjustments can be made if an inventory falls below a prescribed level. All these can be realized with a multilevel control strategy (MCS).

According to this strategy, the set of so-called control levels $\{n_0, n_1,..., n_r\}$, $n_0 < n_1 < n_2 < ... < n_r$ (non-negative integers) is introduced so that the policy of replenishing the inventory and of filling the orders depends on the interval (n_i, n_{i+1}), $i = 0, 1, 2,..., r - 1$, containing the magnitude of the current inventory supply.

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A particular case of using this strategy ("a single-level control") in Moran's problem for a dam was previously considered by the author [1].

In this article a general approach to inventory control problems with an MCS is developed. A special class of Markov chains describes the behavior of the random inventory supply in these problems. A condition of the ergodicity as well as the generating function of the stationary probabilities of states of these Markov chains are determined. The general results are applied to Moran's problem for a dam with state-dependent parameters. Explicit solutions are obtained in some cases.

2. MARKOV CHAINS IN MORAN'S PROBLEM WITH AN MCS.

Suppose that X_k units of volume of water enter a reservoir during the interval $(k,k+1)$ and Y_k units flow out at the instant $k+1$, $k = 0, 1, 2, 3, \ldots$. Denote by Z_k the supply of water in the reservoir at the instant $k+0$. We will introduce an MCS as follows. Let $\{n_0, n_1,\ldots,n_r\}$ and $\{m_0, m_1,\ldots, m_r\}$ be 2 sets of nonnegative integers such that $n_0 < n_1 < \ldots < n_r = n$, $n_0 = 0$, $m_0 \geq 1$, $m_r = m$, $m_i < n_i$, $i = 1, 2, \ldots, r$. Suppose that

$$P\{X_k = j \mid n_i \leq Z_k < n_{i+1}\} = g_{ij}, \qquad \text{and } P\{X_k = j \mid Z_k \geq n\} = g_j,$$

$$i = 0, 1, 2,\ldots, r-1, \quad j = 0, 1, 2, \ldots, \qquad \sum_{j=0}^{\infty} g_{ij} = 1, \ \sum_{j=0}^{\infty} g_j = 1.$$

Suppose also that the release rate m_i, $i = 0, 1, 2, \ldots, r$, per unit time depends on the supply of water Z_k at moment k, $k = 1, 2,\ldots$ in the following way. The output rate is kept at m_0 while $n_0 \leq Z_k < n_1$. As soon as Z_k reaches n_1, the rate changes to m_1 and is kept at this value while $n_1 \leq Z_k < n_2$, and so on. If the content Z_k of the dam reaches the highest level $n_r = n$, the water is released at rate $m_r = m$ as long as $Z_k \geq n$. Similarly, when the supply of water Z_k becomes less than n_i, $i = 1, 2,\ldots,r$ the output rate falls to m_{i-1}. In other words,

$$Y_k = \begin{cases} \min[Z_k + X_k, m_j], & \text{if } n_i \leq Z_k < n_{i+1}, \ i = 0, 1, 2,\ldots, r-1 \\ \min[Z_k + X_k, m], & \text{if } Z_k \leq n \end{cases}$$

Under these conditions the sequence of random variables Z_k, $k = 0, 1, 2,\ldots$, obviously constitutes a homogeneous Markov chain with the recursion relation

$$(1) \qquad Z_{k+1} = \begin{cases} \max\{0,\ Z_k + X_k - m_i\}, \text{ if } n_i \le Z_k < n_{i+1},\ i = 0, 1, 2,\ldots, r-1, \\ \max\{0,\ Z_k + X_k - m\}, \text{ if } Z_k \ge n \end{cases}$$

We define

$$g_i(x) = \sum_{j=0}^{\infty} g_{ij}x^j,\ \ g(x) = \sum_{j=0}^{\infty} g_j x^j,\ \ |x| \le 1,\ \ f_s = \sum_{j=1}^{s} g_{oj},$$

$$p_{ij}^{(k)} = P\{Z_k = j \mid Z_0 = i\},\ \ i,j, = 0, 1, 2,\ldots, k \ge 1.$$

$$p_j = \lim_{k\to\infty} p_{ij}^{(k)}, \qquad\qquad P(x) = \sum_{j=0}^{\infty} p_j x^j.$$

