Nonlinear Second Order System of Neumann Boundary Value Problems at Resonance*

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Abstract
Let \( f : [0, \pi] \times \mathbb{R}^N \to \mathbb{R}^N \), \( (N \geq 1) \) satisfy Caratheodory conditions, \( g(x) \in \mathcal{L}^1([0, \pi]; \mathbb{R}^N) \). This paper studies the system of nonlinear Neumann boundary value problems
\[
\begin{align*}
    x''(t) + f(t, x(t)) &= g(t), & 0 < t < \pi, \\
    x'(0) &= x'(\pi) = 0.
\end{align*}
\]
This problem is at resonance since the associated linear boundary value problem
\[
\begin{align*}
    x''(t) = g(t), & \quad 0 < t < \pi, \\
    x'(0) &= x'(\pi) = 0,
\end{align*}
\]
has \( \lambda = 0 \) as an eigenvalue. Asymptotic conditions on the nonlinearity \( f(t, x(t)) \) are offered to give existence of solutions for the nonlinear systems. The methods apply to the corresponding system of Lienard-type periodic boundary value problems.

Key words and phrases: Second-order system of Neumann boundary value problems, resonance at infinitely many eigenvalues, absence of \( L_\infty \)-resonance, asymptotic resonance conditions, Fredholm operator

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1 Introduction

Let \( \mathbb{R}^N \) denote the \( N \)-dimensional Euclidean space. For \( x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \), and \( y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N \), let \( |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2} \), and \( <x, y> = x_1y_1 + x_2y_2 + \ldots + x_Ny_N \) denote the Euclidean norm of \( x \) and the inner product of \( x \) and \( y \) in \( \mathbb{R}^N \), respectively. Let

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$f = (f_1, f_2, \ldots, f_N) : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a function satisfying Caratheodory’s conditions, and let $e : [0, \pi] \rightarrow \mathbb{R}^N$ be a Lebesgue integrable function.

This paper is devoted to the study of systems of Neumann boundary value problems

$$-x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi,$$
$$x'(0) = x' (\pi) = 0, \quad (1.1)$$

and

$$x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi,$$
$$x'(0) = x' (\pi) = 0. \quad (1.2)$$

We obtain the existence of a solution for (1.1)-(1.2) when $\int_0^\pi e(t) dt = 0$ and when, for each $i = 1, 2, \ldots, N$, there exists a real number $r_i > 0$ such that

$$i. \quad f_i(t, x)x_i \geq 0,$$

for a.e. $t \in [0, \pi]$ and all $x \in \mathbb{R}^N$ with $|x_i| \geq r_i$, and

$$ii. \quad |f_i(t, x)| \leq \alpha_i(t), \quad (1.5)$$

for a.e. $t \in [0, \pi]$ and all $x \in \mathbb{R}^N$ with $|x_i| \leq r_i$. We give asymptotic conditions on the behavior of $x_i^{-1} f_i(t, x), i = 1, 2, \ldots, N,$ at the first two eigenvalues 0 and 1 of the linear problem

$$x''(t) + \lambda x(t) = 0, \quad 0 < t < \pi,$$
$$x'(0) = x'(\pi) = 0, \quad (1.6)$$

for the problem (1.3)-(1.4).

Our methods can be adapted and similar results obtained for Lienard’s system of equations

$$\pm x''(t) + \left[ \frac{d}{dt} \text{grad} F(x(t)) \right] + f(t, x(t)) = e(t), \quad 0 < t < \pi,$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \quad (1.8)$$

where $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is in $C^2(\mathbb{R}^N, \mathbb{R}), f : [0, 2\pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfies Caratheodory’s conditions, and $e : [0, 2\pi] \rightarrow \mathbb{R}^N$ is Lebesgue integrable. The problem (1.9)-(1.10) was studied by Ianacci and Nkashama in [3], where they give sufficient non-resonance conditions for the existence of a solution. We provide in this paper sufficient resonance conditions for the existence of a solution for the problems (1.1)-(1.2) and (1.3)-(1.4) and accordingly for (1.9)-(1.10) in line with our remark above.

Our results and methods are inspired by the results of Gupta and Mawhin [2] for the problem (1.9)-(1.10) when $N = 1$. We present in Section 2 notations and definitions that we need in this paper. In Section 3 we present some lemmas that are extensions to systems of corresponding lemmas in [3]. We present in Section 4 our theorems giving the existence of solutions for the problems (1.1)-(1.2) and (1.3)-(1.4). Our conditions for the existence of solutions for (1.3)-(1.4) allow resonance at infinitely many eigenvalues of the linear problem (1.7)-(1.8). Finally, in Section 5 we present a theorem for the problem (1.3)-(1.4) sharpening the condition for resonance at infinitely many eigenvalues of the linear problem (1.7)-(1.8) in the absence of $L^\infty$-resonance at the second eigenvalue $\lambda = 1$ of (1.7)-(1.8).
2 Notations and Definitions

Let $\mathbb{R}^N, N \geq 1$, denote the $N$-dimensional Euclidean space. For $x = (x_1, x_2, ..., x_n)$, let

$$|x| = (x_1^2 + x_2^2 + ... + x_N)^{1/2} \quad (2.1)$$

denote the Euclidean norm of $x$ in $\mathbb{R}^N$; and for $x = (x_1, x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let

$$< x, y > = \sum_{i=1}^{N} x_i y_i \quad (2.2)$$

denote the inner product of $x$ and $y$ in $\mathbb{R}^N$.

We shall use the following spaces:

(i) the Lebesgue spaces $L^p([0, \pi], \mathbb{R}^N), 1 \leq p \leq \infty$, with the norms defined by

$$||f||_{L^p_N} = \left[ \sum_{i=1}^{N} \left( \frac{1}{\pi} \int_0^\pi |f_i|^p dt \right)^{2/p} \right]^{1/2}, \text{ for } 1 \leq p < \infty,$$

and

$$||f||_{L^\infty_N} = \left( \sum_{i=1}^{N} |f_i|_{L^\infty}^2 \right)^{1/2}, \text{ for } p = \infty;$$

(ii) the space of $C([0, \pi], \mathbb{R}^N)$ of continuous functions with its usual norm, the norm induced by the Lebesgue space $L^\infty([0, \pi], \mathbb{R}^N)$;

(iii) the Sobolev space $H^1([0, \pi], \mathbb{R}^N)$ defined by

$$H^1([0, \pi], \mathbb{R}^N) = \{ x : [0, \pi] \to \mathbb{R}^N | x \text{ is absolutely continuous and } x' \in L^2([0, \pi], \mathbb{R}^N) \},$$

with the inner product defined by

$$(x, y)_{H^1_N} = \left< \frac{1}{\pi} \int_0^\pi x(t) dt, \frac{1}{\pi} \int_0^\pi y(t) dt \right> + \frac{1}{\pi} \int_0^\pi <x'(t), y'(t)> dt,$$

and the corresponding norm $|| \cdot ||_{H^1_N}$ defined by

$$||x||_{H^1_N} = \left( \frac{1}{\pi} \int_0^\pi |x'(t)|^2 dt + \frac{1}{\pi} \int_0^\pi |x(t)|^2 dt \right)^{1/2};$$

