LARGE LEVEL CROSSINGS OF A RANDOM POLYNOMIAL

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ABSTRACT

We know the expected number of times that a polynomial of degree n with independent random real coefficients asymptotically crosses the level K , when K is any real value such that $(K^2/n) \rightarrow 0$ as $n \rightarrow \infty$. The present paper shows that, when K is allowed to be large, this expected number of crossings reduces to only one. The coefficients of the polynomial are assumed to be normally distributed. It is shown that it is sufficient to let $K \geq \exp(nf)$ where f is any function of n such that $f \rightarrow \infty$ as $n \rightarrow \infty$.

Key words and phrases: Number of level crossings, number of real roots, Kac-Rice formula, random algebraic equation.

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1. INTRODUCTION

Let

$$P(x) = \sum_{i=0}^{n-1} a_i x^i,$$

where $a_0, a_1, a_2, \dots, a_{n-1}$ is a sequence of independent, normally distributed random variables assuming real values only with mathematical expectation zero and variance unity. Let $N_K(a, b)$ be the number of real roots of the algebraic equation $P(x) = K$ in the interval (a, b) , where K is a constant independent of x and multiple roots are counted only once. Kac [5] and [6] obtained the first asymptotic result concerning the mathematical expectation of the number of real roots of the equation $P(x) = 0$, establishing that for n sufficiently large, $EN_0(-\infty, \infty) \sim (2/\pi) \log(n)$. We know from the work of Ibragimov and Maslova [3] that the same asymptotic relation exists when the coefficient a_i ($i = 0, 1, 2, \dots, n-1$) belongs to the domain of attraction of the normal law with zero means and $\text{prob}(a_i \neq 0) > 0$. Their method also allowed them [4] to prove that, when $E(a_j) \neq 0$ ($j = 0, 1, 2, \dots, n-1$) the number of real roots is half that of the previous case. Further, in the case of K

not necessarily zero, Farahmand [2] found that the above asymptotic formula for $EN_K(-\infty, \infty)$ remains valid if K is bounded. For unbounded K , he showed that, if (K^2/n) tends to zero for all sufficiently large n ,

$$EN_K(-1, 1) \sim (1/\pi) \log(n/K^2)$$

and

$$EN_K(-\infty, -1) = EN_K(1, \infty) \sim (2\pi)^{-1} \log(n),$$

and if (K^2/n) tends to a non-zero positive constant,

$$EN_K(-\infty, \infty) \sim (1/\pi) \log(n).$$

As expected, this shows that by increasing $|K|$ the number of real roots of $P(x) = K$ decreases with probability one. In this work we study the case when $|K|$ is very large compared to n , which together with the previous result gives a more complete idea of the behaviour of the random algebraic polynomial.

2. LARGE LEVEL CROSSINGS

From looking at Maslova's work [7], we already know that the distribution of the number of real roots of $P(x) = 0$ is asymptotically normal. Thus as $|x|$ increases the polynomial will cut the x -axis an ever-decreasing number of times. In other words, with probability one, the polynomial has the tendency to stay away from the x -axis. But Das [1] showed that the number of turning points of $P(x)$ is $\{(\sqrt{3}+1)/\pi\} \log n$ which is about 1.366 times the real roots, revealing there are $(0.732/\pi) \log n$ oscillations which do not occur between two axis crossings. The following theorem shows that although the polynomial will cut the large level $|K|$ for some value of x , with probability one it will not oscillate after that and it will increase (decrease) to infinity $(-\infty)$.

Theorem. *If $\exp(nf) \leq |K| < \infty$, where f is any function of n such that it tends to infinity as n tends to infinity, then for n sufficiently large $EN_K(-\infty, \infty) \sim 1$.*

Proof. For the equation $P(x) = K$ when $K > 0$ there is, with probability one, at least one real root. For n even ($n - 1$ odd) this occurs on the positive or negative far side of the x -axis depending on whether a_{n-1} is positive or negative. For n odd ($n - 1$ even) there will be at least two real roots when a_{n-1} is positive, each one occurring on the positive and negative far side of the x -axis, and none when a_{n-1} is negative; and since a_{n-1} has equal probability of being positive or negative, on the average there is at least one real root. For

$K < 0$ the above argument is valid when we replace positive a_{n-1} by negative a_{n-1} . Hence $EN_K(-\infty, \infty) \geq 1$, and for the proof of the theorem we only have to show the upper limit is 1 as well.