In accordance with (1) the transition matrix of the Markov chain Z_k, $k = 0, 1, 2,\ldots$ has the form (3) given on the following page for which we define:

$$a_{i0} = f_{m_0 - 1}\ , \qquad i = 0,1,2,\ldots\ldots m_0,$$

$$a_{ij} = \begin{cases} g_{j-i+m_0}^{(0)} & \text{if } j - i + m_0 \ge 0,\ i = 0, 1, 2,\ldots, n_1 - 1 \\ 0 & \text{otherwise,} \qquad j = 1, 2, 3,\ldots \end{cases}$$

$$(2) \qquad a_{ij} = \begin{cases} g_{j-i+m_1}^{(1)} & \text{if } j - i + m_1 \ge 0,\ i = n_1, n_1 + 1,\ldots, n_2 - 1 \\ 0 & \text{otherwise,} \qquad j = 0, 1, 2,\ldots \end{cases}$$

$$a_{ij} = \begin{cases} g_{j-i+m_{r-1}}^{(r-1)} & \text{if } j - i + m_{r-1} \ge 0,\ i = n_{r-1}, n_{r-1} + 1,\ldots, n - 1 \\ 0 & \text{otherwise,} \qquad j = 0, 1, 2,\ldots \end{cases}$$

$$
(3)\quad A = \begin{array}{c}
n_0 = 0 \\ 1 \\ \cdots\cdots \\ m_0 \\ \\ \\ n_1 \\ \\ \\ n_2 \\ \\ \\ \cdots\cdots \\ n = n_r \\ n+1 \\ n+2 \\ \\ \cdots\cdots
\end{array}
\begin{bmatrix}
a_{00}\ a_{01} \cdots\cdots & & & & & & & \\
a_{10}\ a_{11} \cdots\cdots & & & & & & & \\
\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & & \\
a_{m_0,0}\ a_{m_0,1} \cdots & & & & & & & \\
 & a_{m_0+1,1} & & & & & & \\
 & & \cdots\cdots\cdots\cdots & & & & & \\
 & & a_{n_1,n_1-m_1}\ a_{n_1,n_1-m+1} \cdots & & & & & \\
 & & & a_{n_1+1,\,n_1-m+1} \cdots & & & & \\
 & & & & \cdots\cdots\cdots\cdots & & & \\
 & & & & & a_{n_2,\,n_2-m_2}\ a_{n_2,\,n_2-m_2+1} \cdots & & \\
 & & & & & & a_{n_2+1,\,n_2-m_2+1} \cdots & \\
 & & & & & & & \cdots\cdots\cdots\cdots \\
 & & & & & & & \cdots\cdots\cdots\cdots \\
 & & & & & & g_0\ g_1\ g_2 \cdots & \\
 & & & & & & & g_0\ g_1 \cdots \\
 & & & & & & & g_0 \\
 & & & & & & & \ddots \\
\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & & \\
\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & & \\
\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & &
\end{bmatrix}
$$

The Markov chain with the transition matrix of the form (3) is encountered not only in Moran's problem for a reservoir but also in many other systems of inventory control and queueing systems with state-dependent parameters. Because of this, it makes sense to consider some general properties of Markov chains with transition matrix of the form (3).

3. QUASI-TRIANGULAR TRANSITION STEP-MATRICES.

Definition. A stochastic matrix of the form (3) with arbitrary entries a_{ij} , $i = 0, 1, ...,n$, $j = 0, 1, 2, ...$ is said to be a *quasi-triangular step-matrix* (or a step $\Delta_{m,n}$-matrix).

Let us notice that a quasi-triangular step-matrix is a particular case of non-essential Δ_m-matrix introduced in [2], and, therefore, some of the general results of [2] hold true in this case. But, due to its specific structure, the Markov chain with transition quasi-triangular step-matrix possesses additional properties which are considered below.