(iv) the Sobolev space $\tilde{H}^1([0, \pi], \mathbb{R}^N)$ defined by

$$\tilde{H}^1([0, \pi], \mathbb{R}^N) = \{ x \in H^1([0, \pi], \mathbb{R}^N) | \int_0^\pi x(t) dt = 0 \}$$

with the norm induced by $H^1([0, \pi], \mathbb{R}^N)$; and

(v) the Sobolev space $W^{2,1}([0, \pi], \mathbb{R}^N)$ defined by

$$W^{2,1}([0, \pi], \mathbb{R}^N) = \{ x : [0, \pi] \to \mathbb{R}^N | x \text{ and } x' \text{ absolutely continuous} \}$$
with the norm defined by
\[ \|x\|_{W^{2,1}_N} = \sum_{j=0}^{2} \|x^{(j)}\|_{L^1_N}, \]
where \(x^{(0)} = x, x^{(1)} = x', x^{(2)} = x''\).

For the sake of simplicity in the notation of the space, we shall omit \(\mathbb{R}^N\) when \(N = 1\).

We note that for \(x \in H^1([0, \pi]; \mathbb{R}^N), x = (x_1, x_2, ..., x_N)\) if and only if \(x_i \in H^1[0, \pi]\), for \(i = 1, 2, ..., N\). Also, every \(x_i \in H^1[0, \pi]\) can be written in the form
\[ x_i(t) = \bar{x}_i + \bar{\xi}_i(t) \]
with \(\bar{x}_i \in \dot{H}^1[0, \pi]\) and \(\bar{\xi}_i = \frac{1}{\pi} \int_0^\pi x_i(t) dt\). Moreover,
\[ \|x_i\|_{H^1} = (\bar{x}_i^2 + \frac{1}{\pi} \int_0^\pi (\bar{\xi}_i'(t))^2 dt)^{1/2}, \]
so that we have
\[ \|x\|_{H^1_N} = (\sum_{i=1}^{N} \|x_i\|_{H^1})^{1/2}. \]

For \(x = (x_1, x_2, ..., x_N) \in L^1([0, \pi], \mathbb{R}^N)\), we write \(\bar{x} = (\bar{x}_1, ..., \bar{x}_N)\), where \(\bar{x}_i = \frac{1}{\pi} \int_0^\pi x_i(t) dt, i = 1, 2, ..., N\) and \(\bar{x} = x - \bar{x}\).

### 3 Technical Lemmas

**Lemma 1** Let \(\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N) \in L^1([0, \pi], \mathbb{R}^N)\) be such that for a.e. \(t \in [0, \pi]\),
\[ \Gamma_i(t) \leq 1, \tag{3.1} \]
for \(i = 1, 2, ..., N\) with strict inequality holding on a subset of \([0, \pi]\) of positive measure. Then there exists a \(\delta = \delta(\Gamma) > 0\) such that for all \(\bar{x} \in \dot{H}^1([0, \pi], \mathbb{R}^N)\) with \(\bar{x}'(0) = \bar{x}'(\pi) = 0, \tag{3.2} \]
\[ B_{\Gamma}(\bar{x}) = \frac{1}{\pi} \int_0^\pi [||\bar{z}'(t)||^2 - \sum_{i=1}^{N} \Gamma_i(t)\bar{z}_i^2(t)] dt \]
\[ \geq \delta \|\bar{z}\|_{H^1_N}^2. \]

*Proof.* Using (3.1), the method of expanding a scalar function \(\bar{x}_i \in \dot{H}^1[0, \pi]\), with \(\bar{x}_i'(0) = 0\) and \(\bar{x}_i'(\pi) = 0\), into a cosine Fourier series, and Parseval's identities for \(\bar{x}_i\) and \(\bar{x}_i'\), we see that
\[ B_{\Gamma}(\bar{x}) = \frac{1}{\pi} \int_0^\pi [||\bar{z}'(t)||^2 - \sum_{i=1}^{N} \Gamma_i(t)\bar{z}_i^2(t)] dt \]
\[ = \sum_{i=1}^{N} \frac{1}{\pi} \int_0^\pi [(\bar{x}_i'(t))^2 - \Gamma_i(t)\bar{z}_i^2(t)] dt \]
\[ \geq 0, \tag{3.3} \]
for all \( \tilde{x} \in \tilde{H}^1([0, \pi], \mathbb{R}^N) \) with \( \tilde{x}'(0) = \tilde{x}'(\pi) = 0 \). Moreover,

\[
B_\Gamma(\tilde{x}) = 0,
\]

if and only if

\[
\tilde{x}(t) = A \cos t,
\]

for some \( A = (A_1, A_2, \ldots, A_N) \in \mathbb{R}^N \). But we then get from (3.4) and (3.5) that

\[
0 = B_\Gamma(\tilde{x}) = \sum_{i=1}^{N} \frac{A_i^2}{\pi} \int_0^{\pi} (1 - \Gamma_i(t)) \cos^2 t dt;
\]

so that by our assumption (3.1) on \( \Gamma_i \) we have \( A_i = 0 \) for every \( i = 1, 2, \ldots, N \), and hence \( \tilde{x} = 0 \).

Let us next assume that the conclusion of the lemma is false. Then there exists a sequence \( \{\tilde{x}_n\}, \tilde{x}_n \in \tilde{H}^1([0, \pi], \mathbb{R}^N) \), such that

\[
B_\Gamma(\tilde{x}_n) \to 0 \quad \text{as} \quad n \to \infty,
\]

\[
||\tilde{x}_n||_{H^1_N} = 1, \quad \text{for every} \quad n = 1, 2, \ldots. \tag{3.6}
\]

We may also assume, by going to a subsequence if necessary, that there exists an \( \tilde{x} \in \tilde{H}^1([0, \pi], \mathbb{R}^N) \) such that

\[
\tilde{x}_n \to \tilde{x} \quad \text{weakly in} \quad H^1([0, \pi], \mathbb{R}^N),
\]

\[
\tilde{x}_n \to \tilde{x} \quad \text{in} \quad C([0, \pi], \mathbb{R}^N). \tag{3.7}
\]

