MONOTONE METHOD FOR FIRST ORDER SINGULAR SYSTEMS WITH BOUNDARY CONDITIONS *

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ABSTRACT

Existence of maximal and minimal solutions are proved for the first order singular systems with boundary conditions by combining the method of upper and lower solutions and the monotone iterative technique.

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1. INTRODUCTION

It is well known [4] that by combining the method of upper and lower solutions with monotone iterative techniques, one can prove the existence of extremal solutions on nonlinear problems in a closed set, namely, the sector defined by means of the upper and lower solutions.

Recently the result has been extended [6] to singular systems with initial conditions since singular systems do occur in many physical applications. In this paper, we extend this result to singular systems with boundary conditions. This is achieved by developing the necessary comparison result. The crucial part is the consistency condition. An example is given to illustrate that such a condition is attainable.

2. PRELIMINARY RESULTS

Consider the boundary value problem

$$A\dot{x} = f(t,x), \quad Ex(0) = a, \quad Fx(T) = b \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$ is singular matrix, E, F are real $n \times n$ nonsingular matrices, and $f \in C^1[J \times \mathbb{R}^n, \mathbb{R}^n]$, $J = [0, T]$. In this paper we combine the method of upper and lower solutions together with monotone iterative technique to prove the existence of

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extremal solutions of (1.1). For this purpose we need the existence and uniqueness of the solution of the corresponding linear boundary value problem of the form

$$A\dot{x} + Mx = g(t), \qquad Ex(0) = a, \qquad Fx(T) = b \tag{1.2}$$

where M^- is an $n \times n$ matrix and g is n-times differentiable on J. The results relative to (1.2) are well known [2,3].

Here, we recall the results without proof. Here and throughout the paper, we assume that A is a singular matrix and that there exists a $\lambda \in \mathbb{R}$ for which $(\lambda A + M)^{-1}$ exists. Also, B^D shall denote the Drazin inverse of the matrix B, and $\mathscr{B}$ denotes the quantity $(\lambda A + M)^{-1}B$. ($\mathscr{A}$, $\mathscr{M}$, and $\mathscr{g}$ will have the same relationship to A, M, and g)

Theorem 1.1. *For a given a,b,g the boundary value problem* (1.2) *is consistent if and only if the equation*

$$\begin{bmatrix} E\mathscr{A}^D\mathscr{A} \\ Fe^{-\mathscr{A}^D\mathscr{M}T}\mathscr{A}^D\mathscr{A} \end{bmatrix} q = \begin{bmatrix} a - E(I - \mathscr{A}^D\mathscr{A})x(0) \\ b - F(I - \mathscr{A}^D\mathscr{A})x(T) - e^{-\mathscr{A}^D\mathscr{M}T}\int_0^T e^{\mathscr{A}^D\mathscr{M}s}\mathscr{A}^D\mathscr{g}\,(s)\,ds \end{bmatrix}$$

has a solution q.

Theorem 1.2. *The boundary value problem* (1.2), *when consistent, has a unique solution provided that:*

(*a*) *the associated homogeneous problem*

$$A\dot{x} + Mx = 0, \quad Ex(0) = 0 = Fx(T) \tag{1.3}$$

has only the zero solution; or

(*b*) rank (Q) = rank $(\mathscr{A}^D\mathscr{A})$ = rank $(\mathscr{A}^n)$,

where

$$Q = \begin{bmatrix} E\mathscr{A}^D\mathscr{A} \\ Fe^{-\mathscr{A}^D\mathscr{M}T}\mathscr{A}^D\mathscr{A} \end{bmatrix}.$$

One can easily show [2] that the solution of (1.2) is given by

$$\begin{aligned} x(t) &= e^{-\mathscr{A}^D\mathscr{M}t}\mathscr{A}^D\mathscr{A}q + e^{-\mathscr{A}^D\mathscr{M}t}\int_0^t e^{\mathscr{A}^D\mathscr{M}s}\mathscr{A}^D g(s)\,ds \\ &\quad + (I - \mathscr{A}^D\mathscr{A})\mathscr{M}^D\sum_{i=0}^{n-1}(-1)^i[\mathscr{A}\mathscr{M}^D]^i g^{(i)}(t) \end{aligned} \tag{1.4}$$

where q is any vector satisfying the consistency condition.

