AN EXISTENCE THEOREM FOR NONLINEAR DELAY DIFFERENTIAL EQUATIONS*

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ABSTRACT

In this paper we prove a theorem on the existence of solutions of nonlinear delay differential equations with implicit derivatives. The result is established using the measure of noncompactness of a set and Darbo's fixed point theorem.

Key words: Existence of solution, Delay differential equations, Darbo's theorem.

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1. INTRODUCTION

The theory of linear and nonlinear delay differential equations has been studied by several researchers [1,4]. Recently Dacka [5] proved an existence theorem for nonlinear delay differential equations with implicit derivatives using the measure of noncompactness of a set and Darbo's fixed point theorem. By the same method Balachandran and Somasundaram [2] established an existence theorem for nonlinear delay differential equations having implicit derivatives with delay depending on state variable. In [3] Banas proved a theorem about existence of solutions of some nonlinear Volterra integral equations with deviating argument without assuming the Lipschitz condition and using a technique similar to Dacka [5]. In this paper we shall prove an existence theorem for nonlinear delay differential equations with delay depending on implicit derivatives.

2. MATHEMATICAL PRELIMINARIES

Let \((X, \| \cdot \|)\) be a Banach space and \(E\) be bounded set of \(X\). In this paper the following definition of the measure of noncompactness of a set \(E\) is used [7].

\[
\mu(E) = \inf \{r > 0: E \text{ can be covered by a finite number of balls whose radii are smaller than } r\}
\]

The following version of Darbo's fixed point theorem being a generalization of Schauder's fixed point theorem shows the usefulness of the measure of noncompactness [6]. "If \(S\) is a nonempty bounded closed convex subset of \(X\) and \(P : S \rightarrow S\) is a continuous mapping, such that for any set \(E \subseteq S\) we have

\[
\mu(PE) \leq k \mu(E)
\]
where \( k \) is a constant with \( 0 \leq k < 1 \), then \( P \) has a fixed point.''

For the space of continuous functions \( C_n[t_0,t_1] \) with norm

\[
\|x\| = \max\{|x_i(t)| : i = 1,2,...,n, \ t \in [t_0,t_1]\},
\]

the measure of noncompactness of a set \( E \) is given by

\[
\mu(E) = \frac{1}{2} \omega_0(E) = \frac{1}{2} \lim_{h \to 0} \omega(E,h)
\]

where \( \omega(E,h) \) is the common modulus of continuity of the functions which belong to the set \( E \), that is,

\[
\omega(E,h) = \sup_{x \in E} \left[ \sup \{ |x(t) - x(s)| : |t - s| \leq h \} \right]
\]

where, as in the space of continuously differentiable functions \( C[t_0,t_1] \) with norm

\[
\|x\|_{C_n} = \|x\| + \|\dot{x}\|_{C_n},
\]

we have

\[
\mu(E) = (1/2) \omega_0(\text{DE})
\]

where

\[
\text{DE} = \{ \dot{x} : x \in E \}.
\]

3. BASIC ASSUMPTIONS

Consider the following nonlinear delay differential equation with implicit derivative of the form

(1) \[
\dot{x}(t) = f(x(t),x(t-r(x(t),\dot{x}(t),t)),t), \ t \geq t_0
\]

\[
x(t) = \phi(t), \ t \leq t_0
\]

where \( x \in \mathbb{R}^n \) and \( f \) is an \( n \)-vector function. Let \( r(x(t),\dot{x}(t),t) \geq 0 \). Set

\[
\alpha(t) = t-r(x(t),\dot{x}(t),t), \ \text{and}
\]

\[
a = \inf_{(x,\dot{x},t)} \alpha(t) \ \text{and} \ -\infty < a < t_0.
\]

Then \( x(t) = \phi(t) \) on \([a,t_0]\).
Assume that the functions \( f(x,y,t) \) and \( r(x,y,t) \) are continuous and satisfy the following conditions. For \( y,\bar{y},x \in \mathbb{R}^n \) and \( t \in [t_0,t_1] \),

\[
(2) \quad |f(x,y,t)| \leq M
\]

\[
(3) \quad |f(x(t),x(\alpha(t)),t) - f(x(t),x(\alpha(s)),t)| \leq N |\alpha(t) - \alpha(s)|
\]

\[
(4) \quad |r(x,y,t) - r(x,\bar{y},t)| \leq b |y - \bar{y}|
\]

where \( M, N \) and \( b \) are positive constants such that \( 0 < Nb < 1 \). Let \( \phi(t) \) be a continuous \( \mathbb{R}^n \)-valued function defined on \([a,t_0]\).