Using Theorem 5.2 in [4], we have that \( B_\Gamma(\tilde{x}) \geq 0 \), even though \( \tilde{x}'(0) \) and \( \tilde{x}'(\pi) \) may not be zero. Also, from (3.7) and the weak lower semicontinuity of the norm in \( H^1([0, \pi], \mathbb{R}^N) \), we have that

\[
||\tilde{x}||_{H^1_N} \leq \liminf_{n \to \infty} ||\tilde{x}_n||_{H^1_N} = 1,
\]

and hence

\[
0 \leq B_\Gamma(\tilde{x}) \leq \liminf_{n \to \infty} B_\Gamma(\tilde{x}_n) = 0.
\]

Thus \( \tilde{x} = 0 \), from the first part of the proof. We next see that

\[
\frac{1}{\pi} \int_0^{\pi} |\tilde{x}_n'|^2 dt = B_\Gamma(\tilde{x}_n) + \frac{1}{\pi} \sum_{i=1}^{N} \int_0^{\pi} \Gamma_i(t)|\tilde{x}_n(t)|^2 dt
\]

\[
\to \quad \frac{1}{\pi} \sum_{i=1}^{N} \int_0^{\pi} \Gamma_i(t)|\tilde{x}_i(t)|^2 dt = \frac{1}{\pi} \int_0^{\pi} |\tilde{x}'(t)|^2 dt.
\]

Thus, \( \tilde{x}_n \to \tilde{x} \) in \( H^1([0, \pi], \mathbb{R}^N) \) and \( ||\tilde{x}||_{H^1_N} = 1 \), which forms a contradiction. Hence the lemma.

\( \square \)

\textbf{Lemma 2} Let \( \Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_N) \in L^1([0, \pi], \mathbb{R}^N) \) and \( \Gamma_\alpha = (\Gamma_{\alpha_1}, \ldots, \Gamma_{\alpha_N}) \in L^1([0, \pi], \mathbb{R}^N) \). Let \( \Gamma_\beta = (\Gamma_{\beta_1}, \ldots, \Gamma_{\beta_N}) \in L^1([0, \pi], \mathbb{R}^N) \) and \( \Gamma_\infty = (\Gamma_{\infty_1}, \ldots, \Gamma_{\infty_N}) \in L^\infty([0, \pi], \mathbb{R}^N) \) be such that

\[
(i) \quad \Gamma = \Gamma_\alpha + \Gamma_\beta + \Gamma_\infty,
\]

\[
(ii) \quad \text{for a.e. } t \in [0, \pi] \text{ and every } i = 1, 2, \ldots, N, \Gamma_{\alpha_i}(t) \leq 1, \tag{3.8}
\]

for all \( \tilde{x} \in \tilde{H}^1([0, \pi], \mathbb{R}^N) \) with \( \tilde{x}'(0) = \tilde{x}'(\pi) = 0 \). Moreover,
with strict inequality holding on a subset of \([0, \pi]\) of positive measure,

\[
(iii) \quad \frac{\pi^2}{3} ||\Gamma_\beta||_{L_N^\infty} + ||\Gamma_\infty||_{L_N^\infty} < \delta(\Gamma_\alpha),
\]

where \(\delta(\Gamma_\alpha) > 0\) is given by Lemma 1.

Then for every \(\tilde{x} \in \tilde{H}^1([0, \pi], \mathbb{R}^N)\) with \(\tilde{x}'(0) = \tilde{x}'(\pi) = 0\),

\[
B_{\Gamma}(\tilde{x}) \geq \left[ \delta(\Gamma_\alpha) - \frac{\pi^2}{3} ||\Gamma_\beta||_{L_N^\infty} - ||\Gamma_\infty||_{L_N^\infty} \right] ||\tilde{x}||_{H_N^1}^2.
\]

**Proof.** Using the fact that \(H^1([0, \pi], \mathbb{R}^N) \subset C([0, \pi], \mathbb{R}^N)\) and the inequalities (see [8])

\[
||\tilde{x}||_{L_N^2} \leq ||\tilde{x}'||_{L_N^2} \leq ||\tilde{x}||_{H_N^1},
\]

\[
||\tilde{x}||_{L_N^\infty} \leq \frac{\pi}{\sqrt{3}} ||\tilde{x}'||_{L_N^2} \leq \frac{\pi}{\sqrt{3}} ||\tilde{x}||_{H_N^1},
\]

for all \(\tilde{x} \in \tilde{H}^1([0, \pi]; \mathbb{R}^N)\), as well as Lemma 1, we see that

\[
B_{\Gamma}(\tilde{x}) = \frac{1}{\pi} \int_0^\pi ||\tilde{x}'(t)||^2 - \sum_{i=1}^N \Gamma_{\alpha,i}(t)\tilde{x}_i^2(t) dt
\]

\[
= \frac{1}{\pi} \int_0^\pi ||\tilde{x}'(t)||^2 - \sum_{i=1}^N \Gamma_{\alpha,i}(t)\tilde{x}_i^2(t) dt
\]

\[
- \frac{1}{\pi} \sum_{i=1}^N \int_0^\pi \Gamma_{\beta,i}(t) + \Gamma_{\infty,i}(t)\tilde{x}_i^2(t) dt
\]

\[
\geq \delta(\Gamma_\alpha) ||\tilde{x}||_{H_N^1}^2 - ||\Gamma_\beta||_{L_N^\infty} ||\tilde{x}||_{L_N^2}^2 - ||\Gamma_\infty||_{L_N^\infty} ||\tilde{x}||_{L_N^\infty}^2
\]

\[
\geq \left( \delta(\Gamma_\alpha) - \frac{\pi^2}{3} ||\Gamma_\beta||_{L_N^\infty} - ||\Gamma_\infty||_{L_N^\infty} \right) ||\tilde{x}||_{H_N^1}^2.
\]

\[
\Box
\]

**Definition 1** For \(x = (x_1, x_2, ..., x_N)\) and \(y = (y_1, y_2, ..., y_N)\) in \(\mathbb{R}^N\), we say \(x \leq y\) if \(x_i \leq y_i\) for every \(i = 1, 2, ..., N\).