Note 1.1: If the index of A is one, the third term of (1.4) simplifies to $(I - \mathscr{A}^D\mathscr{A})\mathscr{M}^D g(t)$. In this case, differentiability of g is enough to show that (1.4) is in fact the solution of (1.2). Also, if the index of A is one and further, if $(\lambda A + M)^{-1}$ exists, assumption (*a*) of Theorem 1.2 is easily satisfied.

Next, we prove a comparison result which is needed in our main result. For convenience we list the following assumptions:

(A1) There exist $v_0(t), w_0(t) \in C^1[J, \mathbb{R}^n]$ with $v_0(t) \le w_0(t)$ for $t \in J$ such that

$$A\dot{v}_0 \le f(t, v_0), \quad Ev_0(0) \le a, \quad Fv_0(T) \le b \tag{1.5}$$

$$A\dot{w}_0 \ge f(t, w_0), \quad Ew_0(0) \ge a, \quad Fw_0(T) \ge b. \tag{1.6}$$

That is, v_0, w_0 are lower and upper solutions of the boundary value problem (1.1).

(A2) There exists a matrix $M \in \mathbb{R}^{n \times n}$ such that

$$f(t,y) - f(t,x) \ge -M(y - x)$$

whenever

(1.7) $$v_0 \le x \le y \le w_0.$$

(A3) Let A and M be $n \times n$ matrices such that $(\lambda A + M)^{-1}$ exists and is nonnegative for some $\lambda \in \mathbb{R}$. Also, let there exist nonnegative matrices such that

$$P^{-1}\mathscr{A}P = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix},$$

where C is a diagonal square matrix with $C_{jj}^{-1} > \lambda$ if $\lambda > 0$, and $C_{jj}^{-1} > 0$ if $\lambda < 0$.

Note that for any square matrix A, there exist nonsingular matrices P and C, and a nilpotent matrix N such that

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$$

where index of A = index of N. Thus $N \equiv 0 \Leftrightarrow$ index of $A = 1$. This in turn implies that the index of $\mathscr{A} = 1$. In such a case the Drazin inverse of $\mathscr{A}$ is called the group inverse and is denoted by $\mathscr{A}^{\#}$. See [5] for details.

Theorem 1.3: *Let* $A\dot{p} + Mp \le 0$ *such that* A *and* M *satisfy assumption* (A3). *Then* $Ep(0) \le 0$ *and* $Fp(T) \le 0$, *where* $E^{-1}, F^{-1} \ge 0$, *implies* $p(t) \le 0$ *on* $J = [0,T]$.

Proof: Set $p = Pz$. Then the inequality $A\dot{p} + Mp \le 0$ reduces to $AP\dot{z} + MPz \le 0$. Premultiplying this by $P^{-1}(\lambda A + M)^{-1}$, we have

(1.8) $$P^{-1}\mathscr{A}P\dot{z} + P^{-1}\mathscr{M}Pz \le 0$$

since P^{-1} and $(\lambda A + M)^{-1}$ are nonnegative. That is

$$\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} I - \lambda C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \le 0.$$

This leads to $C\dot{z}_1 + (I - \lambda C)z_1 \le 0$ and $z_2 \le 0$. Also $Ep(0) = EPz(0) \le 0$ implies $z(0) \le 0$ since

P^{-1}, E^{-1} are nonnegative. Similarly, $z(T) \le 0$. One can easily prove [1,4] that $z_1(t) \le 0$. Hence $z(t) \le 0$ on J. Consequently, $Pz(t) \le 0$ or $p(t) \le 0$ on J.

3. MAIN RESULT

Here we develop the monotone iterative technique for the singular system of the boundary value problem which yields monotone sequences that converge to the extremal solutions of (1.1) relative to the sector $[v_0, w_0]$.