Definition: The solution of (1) is the function \( x(t) \) such that:

i) \( x(t) \) is defined and continuous on the interval \([a,t_1]\) and is of class \( C^1 \) on \([t_0,t_1]\) such that at the point \( t_0 \) the right side derivative only is taken into account;

ii) The function \( x(t) \) satisfies (1) on the interval \([t_0,t_1]\), whereas on the interval \([a,t_0]\) the function \( x(t) = \phi(t) \).

Next we shall prove that the solution of (1) exists in the sense of the above definition.

4. EXISTENCE THEOREM

Theorem: If the function \( f(x,y,t) \) satisfies the conditions (2) and (3), and if \( r(x(t),\dot{x}(t),t) \geq 0 \) and satisfies the condition (4), then (1) has at least one solution for any initial function \( \phi \in C_n[a,t_0] \).

Proof: Consider the Banach space \( C^1_n[t_0,t_1] \) and the set

\[
H = \{ x: x \in C^1_n[t_0,t_1], x(t_0) = \phi(t_0) \}. 
\]

For any function \( x \in H \), \( x(\alpha(t)) \) will be the function defined in such a way that if \( \alpha(t) < t_0 \) for \( t \in [t_0,t_1] \) then

\[
(6) \quad x(\alpha(t)) = \phi(\alpha(t)). 
\]

Define the mapping \( T \) by

\[
(7) \quad T(x)(t) = \phi(t_0) + \int_{t_0}^{t} f(x(s), x(s - r(x(s), \dot{x}(s), s)), s) ds. 
\]

Moreover, consider the bounded closed set \( B \) in \( H \) as

\[
(8) \quad B = \{ x \in H : \|x\| \leq L, \|\dot{x}\| \leq M \} 
\]
where $L$ and $M$ are positive constants such that

$$L = |\phi(t_0)| + (t_1 - t_0)M.$$  

Since $f$ is continuous, $T$ is continuous and maps $B$ into itself. Next let us estimate the modulus of continuity of the function $DT(x)(t)$ for $t, s \in [t_0, t_1]$. Since the only functions considered belong to some bounded subset of the space $C[t_0, t_1]$, and since these all have uniformly bounded derivatives, it follows that they are equicontinuous.

Thus

$$|DT(x)(t) - DT(x)(s)| = |f(x(t), x(\alpha(t)), t) - f(x(s), x(\alpha(s)), s)|$$

$$\leq |f(x(t), x(\alpha(t)), t) - f(x(s), x(\alpha(t)), s)|$$

$$+ |f(x(s), x(\alpha(t)), s) - f(x(s), x(\alpha(s)), s)|$$

$$\leq |f(x(t), x(\alpha(t)), t) - f(x(s), x(\alpha(t)), s)|$$

$$+ N |t - s| + N |r(x(t), \dot{x}(t), t) - r(x(s), \dot{x}(s), s)|$$

$$+ N |r(x(s), \dot{x}(t), s) - r(x(s), \dot{x}(s), s)|$$

For the first term on the right of inequality (10), one can take the upper estimate as $\beta_0(|t - s|)$, where $\beta_0$ is some non-negative continuous function such that $\lim_{h \to 0} \beta_0(h) = 0$. This follows from the fact that it is formed by the composition of a finite number of functions having a uniformly bounded modulus of continuity. The second and third terms on the right of inequality (10) have the upper bound $Nh + N\beta_1(h)$; and the last term has upper bound $Nb |x(t) - \dot{x}(s)|$. Letting $\beta = \beta_0 + Nh + N\beta_1$ and $k = Nb$, one gets

$$\omega(DT(x), h) \leq k\omega(Dx, h) + \beta(h);$$

hence it follows that

$$\mu(TE) \leq k\mu(E)$$

for any bounded set $E \subset B \subset H$. Consequently, by Darbo's fixed point theorem, the mapping $T$ has a fixed point $x \in C[t_0, t_1]$ such that

$$x(t) = T(x)(t).$$

Clearly the extension of this function to the interval $[a, t_0]$ by means of the function $\phi$ is a solution of equation (1) having the following form:
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(12) \[ x(t) = \phi(t_0) + \int_{t_0}^{t} f(x(s), x(s-r(x(s), c(s), s)), s) ds, \quad t \geq t_0 \]

\[ x(t) = \phi(t), \quad a < t < t_0 \]

Remark: It should be observed that if one assumes that the function \( f \) and \( r \) satisfy also the Lipschitz condition, then the uniqueness of the solution of (1) can be established by standard techniques used in proving the uniqueness theorem.

REFERENCES