**Lemma 3** Let \(\gamma = (\gamma_1, \gamma_2, ..., \gamma_N) \in L^1([0, \pi], \mathbb{R}^N)\) and \(\Gamma = \Gamma_\alpha + \Gamma_\beta + \Gamma_\infty \in L^1([0, \pi], \mathbb{R}^N)\) be as in Lemma 2, and let \(\delta(\Gamma_\alpha)\) be given by Lemma 1. Then for all measurable functions \(p : [0, \pi] \to \mathbb{R}^N\) such that \(\tilde{\gamma} \leq \tilde{p}\), \(p(t) \leq \Gamma(t)\) for a.e. \(t \in [0, \pi]\) and all \(x \in W^{2,1}([0, \pi], \mathbb{R}^N)\) with \(x'(0) = x'(\pi) = 0\),

\[
\frac{1}{\pi} \int_0^\pi < \tilde{x} - \tilde{x}(t), \tilde{x}'(t) + p^T(t)I\tilde{x}(t) > dt
\]

\[
\geq \eta||\tilde{x}||^2 + \left( \delta(\Gamma_\alpha) - \frac{\pi^2}{3} ||\Gamma_\beta||_{L_N^\infty} - ||\Gamma_\infty||_{L_N^\infty} \right) ||\tilde{x}||_{H_N^1}^2.
\]

Here \(\eta = \min\{|\tilde{\gamma}_i| 1 \leq i \leq N\}\), \(I\) denotes the \(N \times N\) identity matrix, \(p^T(t)\) denotes the transpose of the column vector \(\text{col.}\{p_1(t), p_2(t), ..., p_N(t)\}\), and all vectors are understood as column vectors for the purpose of matrix arithmetic.
Proof. For \( x = (x_1, x_2, \ldots, x_N) \in W^{2,1}([0, \pi], \mathbb{R}^N) \) with \( x'(0) = x'(\pi) = 0 \), we have (on integrating by parts and from Lemma 2) that
\[
\frac{1}{\pi} \int_0^\pi < \ddot{x} - \dddot{x}(t), \dddot{x}(t) + p'T(t)I\dddot{x}(t) > dt
\]
\[
= \frac{1}{\pi} \int_0^\pi |\dddot{x}'(t)|^2 dt + \sum_{i=1}^N \frac{1}{\pi} \int_0^\pi p_i(t)(\dddot{x}^2_i - \dddot{x}^2_i(t))dt
\]
\[
= \frac{1}{\pi} \int_0^\pi |\dddot{x}'(t)|^2 - \sum_{i=1}^N p_i(t)\dddot{x}^2_i(t)dt + \sum_{i=1}^N \dddot{p}_i\dddot{x}^2_i
\]
\[
\geq \sum_{i=1}^N \dddot{g}_i\dddot{x}^2_i + [\delta(\Gamma_\alpha) - \frac{\pi^2}{3}||\Gamma_\rho||_{L^1_\mathcal{H}} - ||\Gamma_\infty||_{L^2_\mathcal{H}}]||\dddot{x}||^2_{H^1_\mathcal{H}}
\]
\[
\geq \eta||\dddot{x}||^2 + [\delta(\Gamma_\alpha) - \frac{\pi^2}{3}||\Gamma_\beta||_{L^1_\mathcal{H}} - ||\Gamma_\infty||_{L^2_\mathcal{H}}]||\dddot{x}||^2_{H^1_\mathcal{H}}.
\]
Hence the lemma follows. \( \Box \)

4 Asymptotic Resonance Conditions for the Existence of Solutions

Let \( f : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a function satisfying Caratheodory's conditions, namely,

(i) for each \( x \in \mathbb{R}^N \), the function \( t \in [0, \pi] \rightarrow f(t, x) \in \mathbb{R}^N \) is measurable on \([0, \pi] \);

(ii) for a.e. \( t \in [0, \pi] \), the function \( x \in \mathbb{R}^N \rightarrow f(t, x) \in \mathbb{R}^N \) is continuous; and

(iii) for each \( r > 0 \), there exists a function \( \alpha(t) \in L^1[0, \pi] \) such that \( |f(t, x)| \leq \alpha(t) \) for a.e. \( t \in [0, \pi] \) and all \( x \in \mathbb{R}^N \) with \( |x| \leq r \).

Let \( x = C([0, \pi], \mathbb{R}^N) \) and \( Y = L^1([0, \pi], \mathbb{R}^N) \). Also, let \( Y_1 \subset Y \) denote the subspace of \( Y \) defined by
\[
Y_1 = \{ x \in L^1([0, \pi], \mathbb{R}^N) | x_i(t) \text{ is constant for a.e. } t \in [0, \pi], i = 1, 2, \ldots, N \}; \quad (4.1)
\]
and let \( Y_2 \) be the closed subspace of \( Y \) such that \( Y = Y_1 \oplus Y_2 \). We define the canonical projections \( P : Y \rightarrow Y_1 \) and \( Q : Y \rightarrow Y_2 \) by setting, for \( x \in Y \),
\[
Px(t) = x(t) - \frac{1}{\pi} \int_0^\pi x(t)dt = \ddot{x}(t), \quad (4.2)
\]
\[
Qx(t) = \frac{1}{\pi} \int_0^\pi x(t)dt = \dddot{x}, \quad (4.3)
\]
for \( t \in [0, \pi] \).

We next define a linear operator \( L : D(L) \subset X \rightarrow Y \) by setting
\[
D(L) = \{ x \in W^{2,1}([0, \pi], \mathbb{R}^N) | x'(0) = x'(\pi) = 0 \}; \quad (4.4)
\]
and for \( x \in D(L) \),
\[
Lx = -x''. \quad (4.5)
\]
Now, for $x \in D(L)$, we see, on integrating by parts, that

$$
(Lx, x) = \frac{1}{\pi} \int_0^\pi < -x''(t), x(t) > \, dt
$$

$$
= \frac{1}{\pi} \int_0^\pi |x'(t)|^2 \, dt = ||x||_{L^2}^2 \geq 0. \tag{4.6}
$$

**Lemma 4** For every given $y \in L^1([0, \pi], \mathbb{R}^N)$ with $\bar{y} = 0$, there exists a unique $x \in C([0, \pi], \mathbb{R}^N)$ with $\bar{x} = 0$ such that

$$
x''(t) = y(t), \quad 0 < t < \pi, \tag{4.7}
$$

$$
x'(0) = x'(\pi) = 0. \tag{4.8}
$$

**Proof.** It is easy to see that

$$
x(t) = -\int_0^t (t - \tau)y(\tau)d\tau + \frac{1}{2\pi} \int_0^\pi (\pi - \tau)^2 y(\tau)d\tau,
$$

for $t \in [0, \pi]$, is the unique solution for (4.7)-(4.8) with $\bar{x} = 0$. Hence the lemma follows. $\square$

It follows from Lemma 4 that there is a bounded linear operator $K : Y_1 \rightarrow X$ such that for $y \in Y$,

$$
KPy \in D(L), LKPy = Py, \quad (KPy, Py) \geq 0. \tag{4.10}
$$

Now let $N : X \rightarrow Y$ be a nonlinear operator defined by

$$
(Nx)(t) = f(t, x(t)), t \in [0, \pi], \tag{4.11}
$$

where $f : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a given function satisfying Caratheodory's conditions. It follows easily from the Arzela-Ascoli theorem that the operator $KPN : X \rightarrow X$ is a compact operator (i.e., it maps bounded subsets in $X$ into relatively compact subsets of $X$) and $QN : X \rightarrow X$ is a bounded operator (i.e., $QN$ maps bounded subsets in $X$ into bounded subsets in $X$).