Let

$$\alpha = \sum_{i=0}^{n-1} x^{2i} \quad , \quad \beta = \sum_{i=1}^{n-1} i^2 x^{2i-2}$$

$$(2.1) \quad \gamma = \sum_{i=1}^{n-1} ix^{2i-1} \quad \text{and} \quad \Delta = \alpha\beta - \gamma^2;$$

then the Kac-Rice formula [8] for the expected number of real roots of the equation $P(x) - K = 0$ gives us

$$\begin{aligned} EN_K(a, b) &= \int_a^b \{ (\Delta^{1/2} / \pi \alpha) \exp(-\beta K^2 / 2\Delta) \\ &\quad + (|K\gamma\alpha^{-3/2}| / \pi) \exp(-K^2 / 2\alpha) \operatorname{erf}(|K\gamma| \sqrt{2\alpha\Delta}) \} dx \\ (2.2) \quad &= \int_a^b \{ I_1(x) + I_2(x) \} dx, \end{aligned} \quad (\text{say}).$$

First we prove that $EN_K(-1, 1)$ tends to zero as n tends to infinity. Since for $0 \leq x \leq 1 - 1/n^2$ and for all sufficiently large n we have

$$\operatorname{Max}\{x^{2n-2}(1 - x^2)^2 + x^{2n}\} \leq 2e^{-1}$$

and

$$\operatorname{Max}\{2nx^{2n-2}(1 - x^2) + x^{2n}\} < 2e^{-1/2}$$

from (2.1), we can show that

$$\begin{aligned}
\beta/\Delta &= (1-x^2) \{ (1-x^{2n})^2 - n^2 x^{2n-2} (1-x^2)^2 \}^{-1} \{ -n^2 x^{2n-2} (1-x^2)^2 \\
&\quad - 2nx^{2n}(1-x^2) + (1+x^2)(1-x^{2n}) \} \\
&\geq (1-x^2) [-n^2 x^{2n-2} (1-x^2)^2 - x^{2n} + 1 + x^2 \{ -2nx^{2n-2} (1-x^2) - x^{2n} + 1 \}] \\
&> 2(1-e^{-1}-e^{-1/2})(1-x^2) > (0.05) .
\end{aligned}$$

Let $\lambda = (0.025) K^2$. Then with the same method of proof for (2.11) of [2], we can show

$$\begin{aligned}
&\int_0^{1-1/n^2} (\Delta^{1/2}/\alpha) \exp(-K^2 \beta/2\Delta) dx \\
&\leq \left(\frac{1}{2}\right) \log \left\{ n^2 \left(2 - \frac{1}{n^2}\right) \right\} - \left(\frac{1}{2}\right) (1+1/\lambda)^{-1/2} \log \left\{ \frac{(1+1/\lambda)^{1/2} + 1 - 1/n^2}{(1+1/\lambda)^{1/2} - 1 + 1/n^2} \right\} \\
&= \left(\frac{1}{2}\right) \log \left\{ n^2 \left(2 - \frac{1}{n^2}\right) \right\} - \left(\frac{1}{2}\right) (1+1/\lambda)^{-1/2} \log \left\{ \frac{2+1/2\lambda - 1/n^2 + o(1/\lambda^2)}{1/2\lambda + 1/n^2 + o(1/\lambda^2)} \right\} \\
&= \left(\frac{1}{2}\right) \{ 1 - (1+1/\lambda)^{-1/2} \} \log(2n^2) + o(1/n) \\
(2.3) \quad &= \frac{1}{2} (1+1/\lambda)^{-1/2} o(1/\lambda) \log(2n^2) + o(1/n),
\end{aligned}$$

which tends to zero as n tends to infinity. On the other hand, by Kac [5, page 318], we have

$$\begin{aligned}
& \int_{1-1/n^2}^1 (\Delta^{1/2} / \alpha) \exp(-K^2 \beta / 2\Delta) dx \\
& \leq \int_{1-1/n^2}^1 (\Delta^{1/2} / \alpha) dx \leq \int_{1-1/n^2}^1 (2n-1)^{1/2} (1-x)^{-1/2} dx \\
(2.4) \quad & = (2n-1)^{1/2} (2/n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Hence from (2.3) and (2.4) we have