Let Z_k, $k = 1, 2,...$ be a Markov chain with transition quasi-triangular step-matrix A. Denote by $A_i(x)$ the generating function of the entries a_{ij} of the i-th row of the matrix A, $|x| < 1$, $i, j, = 0, 1, 2,...$. Our goal is to find the stationary distribution of the chain (Z_k).

The following theorem is valid.

Theorem. A Markov chain (Z_k) with transition irreducible and nonperiodic quasi-triangular step-matrix A is ergodic if

$$A_i(1) < \infty,\ i = 0, 1, 2,..., n \tag{4}$$

and

$$g'(1) < m.$$

Under these conditions:

(A) the generating function $P(x)$ of the stationary distribution $\{p_0, p_1, p_2,...\}$ of the chain (Z_k) satisfies the relation:

$$P(x) = \sum_{i=0}^{\infty} p_i x^i = \frac{\sum_{i=0}^{n} p_i(A_i(x)x^m - g(x)x^i)}{x^m - g(x)} \tag{5}$$

(B) the function $x^m - g(x)$ has exactly m roots (counting multiplicities) in the disk $|x| \le 1$; those roots, which lie on the boundary Γ of the disk, are simple and, for some s, they are s-th roots of 1.

Proof. (A) Let us prove the ergodicity of the chain (Z_k). Taking into account the structure of the matrix A and setting $x_j = j$, it is easy to show that

$$\left(\sum_{j=0}^{\infty} a_{ij}x_j - x_i\right)\Big|_{x_j = j} = \begin{cases} A_i'(1) - i, & \text{if } i = 0, 1, \ldots, n-1, \\ g'(1) - m, & \text{if } i = n, n+1, \ldots . \end{cases}$$

It is obvious that if $g'(1) < \infty$ then $A_i'(1) - i < \infty$. Therefore, by the criterion of Moustafa [3], we can conclude that the chain (Z_k) is ergodic. In this case the limits

$$\lim_{k\to\infty} p_{ij}^{(k)} = \lim_{k\to\infty} P\{Z_{k+1} = j \mid Z_k = i\} = p_j,\ i, j = 0, 1, 2, \ldots$$

exist and the vector $\mathbf{P} = (p_0, p_1, p_2 \ldots)$ of the stationary probabilities satisfies the matrix equation

(6) $$\mathbf{P} = \mathbf{P}A$$

Applying usual generating function techniques to this equation and making use of the structure of the matrix A, we obtain (5).

(B) Let us prove the second part of the theorem. Denote by s the greatest common divisor of m and all k, such that $g_k \neq 0$ in the expression

$$\sum_{k=0}^{\infty} g_k x^k = g(x).$$

We will prove that if $x^m - g(x)$ has roots on the unit circle $|x| = 1$ then these roots are all s-th roots of 1. First suppose that x_0 is a root of $x^m - g(x)$ such that $|x_0| = 1$. Then $|g(x_0)| = |x_0^m| = 1$. On the other hand

$$|g(x_0)| = \left|\sum_{k=0}^{\infty} g_k x_0^k\right| \leq \sum g_k |x_0|^k = 1$$

and the equality is attained if and only if $x_0^k = |x_0^k| = 1$ for any k such that $g_k \neq 0$ and $x_0^m = 1$. In this case, obviously, $x^s = 1$. Otherwise, if x_0 is any root of the equation $x^s - 1 = 0$, we obtain $g(x_0) = 1 = x_0^m$ which implies that x_0 is a root of $x^m - g(x) = 0$.

Now we return to the roots of $x^m - g(x)$ inside Γ. First we suppose that $s = 1$, that is, that $g(x)$ cannot be expressed as a function of x^k for any integer k, $k > 1$. Then the function $x^m - g(x)$ has only one root on the unit circle. This root is equal to 1 and it is simple since $g'(1) = m$. We will prove that in this case $x^m - g(x)$ has exactly $m - 1$ roots inside Γ. Consider the auxiliary function

$$f(x) = \frac{1 - x^{-m}g(x)}{1 - x^{-1}}.$$

Clearly, $f(x) \neq 0$ for all $x \in \Gamma$ since the numerator of $f(x)$ may be zero only if $x = 1$, but $f(1) = m - g'(1) > 0$. Suppose now that $\mathrm{Ind}_r\ f(x)$ denotes the difference between the number of the roots and the number of the poles of the function $f(x)$ inside Γ.