**Theorem 1** Let $f = (f_1, f_2, ..., f_N) : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a function satisfying Caratheodory's conditions. Suppose that for each $i = 1, 2, ..., N$ there exist real numbers $r_i, R_i, a_i$ and $A_i$ with $r_i < 0 < R_i$ and $a_i \leq A_i$ such that

(i) for a.e. $t \in [0, \pi]$ and all $x \in \mathbb{R}^N$ with $x_i \geq R_i$,

$$
f_i(t, x) \geq A_i, \tag{4.12}
$$

(ii) for a.e. $t \in [0, \pi]$ and all $x \in \mathbb{R}^N$ with $x_i \leq r_i$,

$$
f_i(t, x) \leq a_i. \tag{4.13}
$$

Suppose further that for every real number $r \geq 0$ and each $i = 1, 2, ..., N$ there exist functions $\alpha_i(t) \in L^1[0, \pi]$ such that

$$
|f_i(t, x)| \leq \alpha_i(t), \tag{4.14}
$$

where $\alpha_i(t) \in L^1[0, \pi]$.
for a.e. \( t \in [0, \pi] \) and all \( x \in \mathbb{R}^N \) with \( |x_i| \leq r \). Then for every \( e \in L^1([0, \pi], \mathbb{R}^N) \) with \( a_i \leq \bar{e}_i \leq A_i \) for each \( i = 1, 2, \ldots, N \) the boundary value problem

\[
-x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi,
\]

(4.15)

\[
x'(0) = x'{}'(\pi) = 0,
\]

(4.16)

has at least one solution.

**Proof.** Define \( F = (F_1, F_2, \ldots, F_N) : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) by

\[
F_i(t) = f_i(t) - \frac{A_i + a_i}{2},
\]

(4.17)

for \( (t, x) \in [0, \pi] \times \mathbb{R}^N \) and \( i = 1, 2, \ldots, N \). Also define \( E : [0, \pi] \rightarrow \mathbb{R}^N \), \( E = (E_1, E_2, \ldots, E_N) \), by

\[
E_i(t) = e_i(t) - \frac{A_i + a_i}{2},
\]

(4.18)

for \( t \in [0, \pi] \) and \( i = 1, 2, \ldots, N \). Clearly \( F \) satisfies Caratheodory's conditions, and for \( i = 1, 2, \ldots, N \) and for a.e. \( t \in [0, \pi] \),

\[
F_i(t, x) \geq \frac{A_i - a_i}{2} \geq 0,
\]

(4.19)

for all \( x \in \mathbb{R}^N \) with \( x_i \geq R_i \), while

\[
F_i(t, x) \leq \frac{A_i - A_i}{2} \leq 0,
\]

(4.20)

for all \( x \in \mathbb{R}^N \) with \( x_i \leq r_i \). Further, for every real number \( r \geq 0 \) and each \( i = 1, 2, \ldots, N \) there exist functions \( \beta_{r_i} \in L^1[0, \pi] \) such that

\[
|F_i(t, x)| \leq \beta_{r_i}(t),
\]

(4.21)

for a.e. \( t \in [0, \pi] \) and all \( x \in \mathbb{R}^N \) with \( |x_i| \leq r \). Indeed, in view of (4.14), we have \( \beta_{r_i}(t) = \alpha_{r_i}(t) + \frac{1}{2}|A_i + a_i| \) for \( t \in [0, \pi] \). We also have for \( i = 1, 2, \ldots, N \),

\[
\frac{1}{2}(a_i - A_i) \leq \bar{e}_i \leq \frac{1}{2}(A_i - a_i).
\]

(4.22)

Clearly, (4.15) is equivalent to

\[
-x''(t) + F(t, x(t)) = E(t), \quad 0 < t < \pi.
\]

(4.23)

Let us next define the nonlinear operator \( N : X \rightarrow Y \) by

\[
(Nx)(t) = F(t, x(t)), \quad t \in [0, \pi],
\]

while \( x(t) \in X \). It is easy to see, in view of (4.19), (4.20), and (4.21), that for every \( k \geq 0 \) there exists a constant \( C(k) \geq 0 \) such that

\[
(Nx, x) \geq k||Nx||_Y - C(k),
\]

(4.24)
for \( x \in X \). Now, if \( L : D(L) \subset X \to Y \) is the linear operator defined by (4.4) and (4.5) and \( K : Y_1 \to X \) is the linear operator as in (4.10), then the boundary value problem of (4.23) with (4.16) is equivalent to the operator equation
\[
Lx + Nx = E, \quad x \in X,
\]  
which in turn is equivalent to the system of equations
\[
\begin{align*}
Px + KPNx &= KPE \\
QNx &= QE,
\end{align*}
\]  
where \( P \) and \( Q \) are as defined by (4.2) and (4.3). Now, (4.26) is clearly equivalent to the single equation
\[
Px + QNx + KPNx = KPE + QE,
\]  
which has the form of a compact perturbation of the Fredholm operator \( P \) of index zero. We can, therefore, apply the version given in [6: Theorem 1, Corollary 1] or [5: Theorem IV.4] or [7] of the Leray-Schauder Continuation theorem which ensures the existence of a solution for (4.27) if the set of all possible solutions of the family of equations
\[
Px + (1 - \lambda)Qx + \lambda QNx + \lambda KPN = \lambda KPE + \lambda QE,
\]  
\( \lambda \in (0, 1) \), is a priori bounded in \( X \), independently of \( \lambda \). Notice that (4.28) is equivalent to the system of equations
\[
\begin{align*}
Px + \lambda KPNx &= \lambda KPE \\
(1 - \lambda)Qx + \lambda QNx &= \lambda QE.
\end{align*}
\]  
If \( x_\lambda \in X \) is a solution for (4.29) for some \( \lambda \in (0, 1) \), then \( x_\lambda \in D(L) \) and
\[
\begin{align*}
Lx_\lambda + \lambda PNx_\lambda &= \lambda PE, \\
(1 - \lambda)Qx_\lambda + \lambda QNx_\lambda &= \lambda QE.
\end{align*}
\]  
Now we get from (4.30) that
\[
(Lx_\lambda, Px_\lambda) + \lambda(PNx_\lambda, Px_\lambda) = \lambda(PE, Px_\lambda),
\]  
\[
(1 - \lambda)(Qx_\lambda, Qx_\lambda) + \lambda(QNx_\lambda, Qx_\lambda) = \lambda(QE, Qx_\lambda).
\]  
Since \( (Lx_\lambda, Px_\lambda) = (Lx_\lambda, x_\lambda) \), and given (4.6) and (4.24), we obtain that
\[
||Px_\lambda||^2_H + k||Nz_\lambda||_Y - C(k) \leq ||PE||_Y \cdot ||Px_\lambda||_X + |QE| \cdot |Qx_\lambda|.
\]  
Now, the second equation in (4.30) gives for each \( i = 1, 2, ..., N \) that
\[
(1 - \lambda)\frac{1}{\pi} \int_0^\pi x_{\lambda i}(t)dt + \lambda \frac{1}{\pi} \int_0^\pi F_i(t, x_\lambda(t))dt = \frac{1}{\pi} \int_0^\pi E_i(t)dt.
\]  
If \( x_{\lambda i}(t) \geq R_i \) for every \( t \in [0, \pi] \), we get from (4.32), in view of (4.19) and (4.22), that
\[
(1 - \lambda)R_i + \frac{\lambda}{2}(A_i - a_i) \leq \frac{\lambda}{2}(A_i - a_i).
Thus, $(1 - \lambda)R_i \leq 0$, and we have a contradiction. Similarly, $x_{\lambda i}(t) \leq r_i$ for every $t \in [0, \pi]$ leads to a contradiction. So for every $i = 1, 2, ..., N$ there exists a $r_i \in [0, \pi]$ such that $r_i \leq x_{\lambda i}(r_i) \leq R_i$.