Theorem 2.1. *Let assumptions* (A1), (A2), (A3) *hold such that* E,F *possess nonnegative inverses. Further, let the equation*

$$(2.1)\quad \begin{bmatrix} E\mathscr{A}^D\mathscr{A} \\ Fe^{-\mathscr{A}^D\mathscr{M}T}\mathscr{A}^D\mathscr{A} \end{bmatrix} q(\eta,\mu) = \begin{bmatrix} a - E(I - \mathscr{A}^D\mathscr{A})\mu(0) \\ b - F(I - \mathscr{A}^D\mathscr{A})\mu(T) - Fe^{-\mathscr{A}^D\mathscr{M}T}\mathscr{A}^D\int_0^T e^{-\mathscr{A}^D\mathscr{M}s}\hat{\sigma}(s)ds \end{bmatrix},$$

where $\hat{\sigma}(t) = (\lambda A + M)^{-1}[f(t,\eta) + M\eta]$, have a solution q for each pair η,μ such that $v_0(t) \le \eta(t) \le w_0(t)$, $v_0(0) \le \mu(0) \le w_0(0)$ and $v_0(T) \le \mu(T) \le w_0(T)$. Then there exist sequences $\{v_n\},\{w_n\}$ which converge uniformly and monotonically on $[0,T]$ to $\rho(t)$ and $\gamma(t)$, where ρ and γ are minimal and maximal solutions of (1.1) respectively. That is, if $x(t)$ is any solution of (1.1) such that $v_0(t) \le x(t) \le w_0(t)$ on J, then $\rho(t) \le x(t) \le \gamma(t)$ on J.

Proof: Consider the linear boundary value problem

$$(2.2)\qquad A\dot{x} = f(t, \eta(t)) - M(x - \eta(t)),\quad Ex(0) = a,\quad Fx(T) = b$$

where η belongs to the sector

$$[v_0, w_0] \equiv \{u \in C^1[J,\mathbb{R}^n]:\ v_0 \le u \le w_0\}.$$

Now (2.2) can be rewritten as

$$(2.3)\qquad A\dot{x} + Mx = f(t,\eta) + M\eta,\quad Ex(0) = a,\quad Fx(T) = b.$$

Choosing $x(0) = \mu(0)$, $x(T) = \mu(T)$, the boundary value problem (2.3) satisfies the consistency condition (2.1) and has a unique solution since the associated homogeneous problem has only the zero solution. It is easy to see this using (A3).

Define a mapping R such that

$$R\eta = x$$

where x is the unique solution of (2.3). This mapping defines the sequences $\{v_n\}$, $\{w_n\}$. First we prove that

(a) $v_0 \le Rv_0$, $w_0 \ge Rw_0$

(b) R is monotone nondecreasing on the sector $[v_0, w_0]$.

To prove (a), set $Rv_0 = v_1$, where v_1 is the unique solution of (2.3) with $\eta = v_0$. Setting $p = v_0 - v_1$, one can see that

$$A\dot{p} = A\dot{v}_0 - A\dot{v}_1 \le f(t, v_0) - f(t, v_0) + M(v_1 - v_0) = -Mp$$

and $p(0) = v_0(0) - v_1(0) \le 0$, $p(T) \le 0$ which by using Theorem 1.3 implies $p(t) \le 0$. That is, $v_0(t) \le v_1(t)$ on J. Similarly, one can prove that $w_0 \ge Rw_0$ on J.

In order to prove (b), consider η_1, η_2 belonging to the sector $[v_0, w_0]$ such that $\eta_1(t) \le \eta_2(t)$ on J. Set $x_1 = R\eta_1$, $x_2 = R\eta_2$ and $p(t) = x_1(t) - x_2(t)$. Then

$$\begin{aligned} A\dot{p} + Mp &= A\dot{x}_1 - A\dot{x}_2 + Mx_1 - Mx_2 \\ &= f(t, \eta_1) + M\eta_1 - f(t, \eta_2) - M\eta_2 \\ &\le 0 \end{aligned}$$

using (A2). Also, $p(0) = 0 = p(T)$. By Theorem 1.3 this implies $p(t) \le 0$ on J. That is, $x_1(t) \le x_2(t)$ on J.

Now define the sequences $\{v_n\}$, $\{w_n\}$ by

$$v_{n+1} = Rv_n,\ w_{n+1} = Rw_n \text{ for } n = 0, 1, 2, \ldots .$$

Using standard arguments it is easy to prove that $\lim_{n\to\infty} v_n = \rho(t)$ and $\lim_{n\to\infty} w_n = \gamma(t)$ uniformly and monotonically on J. It is also easy to show that $\rho(t)$ and $\gamma(t)$ are solutions of (1.1) in view of the fact that v_n, w_n satisfy

$$A\dot{v}_n = -Mv_n + f(t, v_{n-1}) + Mv_{n-1}, \qquad Ev_n(0) = a,\ Fv_n(T) = b$$