By the principle of the argument:

$$\mathrm{Ind}_r\ f(x) = \frac{1}{2\pi}\ \Delta_r\ \mathrm{Arg}\ f(x) = \mathrm{Ind}_r[1 - x^{-m}g(x)] - \mathrm{Ind}_r(1 - x^{-1}) \tag{7}$$

where $\Delta_r\ \mathrm{Arg}\ f(x)$ is the increment of the argument of $f(x)$ when the argument φ of $x = e^{i\varphi}$ increases from 0 to 2π.

Let us consider the right side of (7). Since $\mathrm{Arg}\ f(1) = 0$, it is readily seen that

$$\lim_{\varphi\to\pm 0}\ \mathrm{Arg}[1 - e^{-im\varphi}g(e^{i\varphi})] = \lim_{\varphi\to\pm 0}\ [1 - e^{-i\varphi}] = \pm\frac{\pi}{2}$$

At the same time if $\varphi \in (0, 2\pi)$, then $|e^{-im\varphi}g(e^{i\varphi})| < 1$ and, hence,

$$\mathrm{Arg}\ [1 - e^{-im\varphi}g(e^{i\varphi})] \in (-\pi/2, \pi/2).$$

It follows that

$$\mathrm{Ind}_r[1 - x^{-m}g(x)] = -\frac{1}{2} = \mathrm{Ind}_r[1 - x^{-1}]$$

and, therefore,

$$\mathrm{Ind}_r\ f(x) = 0.$$

From the fact that $f(x) = \dfrac{x^m - g(x)}{x^{m-1}(x - 1)}$ has exactly $m - 1$ poles inside Γ, we conclude that the number of roots of $f(x)$ inside Γ is also $m - 1$.

Now consider the general case: $s > 1$. Let

$$m_1 = \frac{m}{s} \text{ and } g_1(x) = \sum_{i=0}^{\infty} g_i x^{i/s}.$$

By definition of s it follows that m_1 and all exponents of x in the expression for $g_1(x)$ are integers and their greatest common divisor is 1. Moreover, it is obvious that $g_1(1) < m_1$.

Therefore, by the previous reasonings, we conclude that $x^{m_1} - g_1(x)$ has exactly $m_1 - 1$ roots in the region $|x| < 1$ and one root $x = 1$ on the boundary. The set of all s-th roots of the roots of $x^{m_1} - g_1(x)$ contains all roots of $x^m - g(x)$. On the other hand, it is obvious that any root of $x^m - g(x)$ raised to the s-th power is a root of $x^{m_1} - g_1(x)$. Therefore, the set of all roots of $x^m - g(x)$ is described completely. The number of them is m: m - s are inside the region and s simple roots are on the boundary $|x| = 1$, $|x| \leq 1$. The theorem is proved.

4. FINDING THE STATIONARY PROBABILITIES OF STATES OF MARKOV CHAIN WITH A TRANSITION QUASI-TRIANGULAR STEP-MATRIX.

The presence of the unknown probabilities p_k, $k = 0, 1, 2,...,n$ in the right side of the relation (5) prevent us from finding the generating function $P(x)$. We can, of course, take advantage of the existence of m roots of the function $x^m - g(x)$ in the unit disc $|x| \leq 1$ (together with $x = 1$) and of the analyticity of $P(x)$ in this region to obtain m relations for determining p_k, $k = 0, 1,...,n$. However, since $n > m^*$, the function $P(x)$ will still be undetermined.