It follows that there exist constants $C_1 \geq 0$ and $C_2 \geq 0$, independent of $\lambda \in (0, 1)$ such that

$$||x_\lambda||_X \leq C_1 + C_2||Px_\lambda||_{H_N^2}. \tag{4.33}$$

Finally, using the facts that $||Px_\lambda||_X \leq 2||x_\lambda||_X$ and $|Qx_\lambda| \leq ||x_\lambda||_X$, we have that

$$||Px_\lambda||_{H_N^2}^2 + k||Nx_\lambda||_Y - C(k) \leq C_3(C_1 + C_2||Px_\lambda||_{H_N^2}),$$

where $C_3 = 2||PE||_Y + |QE|$. Hence, there exists a constant $C > 0$, independent of $\lambda \in (0, 1)$, such that

$$||Px_\lambda||_{H_N^2} \leq C,$$

which implies, from (4.33), that

$$||x_\lambda||_X \leq C_1 + C_2C.$$

We have thus shown that the set of solutions of (4.28) is bounded in $X$ independently of $\lambda \in (0, 1)$. Hence the theorem follows. $\square$

**Theorem 2** Let $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N) \in L^1([0, \pi], R^N)$ be as in Lemma 2. Let $f = (f_1, f_2, ..., f_N) : [0, \pi] \times R^N \rightarrow R^N$ be as in Theorem 1. Assume, further, for each $i = 1, 2, ..., N$

$$\lim \sup_{|x_i| \rightarrow \infty} \frac{f_i(t, x)}{x_i} \leq \Gamma_i(t), \tag{4.34}$$

uniformly a.e. in $t \in [0, \pi]$. Then, for every $e \in L^1([0, \pi], R^N)$ with $a_i \leq \bar{e}_i \leq A_i$ for each $i = 1, 2, ..., N$ the boundary value problem

$$x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi, \tag{4.35}$$

$$x'(0) = x'(\pi) = 0, \tag{4.36}$$

has at least one solution.

**Proof.** Define $F = (F_1, F_2, ..., F_n) : [0, \pi] \times R^N \rightarrow R^N$ and $E : [0, \pi] \rightarrow R^N$, $E = (E_1, E_2, ..., E_N)$, as in the proof of Theorem 1, so that (4.19), (4.20), (4.21), and (4.22) hold. We have from (4.34) for each $i = 1, 2, ..., N$ that

$$\lim \sup_{|x_i| \rightarrow \infty} \frac{F_i(t, x)}{x_i} \leq \Gamma_i(t), \tag{4.37}$$

uniformly a.e. in $t \in [0, \pi]$. We also see for each $i = 1, 2, ..., N$ for all $x \in R^N$ with $|x_i| \geq \max(R_i, -r_i)$ that $F_i(t, x)x_i \geq 0$, so that $\Gamma_i(t) \geq 0$ a.e. in $[0, \pi]$. Moreover, the equation (4.35) is equivalent to

$$x''(t) + F(t, x(t)) = E(t), \quad 0 < t < \pi. \tag{4.38}$$

Now let $\eta = \frac{1}{2N}[\delta(\Gamma_\alpha) - \frac{\pi^2}{3||\Gamma_\alpha||_{L_N^2}} - ||\Gamma_\infty||_{L_N^2}] > 0$. Then for each $i$ there exists a $\rho_i > 0$ such that for a.e. $t \in [0, \pi]$ and all $x \in R^N$ with $|x_i| \geq \rho_i$,

$$0 \leq \frac{F_i(t, x)}{|x_i|} \leq \Gamma_i(t) + \eta. \tag{4.39}$$
Next, set \( \rho = \max\{\rho_i | 1 \leq i \leq N\} \) and define \( \tilde{\gamma} : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) by setting, for \( (t, x) \in [0, \pi] \times \mathbb{R}^N \) and each \( i = 1, 2, \ldots, N \),

\[
\tilde{\gamma}_i(t, x) = \begin{cases} \frac{F_i(t, x)}{x_i} & \text{if } |x_i| \geq \rho, \\ \frac{F_i(t, x_1, \ldots, x_{i-1}, \rho, x_{i+1}, \ldots, x_N)}{\rho} (x_i) & \text{if } 0 \leq x_i < \rho, \\ + (1 - \frac{x_i}{\rho}) \Gamma_i(t) & \text{if } -\rho \leq x_i < 0. 
\end{cases}
\]

Then \( \tilde{\gamma} \) satisfies Caratheodory's conditions and

\[
0 \leq \tilde{\gamma}_i(t, x) \leq \Gamma_i(t) + \eta, \tag{4.40}
\]

for a.e. \( t \in [0, \pi] \), all \( x \in \mathbb{R}^N \), and \( i = 1, 2, \ldots, N \). If we next set \( h = (h_1, h_2, \ldots, h_N) \) with

\[
h_i(t, x) = F_i(t, x) - \tilde{\gamma}_i(t, x)x_i,
\]

for \( t \in [0, \pi] \), \( x \in \mathbb{R}^N \), and \( i = 1, 2, \ldots, N \), then we see from (4.21) and the definition of \( \tilde{\gamma}_i \) that there exist functions \( m_i(t) \in L^1[0, \pi] \) such that

\[
|h_i(t, x)| \leq m_i(t), \tag{4.41}
\]

for a.e. \( t \in [0, \pi] \) and all \( x \in \mathbb{R}^N \). Defining \( g : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \), \( g = (g_1, g_2, \ldots, g_N) \) by setting

\[
g_i(t, x) = \tilde{\gamma}_i(t, x)x_i
\]

for \( (t, x) \in [0, \pi] \times \mathbb{R}^N \), \( i = 1, 2, \ldots, N \), we see that the equation (4.38) is equivalent to

\[
x''(t) + g(t, x(t)) + h(t, x(t)) = E(t). \tag{4.42}
\]

We can next apply Theorem IV.4 of [5] to the boundary value problem posed by (4.42) and (4.36). It suffices to show that the set of solutions of the family of equations

\[
x''(t) + (1 - \lambda)\tilde{T}(t)x(t) + \lambda g(t, x(t)) + \lambda h(t, x(t)) = \lambda E(t), \tag{4.43}
\]

\[
x'(0) = x'(-\pi) = 0,
\]

\( \lambda \in (0, 1) \), is a priori bounded in \( X = C([0, \pi], \mathbb{R}^N) \) independently of \( \lambda \), where \( \tilde{T}(t) = (\tilde{T}_1(t), \ldots, \tilde{T}_N(t)) \) with \( \tilde{T}_i(t) = \Gamma_i(t) + \eta, i = 1, 2, \ldots, N \).