$$(2.5) \quad \int_0^1 I_1(x) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Also since

$$\gamma = x(1-x^{2n})(1-x^2)^{-2} - nx^{2n-1}(1-x^2)^{-1}$$

for $0 \leq x \leq 1-1/n$ and all sufficiently large n , we have

$$(2.6) \quad \gamma \alpha^{-3/2} \leq x(1-x^{2n})^{-1/2}(1-x^2)^{-1/2} \leq x(1-e^{-2})^{-1/2}(1-x^2)^{-1/2}.$$

On the other hand, for $1-1/n \leq x \leq 1$ and all sufficiently large n , we have

$$\begin{aligned}
 \gamma \alpha^{-3/2} &\leq \alpha^{-3/2} (n/x) \sum_{i=0}^{n-1} x^{2i} \\
 &\leq (n/x) (1-x^2)^{1/2} (1-x^{2n})^{-1/2} \\
 (2.7) \qquad &\leq (2n^{1/2}/x) (1-e^{-2})^{-1/2}.
 \end{aligned}$$

Now let $t = K(1-x^2)^{1/2}$ then from (2.6) and (2.7), and from the fact that $\operatorname{erf}(z) \leq \sqrt{\pi/2}$ for any z , we have

$$\begin{aligned}
 &\int_0^1 I_2(x) dx \\
 &\leq |K| \{2\pi(1-e^{-2})\}^{-1/2} \left[\int_0^{1-1/n} x(1-x^2)^{-1/2} \exp\{-K^2(1-x^2)/(1-x^{2n})\} dx \right. \\
 &\qquad \qquad \qquad \left. + 2\sqrt{n} \exp(-K^2/3n) \int_{1-1/n}^1 (1/x) dx \right] \\
 &< \left(\frac{1}{2}\right) \{2\pi(1-e^{-2})\}^{-1/2} \int_{K(2/n-1/n^2)^{1/2}}^K \exp(-t^2) dt \\
 &\qquad \qquad \qquad + \sqrt{2n/\pi} |K| (1-e^{-2})^{-1/2} (n-1)^{-1} \exp(-K^2/3n) \\
 &< |K| \{2\pi(1-e^{-2})\}^{-1/2} \{1-(2/n-1/n^2)^{1/2}\} \exp\{-K^2(2/n-1/n^2)\} \\
 (2.8) \qquad &+ \sqrt{2n/\pi} |K| (1-e^{-2})^{-1/2} (n-1)^{-1} \exp(-K^2/2n),
 \end{aligned}$$

which tends to zero as n tends to infinity. Since $I_1(-x) = I_1(x)$ and $I_2(-x) = I_2(x)$, from (2.5) and (2.8), we have

$$(2.9) \quad EN_K(-1, 1) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

To find $EN_K(1, \infty)$ we let $y = 1/x$. From (2.2) we have

$$(2.10) \quad EN_K(1, \infty) = \int_0^1 \{I_1(1/y) + I_2(1/y)\} y^{-2} dy.$$

Then, in order to show that the first integral tends to zero as n tends to infinity, let $h(y) = ny^{n-1}(1-y^2) / (1 - y^{2n})$. From (2.2) we find

$$(2.11) \quad \begin{aligned} (\beta / \Delta) &= y^{2n-4} (1 - y^2) (1 - y^{2n})^{-2} \{1 - h^2(y)\}^{-1} \{(1 + y^2) (1 - y^{2n}) \\ &\quad + n^2(1 - y^2)^2 - 2n(1 - y^2)\}. \end{aligned}$$

Assuming $b = \{\log(n) / K^2\}^{1/(2n-4)}$ and $a = \{(3/8) \log \log n\}^{1/3}$, we divide the interval $(0, 1)$ into three subintervals, $(0, b)$, $(b, 1-1/na)$, and $(1-1/na, 1)$. For $0 \leq y \leq 1-1/na$ by looking at the functions $y^2 - y^{2n} - y^{2n+2}$ and $n^2(1 - y^2)^2 - 2n(1 - y^2)$ separately, we can easily show that both functions are minimum at $y = 1 - 1/na$. Thus

$$(2.12) \quad \begin{aligned} &\text{Min } \{(1 + y^2) (1 - y^{2n}) + n^2(1 - y^2)^2 - 2n(1 - y^2)\} \\ &= 2 + 4/a^2 - 4/a - \exp(-2/a) - \exp(-2/a + 2/na) + o(1/na) \\ &= (8/3)a^{-3} + o(a^{-4}). \end{aligned}$$