In order to decrease the number of unknown probabilities in the right side of (5), we will take advantage of a specific structure of the matrix A. From the coordinate-wise form of the equation (6) it follows that all probabilities P_j, $j \notin J$, $J = \{0, 1, 2, 3,..., m_0 - 1, n_1 - m_1 + m_0, n_1 - m_1 + m_0 + 1,...,n_1 - 1, n_2 - m_2 + m_1, n_2 - m_2 + m_1 + 1,..., n_2 - 1,..., n_r - m_r + m_{r-1}, n_r - m_r + m_{r-1} + 1,...,n_r - 1\}$ are uniquely expressed in terms of p_j, $j \in J$.

Indeed, from the coordinate-wise equations (6) we obtain:

$$p_{m_0} = \sum_{j \in J} L_{m_0,j}\, p_j = -\frac{1}{a_{m_0,0}} [(a_{00} - 1)\, p_0 + a_{10}p_1 + a_{20}p_2 + \dots + a_{m_0-1,0}p_{m_0-1}]$$

$$p_{m_0+1} = \sum_{j \in J} L_{m_0+1,j}\, p_j = -\frac{1}{a_{m_0+1,1}} [a_{01}p_0 + (a_{1,1} - 1)p_1 + a_{21}p_2 + \dots + a_{m_0,1}\, p_{m_0}]$$

$$= -\frac{1}{a_{m_0+1,1}\, a_{m_0 0}} [(a_{01}a_{m_0,0} - a_{m_0,1}a_{00})p_0$$

* This condition is natural and corresponds to real situations.

$$+ (a_{1,1}a_{m_0 0} - a_{m_0 0} - a_{m_0 1}a_{10})p_1 + (a_{21} - a_{m_0 1}a_{20})p_2$$

$$+ \dots + (a_{m_0-1,1}a_{m_0,0} - a_{m_0,1}a_{m_0-1,0})p_{m_0-1}]$$

and so on.

Thus we have:

$$p_i = \sum_{j \in J} L_{ij}p_j, \qquad i = 0, 1, 2,\dots \tag{8}$$

and if $i \in J$, then

$$L_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

Therefore,

$$\sum_{i=0}^{n} p_i\,[A_i(x)x^m - g(x)x^i] = \sum_{i=0}^{n} \sum_{j \in J} L_{ij}p_j[A_i(x)x^m - g(x)x^i]$$

$$= \sum_{j \in J} p_j \sum_{i=0}^{n} L_{ij}\,[A_i(x)x^m - g(x)x^i] = \sum_{j \in J} p_j h_j(x)$$

where

$$h_j(x) = \sum_{i=0}^{n} L_{ij}[A_i(x)x^m - g(x)x^i].$$

Thus,

$$P(x) = \frac{\sum_{j \in J} p_j h_j(x)}{x^m - g(x)} \tag{9}$$

Now, the number of unknown probabilities in the right side of (9) is $(m_0 - 1) + 1 + (n_1 - 1) - (n_1 - m_1 + m_0) + 1 + \dots + (n_r - 1) - (n_r - m_r + m_{r-1}) + 1 = m_0 = m$ and, therefore, is equal to the number of roots of the function in the right side of (9) in the unit disc $|x| \leq 1$. Since $P(x)$ should be analytic in this region we obtain m relations for determining p_j,

$j \in J^*$, and, therefore, obtain the function P(x). Some examples of this approach are given in section 5; see also [1] and [2].

5. A MULTILEVEL CONTROL STRATEGY IN MORAN'S PROBLEM FOR A DAM.

An explicit solution for r = 0, 1. Now let us come back to the problem of finding the stationary distribution of the chain (Z_k) described in section 1. Since the transition matrix of the chain (Z_k) is a quasi-triangular step-matrix, the general approach of section 3 can be applied in this case. To find the generating function P(x) it is necessary to use the coordinate-wise form of the equation (6) taking into account (3). It enables us to determine all coefficients L_{ij}, $i = 0, 1, 2,...,n$, $j \in J$ in (8) and, by the same token, to determine the functions $h_j(x)$, $j \in J$. Then, after obtaining all m roots of the function $x^m - g(x)$ in the disc $|x| \le 1$, we can find m probabilities p_j, $j \in J$ and obtain the generating function P(x) in (9).