If, now, \( x(t) \) is a possible solution of (4.43) for some \( \lambda \in (0, 1) \), we see on integrating the equation obtained by taking the inner product of the equation in (4.43) with \( \frac{1}{\pi}(\tilde{x} - \tilde{x}(t)) \) and using Lemma 3 with \( \Gamma_{\infty} \) replaced by \( \Gamma_{\infty} + \eta \), for \( i = 1, 2, \ldots, N \) and \( \gamma = (\gamma_1, \ldots, \gamma_N) \equiv 0 \) that

\[
0 = \frac{1}{\pi} \int_0^\pi \langle x''(t), \tilde{x} - \tilde{x}(t) \rangle dt
\]
where $C > 0$ is a constant independent of $\lambda \in (0, 1)$. Hence,

\[
\|\ddot{z}\|^2_{H^1_N} \leq \left( \frac{C}{\eta N} \right) (|\ddot{z}| + \|\ddot{z}\|_{H^1_N}). \tag{4.44}
\]

Next, integrating each of the component equations in (4.43) over $[0, \pi]$, we see that

\[
(1 - \lambda) \frac{1}{\pi} \int_0^\pi (\Gamma_i(t) + \eta)x_i(t)dt + \lambda \frac{1}{\pi} \int_0^\pi F_i(t, x(t))dt = \lambda \frac{1}{\pi} \int_0^\pi E_i(t)dt,
\]

$i = 1, 2, \ldots, N$. As in the proof of Theorem 1, we see that there exist constants $C_1 \geq 0$ and $C_2 \geq 0$, such that

\[
|\ddot{z}| \leq \|\ddot{z}\|_{X} \leq C_1 + C_2\|\ddot{z}\|_{H^1_N}. \tag{4.45}
\]

It follows from (4.44) and (4.45) that there exists a constant $C_3$ independent of $\lambda \in (0, 1)$ such that

\[
\|\ddot{z}\|_{H^1_N} \leq C_3,
\]

and hence

\[
\|x\|_{X} \leq C_1 + C_2C_3.
\]

Thus we have shown that the set of solutions of (4.43) is bounded in $X$ independently of $\lambda$. Hence the theorem holds. \(\Box\)

**Remark 1.** We say that the boundary value problem (4.35)-(4.36) has “no $L^\infty$-resonance” at the second eigenvalue $\lambda = 1$ of the linear eigenvalue problem (1.7)-(1.8) if $\Gamma_\alpha = \Gamma_\infty = 0$ in Theorem 2. In the case of no $L^\infty$-resonance, Theorem 2 implies the existence of a solution for the boundary value problem (4.35)-(4.36) if $\|\Gamma_\beta\|_{L^1_N} < \frac{3}{2\pi}$. We give a sharpening of this result in Section 5.

## 5 Resonance Condition When No $L^\infty$-Resonance Exists

We need the following lemma for a sharper resonance condition that gives the existence of a solution for the boundary value problem (4.35)-(4.36) when there is no $L^\infty$-resonance.

**Lemma 5** Let $e \in L^1([0, \pi], \mathbb{R}^N)$ and $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_N) \in L^1([0, \pi], \mathbb{R}^N)$ with $\bar{\Gamma}_i = \frac{1}{\pi} \int_0^\pi \Gamma_i(t)dt \geq 0$ for every $i = 1, 2, \ldots, N$. Then every possible solution $x(t)$ of the linear boundary value problem

\[
x''(t) + p(t)^T I x(t) = e(t), 0 < t < \pi
\]
\[ x'(0) = x'(\pi) = 0, \quad (5.1) \]

with \( p = (p_1, p_2, \ldots, p_N) \in L^1([0, \pi], \mathbb{R}^N) \) such that
\[ \tilde{\rho}_i \leq \bar{\Gamma}_i, \quad 0 \leq p_i(t) \quad (5.2) \]

for a.e. \( t \in [0, \pi], \ i = 1, 2, \ldots, N, \) satisfies the inequality
\[ (1 - \frac{\pi^2}{4} |\Gamma|) \|x''\|_{L^2_N}^2 \leq 2 \|\epsilon\|_{L^1_N} \|x''\|_{L^1_N} + \|\bar{\Gamma}\| \|\epsilon\|_{L^1_N} \|x\|_{L^\infty}. \quad (5.3) \]

(Here \( \bar{\Gamma} = (\bar{\Gamma}_1, \bar{\Gamma}_2, \ldots, \bar{\Gamma}_N). \)

**Proof.** It follows from Lemma 4 of [1] that each solution \( x_i(t) \) of the \( i \)-th component boundary value problem of (5.1), namely,
\[ x''_i(t) + p_i(t)x_i(t) = e_i(t), \quad 0 < t < \pi \]
\[ x'_i(0) = x'_i(\pi) = 0, \]

satisfies the inequality
\[ (1 - \frac{\pi^2}{4} |\Gamma_i|) \|x''_i\|_{L^2_N}^2 \leq 2 \|e_i\|_{L^1_N} \|x''_i\|_{L^1_N} + |\bar{\Gamma}_i| \|e_i\|_{L^1_N} \|x_i\|_{L^\infty} \]

for each \( i = 1, 2, \ldots, N. \) Noting that \( \max_{1 \leq i \leq n} \bar{\Gamma}_i \leq |\bar{\Gamma}|, \) we get
\[ (1 - \frac{\pi^2}{4} |\bar{\Gamma}|) \|x''_i\|_{L^2_N}^2 \leq 2 \|e_i\|_{L^1_N} \|x''_i\|_{L^1_N} + |\bar{\Gamma}| \|e_i\|_{L^1_N} \|x_i\|_{L^\infty} \]

for each \( i = 1, 2, \ldots, N. \) On adding all these inequalities and using the Cauchy-Schwarz inequality in \( \mathbb{R}^N \) we get that
\[ (1 - \frac{\pi^2}{4} |\bar{\Gamma}|) \|x''\|_{L^2_N}^2 \leq 2 \|e\|_{L^1_N} \|x''\|_{L^1_N} + |\bar{\Gamma}| \|e\|_{L^1_N} \|x\|_{L^\infty}. \]

Hence the lemma follows. \( \square \)

**Theorem 3** Let \( \Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_N) \in L^1([0, \pi], \mathbb{R}^N) \) be such that \( |\Gamma| < \frac{4}{\pi^2}. \) Let \( f = (f_1, f_2, \ldots, f_N) : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) be as in Theorem 2. Then for every \( e \in L^1([0, \pi], \mathbb{R}^N) \) with \( a_i \leq \tilde{e}_i \leq A_i \) for each \( i = 1, 2, \ldots, N, \) the boundary value problem
\[ x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi, \quad (5.4) \]
\[ x'(0) = x'(\pi) = 0, \]

has at least one solution.