Hence from (2.11) and (2.12), we have

$$(2.13) \quad (\beta / \Delta) \geq y^{2n-4} (1 - y^2) \{ (8/3) a^{-3} + o(a^{-4}) \}.$$

Let $\xi = (8/3)K^2a^{-3}$. Then from (2.2) and (2.13), we have

$$\begin{aligned}
 & \int_0^1 y^{-2} I_1(1/y) dy \\
 & \leq \int_0^b (\Delta^{1/2} / \alpha) dy + \int_b^{1-1/na} (1-y^2)^{-1} \exp \{-\xi y^{2n-4}(1-y^2)\} dy \\
 (2.14) \quad & + \int_{1-1/na}^1 (\Delta^{1/2} / \alpha) dy.
 \end{aligned}$$

Now we show that each of these three terms tends to zero as n tends to infinity. First,

$$\begin{aligned}
 & \int_0^b (\Delta^{1/2} / \alpha) dy \\
 (2.15) \quad & \leq \int_0^b (1-y^2)^{-1} dy = \log \{ (1+b) / (1-b) \} \rightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

since b tends to zero as n tends to infinity. Secondly,

$$\begin{aligned}
 & \int_b^{1-1/na} (1-y^2)^{-1} \exp \{-\xi y^{2n-4}(1-y^2)\} dy \\
 & \leq \exp \{-\xi \log(n) / K^2\} \int_b^{1-1/na} (1-y^2)^{-1} dy \\
 (2.16) \quad & \leq \frac{1}{2} \log(2na-1) \exp \{-\log n / \log \log n\},
 \end{aligned}$$

which also tends to zero as n tends to infinity. Finally, as for (2.4),

$$\begin{aligned}
 \int_{1-1/na}^1 (\Delta^{1/2} / \alpha) dy &\leq \int_{1-1/na}^1 (2n-1)^{1/2} (1-y)^{-1/2} dy \\
 (2.17) \qquad \qquad \qquad &< 2a^{-1/2} (1-1/n)^{1/2} \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Hence from (2.10), (2.14), (2.15), (2.16), and (2.17) we have

$$(2.18) \qquad \int_0^1 y^{-2} I_1(1/y) dy \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since changing y to $-y$ leaves the function $y^{-2} I_1(1/y)$ invariant from (2.18) we have

$$(2.19) \qquad \int_{-1}^0 y^{-2} I_1(1/y) dy \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In the following we obtain an upper estimate for $\int_1^\infty I_2(x) dx$. Let $K^2 / \alpha = u^2$. Since from (2.1) $d\alpha / dx = 2\gamma$, (2.2) gives

$$\begin{aligned}
 \int_1^\infty I_2(x) dx &\leq (2\pi)^{-1/2} \int_1^\infty |K\gamma\alpha^{-3/2}| \exp(-K^2 / 2\alpha) dx \\
 (2.20) \qquad &= (2\pi)^{-1/2} \int_0^{K/\sqrt{2n}} \exp(-u^2 / 2) du \leq 1/2.
 \end{aligned}$$

Hence from (2.19) and (2.20) and from the fact that

$$\int_1^\infty I_2(x) dx = \int_{-\infty}^{-1} I_2(x) dx,$$

we have the proof of the theorem.

Consider the following remark and open problem.

The largest number of K -level crossings of the polynomial $P(x)$ occurs for $K = 0$; after this by increasing $|K|$, the number of crossings decreases, until for $(K^2/n) = o(1)$ it is reduced by half [2, page 709]. While in this paper we dealt with $K \geq \exp(nf)$, the behaviour of the number of crossings between these two limits is not known. But one can expect, with probability one, this number will be no larger than $(1/\pi) \log n$ and not smaller than one. The proof is left as a problem for the reader.

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