If the number r of levels in the multilevel control strategy is not too large, this approach allows us to obtain the functions $h_j(x)$, $j \in J$ and the generating function P(x) explicitly. For example, if r = 0 (it means that $m_0 = m_1 = ... = m_r = m$), the matrix A is a quasi-triangular Δ_m-matrix considered in [2] and, according to [2],

$$P(x) = \frac{\sum_{j=0}^{m-1} p_j h_j(x)}{x^m - g(x)}, \text{ where } h_j(x) = (x - 1) \Big|_{m-k-1} \frac{g(x)}{1 - x}$$

(the symbol $|_k F(x)$ denotes the k-th truncation of the Taylor's series of F(x)). If r = 1 (a single-level control strategy), the generating function P(x) also can be found explicitly [1]:

$$P(x) = \frac{\sum_{j=0}^{m_0-1} p_j h_j(x) + \sum_{j=n-m+m_0}^{n-1} p_j h_j(x)}{x^m - g(x)}$$

where

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* It can be proved that the corresponding system of m equations in m unknown probabilities p_j, $j \in J$ has a unique solution.

$$h_j(x) = \begin{cases} x^{m-m_0}(x-1)\left[\left|_{m_0-j-1} \frac{x^j g(x)}{1-x} + (g_0(x)x^{m-m_0} - g(x)) \right.\right. \\ \times \left|_{m_0-j-1} \frac{x^j g(x)}{1-x} \right] \frac{1}{x^{m_0} - g_0(x)}, \quad j = 0, 1, 2, \ldots, m_0 - 1 \\ x^j(g_0(x)x^{m-m_0} - g(x)) \qquad\qquad ,j = n - m + m_0, \ldots, n - 1 \end{cases}$$

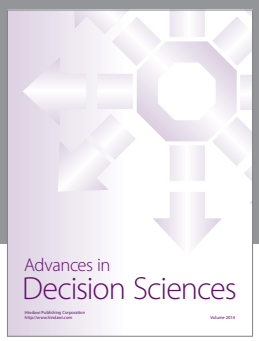
Using a similar approach with [1], it is possible to find the explicit expressions for P(x) in case of $r \geq 2$. However, these expressions are too cumbersome and inconvenient in practice, so that in determining r-level control strategy in Moran's problem for $r \geq 2$ it is better to use general algorithm described in section 4.

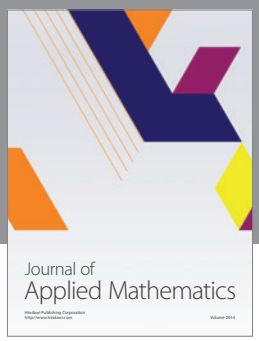
REFERENCES

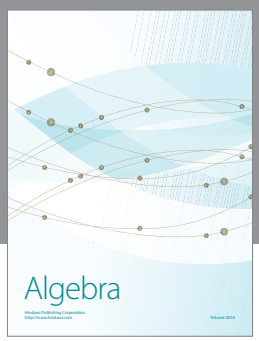
1. Abolnikov, L. A Single-Level Control in Moran's Problem with Feedback, *"Operations Research Letters."* V.2, N1, 1983, 16-19.

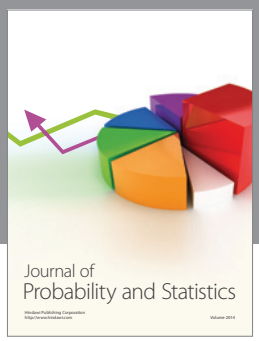
2. Abolnikov, L. M. Investigation of a Class of Discrete Markov Processes, *"Izvestia Academii Nauk USSR, Tekhnicheskaya Kibernetika,"* N2, 1977, 69-82. (English Translation: Engineering Cybernetics, 15, 1977, 51-73)

3. Moustafa, M. D. Input-Output Markov Processes. *"Proc. Koninkiyke Nederlande Akad. Wetenchappen* 1957, 60, 112-118.









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