**Proof.** Define \( F = (F_1, F_2, \ldots, F_N) : [0, \pi] \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \) \( E : [0, \pi] \rightarrow \mathbb{R}^N \) and \( E = (E_1, E_2, \ldots, E_N) \) as in the proof of Theorem 2. Then the boundary value problem (5.4) is equivalent to the boundary value problem
\[ x''(t) + F(t, x(t)) = E(t), \quad 0 < t < \pi, \quad (5.5) \]
Also
\[
x'(0) = x'(\pi) = 0.
\]

Also
\[
\lim_{|x_i| \to \infty} \sup_{|x| \leq 1} \frac{F_i(t, x)}{x_i} \leq \Gamma_i(t),
\]
uniformly a.e. in \( t \in [0, \pi] \) and
\[
F_i(t, x)x_i \geq 0,
\]
for a.e. \( t \in [0, \pi] \) and all \( x \in \mathbb{R}^N \) with \( |x_i| \geq \max(R_i, -r_i) \) so that \( \Gamma_i(t) \geq 0 \) for a.e. \( t \in [0, \pi] \). Let
\[
\eta = \frac{1}{2N}(\frac{4}{\pi^2} - |\bar{\Gamma}|) \quad \text{so that} \quad |\Gamma_i| + N\eta < \frac{4}{\pi^2}.
\]
Proceeding as in the proof of Theorem 2, we can write the boundary value problem (5.5) in the equivalent form
\[
x''(t) + g(t, x(t)) + h(t, x(t)) = E(t),
\]
\[
x'(0) = x'(\pi) = 0.
\]

The same degree arguments will imply the existence of a solution for (5.6) if the set of all possible solutions of the family of equations
\[
x''(t) + (1 - \lambda)\bar{\Gamma}^*T(t)x(t) + \lambda g(t, x(t)) + \lambda h(t, x(t)) = \lambda E(t)
\]
\[
x'(0) = x'(\pi) = 0,
\]
\( \lambda \in (0, 1) \), is a priori bounded in \( X = C([0, \pi], \mathbb{R}^N) \) independently of \( \lambda \). Here \( \bar{\Gamma}^*(t) = (\bar{\Gamma}_1^*(t), \ldots, \bar{\Gamma}_N^*(t)) \) with \( \bar{\Gamma}_i^*(t) = \Gamma_i(t) + \eta, i = 1, 2, \ldots, N, t \in [0, \pi] \).

We note that \( g = (g_1, g_2, \ldots, g_N) \) in (5.6) is such that \( g_i(t, x) = \bar{\gamma}_i(t, x)x_i \). If we write \( \bar{\gamma}(t, x) = (\bar{\gamma}_1(t, x), \ldots, \bar{\gamma}_N(t, x)) \), then
\[
g(t, x) = \bar{\gamma}(t, x)^T Ix.
\]

We see that
\[
0 \leq (1 - \lambda)\bar{\Gamma}^*_i(t) + \lambda \bar{\gamma}_i(t, x(t)) \leq \Gamma^*_i(t)
\]
for \( i = 1, 2, \ldots, N \) in view of (4.40) with
\[
|\bar{\Gamma}^*| = \left\{ \sum_{i=1}^{N} (\bar{\Gamma}_i + \eta)^2 \right\}^{1/2} \leq \left( \sum_{i=1}^{N} \bar{\Gamma}_i^2 \right)^{1/2} + \sqrt{N}\eta
\]
\[
\leq |\bar{\Gamma}| + \eta N < \frac{4}{\pi^2}.
\]

Also, since
\[
||E(t) - h(t, x(t))||_{L^1_N} \leq ||E||_{L^1_N} + \sum_{i=1}^{N} ||m_i||_{L^1},
\]
it follows from Lemma 5 that
\[
(1 - \frac{\pi^2}{4} |\bar{\Gamma}^*|)||x''||_{L^1_N} \leq 2(||E||_{L^1_N} + \sum_{i=1}^{N} ||m_i||_{L^1})||x'||_{L^1_N}
\]
\[
+ |\bar{\Gamma}^*(||E||_{L^1_N} + \sum_{i=1}^{N} ||m_i||_{L^1})||x||_{L^\infty_N}.
\]

(5.8)
As in the proof of Theorem 2, we have that there exist constants \( C_1 \geq 0 \) and \( C_2 \geq 0 \), independent of \( \lambda \in (0, 1) \) such that

\[
|\tilde{x}| \leq ||x||_{L_N^\infty} \leq C_1 + C_2 ||\tilde{x}||_{H_N^1} = C_1 + C_2 ||x'||_{L_N^3} \leq C_1 + C_2 \frac{\pi}{2} ||x'''||_{L_N^1}.
\]  

(5.9)

It then follows from (5.8) and (5.9) that there exists a constant \( C_3 \geq 0 \), independent of \( \lambda \in (0, 1) \), such that

\[
||x'''||_{L_N^1} \leq C_3
\]

and, hence, from (5.9) again,

\[
||x||_{L_N^\infty} \leq C_1 + C_2 \frac{\pi}{2} C_3.
\]

Hence the theorem holds. \(\Box\)

Remark 2. If there is no \( L^\infty \)-resonance (i.e., \( \Gamma_\alpha = \Gamma_\infty = 0 \)), Theorem 3 improves the condition on \( \Gamma \) to \( |\Gamma| \leq \frac{3}{4} \) compared to Theorem 2, where \( |\Gamma| < \frac{3}{2} \).

Remark 3. If \( p(t) = (p_1(t), ..., p_N(t)) \in L^1([0, \pi], \mathbb{R}^N) \) in Lemma 5 satisfies, additionally, for a given \( \eta > 0 \), \( p_i(t) \geq \eta > 0 \) for a.e. \( t \in [0, \pi] \), \( i = 1, 2, ..., N \) and \( |\Gamma| < \frac{3}{2} \), it follows easily from the inequality (5.3) that the boundary value problem (5.1) has at most one solution.

References


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