$$A\dot{w}_n = -Mw_n + f(t, w_{n-1}) + Mw_{n-1},\ Ew_n(0) = a,\ Fw_n(T) = b.$$

To prove that ρ, γ are the minimal and maximal solutions of (1.1), it is enough to show that if $x(t)$ is any solution of (1.1) such that $v_0 \le x \le w_0$ on J, then $v_0 \le \rho \le x \le \gamma \le w_0$ on J. To do this, suppose that for some n, $v_n \le x \le w_n$ on J. Set $p = x - v_{n+1}$ so that

$$\begin{aligned} A\dot{p} &= A\dot{x} - A\dot{v}_{n+1} \\ &= f(t, x) - [-Mv_{n+1} + f(t, v_n) + Mv_n] \\ &\ge -M(x - v_n) + Mv_{n+1} - Mv_n \\ &= -M(x - v_{n+1}) = -Mp \end{aligned}$$

and $p(0) = x(0) - v_{n+1}(0) = 0$, $p(T) = x(T) - v_{n+1}(T) = 0$. Using Theorem 1.3, $p(t) \geq 0$ or $x(t) \geq v_{n+1}(t)$ on J. Similarly, $x(t) \leq w_{n+1}(t)$ on J and hence $v_{n+1} \leq x \leq w_{n+1}$ on J. Since $v_0 \leq x \leq w_0$ on J, this proves by induction that $v_n \leq x \leq w_n$ on J for all n. Taking the limit as $n \to \infty$, the theorem follows.

Below is an example to illustrate that the condition (2.1) of Theorem 2.1 is attainable.

Example 2.1: Consider the nonlinear problem

$$A\dot{x} = f(t, x), \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x(1) = \begin{bmatrix} e^{-2} \\ e^{-6} \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f(t, x) = \begin{bmatrix} e^{-2t} - 2x_1 \\ x_1^3 - x_2 \end{bmatrix}.$$

For this f it is easy to see that M can be chosen as $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ so that assumption (A2) is satisfied. The corresponding linear problem (2.3) is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ \eta_1^3 \end{bmatrix},$$

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x(1) = \begin{bmatrix} e^{-2} \\ e^{-6} \end{bmatrix}.$$

By direct computation, condition (2.1) reduces to the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ e^{-2} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} q_1(\eta, \mu) \\ q_2(\eta, \mu) \end{pmatrix} = \begin{bmatrix} 0 \\ -\mu_2(0) \\ 0 \\ e^{-6} - \mu_2(1) \end{bmatrix}$$

with solution $q_1 = 0$ and q_2 arbitrary.

Remark 2.1: In the case where $a = 0 = b$, and $T = 1$, the solution of the linear problem (1.2) can be written in the form (see [2] for details)

$$x(t) = \int_0^1 G(t, s)\rho(s)\, ds + \int_0^1 H(t, s)\rho(s)\, ds \tag{2.4}$$

where the matrix G is given by

$$G(t, s) = \begin{cases} Y(t)R^{+}FY(1)Y^{-1}(s)\mathscr{A}^{D}, & s < t \\ -Y(t)R^{+}FY(1)Y^{-1}(s)\mathscr{A}^{D}, & s > t \end{cases}$$

and

$$H(t, s) = (I - \mathscr{A}^{D}\mathscr{A})\,\mathscr{M}^{D}\delta(s - t) + Y(t)R^{+}E(I - \mathscr{A}^{D}A)\mathscr{M}^{D}\delta(s)$$
$$+ Y(t)R^{+}F(I - \mathscr{A}^{D}\mathscr{A})\,\mathscr{M}^{D}\delta(s - 1),$$

where $R = E\mathscr{A}^{D}\mathscr{A} + FY(1)\mathscr{A}^{D}\mathscr{A}$, $\delta(t)$ is the delta function, R^{+} denotes the generalized inverse of R and $y = e^{-\mathscr{A}^{D}\mathscr{M}t}$ is the fundamental matrix for the homogeneous problem:

$$A\dot{x} + Mx = 0, \quad Ex(0) = 0 = Fx(1).$$

It is to be noted in this special case that it suffices if $f \in C[J \times \mathbb{R}^{n}, \mathbb{R}^{n}]$. Now (2.4) can be used to develop the monotone method and show the existence of extremal solutions of (1.1) when $a = b = 0$.

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