AN EXISTENCE THEOREM FOR A VOLTERRA INTEGRAL EQUATION WITH DEVIATING ARGUMENTS*

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ABSTRACT

An existence theorem is proved for a nonlinear Volterra integral equation with deviating arguments.

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1. INTRODUCTION

The theory of nonlinear Volterra integral equations with deviating arguments and functional integral equations have been studied by many authors [1,2,5,6]. Banas [4] has proved an existence theorem for functional integral equations and Balachandran [1] has proved an existence theorem for a nonlinear Volterra integral equation with deviating argument. In this paper we shall derive a set of sufficient conditions for the existence of a solution of nonlinear Volterra integral equations with deviating arguments. This result is a generalization of the results in [1,4].

2. BASIC ASSUMPTIONS

Let $p(t)$ be a given continuous function defined on the interval $[0, \infty)$ and taking real positive values. Denote $C([0, \infty), p(t); \mathbb{R}^n)$ by C_p , the set of all continuous functions from $[0, \infty)$ into $\mathbb{R}^n$ such that

$$\sup \{ |x(t)|p(t) : t \geq 0 \} < \infty.$$

It has been proved [7] that C_p forms a real Banach space with regard to the norm

$$\|x\| = \sup \{ |x(t)|p(t) : t \geq 0 \}.$$

If $x \in C_p$ then we will denote by $\omega^T(x, \varepsilon)$ the usual modulus of continuity of x on the interval $[0, T]$, that is,

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$$\omega^T(x, \varepsilon) = \sup \{ |x(t) - x(s)| : |t - s| \leq \varepsilon, t, s \in [0, T] \}.$$

Our existence theorem is based on the following lemma.

Lemma (See [3].): *Let E be a bounded set in the space C_p . If all functions belonging to E are equicontinuous on each interval $[0, T]$ and*

$$\lim_{T \rightarrow \infty} \{ |x(t)|p(t) : t \geq T \} = 0$$

uniformly with respect to E , then E is relatively compact in C_p .

Consider the nonlinear Volterra integral equation with deviating arguments

$$(1) \quad x(t) = G(x)g(t, x(h_1(t)), x(h_2(t)), \dots, x(h_n(t))) + \int_0^t K(t, s, x(H_1(s)), \dots, x(H_m(s)))ds$$

where x, H and K are n -vectors and G is a real-valued function. Assume the following conditions.

(i) Let $\Delta = \{(t, s) : 0 \leq s \leq t < \infty\}$. The kernel $K : \Delta \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^n$ is continuous and there exist continuous functions $m : \Delta \rightarrow [0, \infty)$, $a : [0, \infty) \rightarrow (0, \infty)$, and $b : [0, \infty) \rightarrow [0, \infty)$ such that

$$|K(t, s, x_1, x_2, \dots, x_m)| \leq m(t, s) + a(t)b(s) \sum_{i=1}^m |x_i|$$

for all $(t, s) \in \Delta$ and $(x_1, x_2, \dots, x_m) \in \mathbb{R}^{nm}$.

In order to formulate other assumptions let us define $L(t) = \int_0^t a(s)b(s)ds$, $t \geq 0$;

furthermore, let us take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t)e^{ML(t)+t}]^{-1}$.

(ii) There exists a constant $A > 0$ such that for any $t \in [0, \infty)$ the following inequality holds.

$$\int_0^t m(t, s)ds \leq Aa(t)e^{ML(t)}$$

(iii) For $i = 1, 2, \dots, n$ the functions $h_i : [0, \infty) \rightarrow [0, \infty)$ are continuous, $h_i(0) = 0$, $h_i(t) \leq t$ for $t \geq 0$ and there exists a positive real number B_i such that $a(h_i(t)) \leq B_i a(t)$.

(iv) $G : C_p \rightarrow [0, \infty)$ is continuous and bounded. Assume $|G(x)| \leq k_1$ where k_1 is a positive constant.

(v) The function $g : [0, \infty) \times \mathbb{R}^{n^2} \rightarrow \mathbb{R}^n$ is continuous and satisfies the conditions

$$|g(t, x_1, x_2, \dots, x_n) - g(t, y_1, y_2, \dots, y_n)| \leq \sum_{i=1}^n \alpha_i(t) |x_i - y_i|$$

where $\alpha_i(t)$ is continuous such that

$$\alpha_i(t) \leq e^{M[L(t) - L(h_i(t))]} \text{ for } t \geq 0, \text{ for } i = 1, 2, \dots, n \text{ and } |g(t, 0, \dots, 0)| \leq a(t)e^{ML(t)}.$$

(vi) For $i = 1, 2, \dots, m$, the functions $H_i: [0, \infty) \rightarrow [0, \infty)$ are continuous and satisfy the following conditions:

$$L(H_i(t)) - L(t) \leq N_i$$

where N_i is a positive constant, and

$$a(H_i(t))/a(t) \leq (M/m)[1 - k_1(1 + B) - A]e^{-MN_i}$$

where $B = \sum_{i=1}^n B_i$ and we assume $k_1(1 + B) + A < 1$.

3. EXISTENCE THEOREM

Theorem: Assume that the hypotheses (i) through (vi) hold; then equation (1) has at least one solution x in the space C_p such that $|x(t)| \leq a(t)e^{ML(t)}$ for any $t \geq 0$.

Proof: Define a transformation F in the space C_p by

$$(2) \quad \begin{aligned} (Fx)(t) = & G(x)g(t, x(h_1(t)), x(h_2(t)), \dots, x(h_n(t))) \\ & + \int_0^t K(t, s, x(H_1(s)), \dots, x(H_m(s)))ds. \end{aligned}$$

From our assumptions we observe that $(Fx)(t)$ is continuous on the interval $[0, \infty)$. Define the set E in C_p by

$$E = \{x \in C_p: |x(t)| \leq a(t)e^{ML(t)}\}.$$

Clearly E is nonempty, bounded, convex, and closed in C_p . Now we prove that F maps the set E into itself. Take $x \in E$. Then from our assumptions we have

$$\begin{aligned} |(Fx)(t)| &\leq |G(x)| |g(t, x(h_1(t)), \dots, x(h_n(t)))| + \int_0^t |K(t, s, x(H_1(s)), \dots, x(H_m(s)))| ds \\ &\leq k_1 |g(t, x(h_1(t)), \dots, x(h_n(t))) - g(t, 0, \dots, 0)| + k_1 |g(t, 0, \dots, 0)| + \int_0^t m(t, s) ds \\ &\quad + a(t) \int_0^t b(s) \sum_{i=1}^m |x(H_i(s))| ds \\ &\leq k_1 \sum_{i=1}^n \alpha_i(t) |x(h_i(t))| + k_1 a(t) e^{ML(t)} + A a(t) e^{ML(t)} \\ &\quad + a(t) \int_0^t b(s) \sum_{i=1}^m a(H_i(s)) e^{ML(H_i(s))} ds \\ &\leq k_1 \sum_{i=1}^n e^{M[L(t) - L(h_i(t))]} a(h_i(t)) e^{ML(H_i(t))} + k_1 a(t) e^{ML(t)} + A a(t) e^{ML(t)} \end{aligned}$$

$$\begin{aligned}
& + a(t)([1 - k_1(1+B) - A]/m) \sum_{i=1}^m e^{-MN_i} \int_0^t M b(s) a(s) e^{MN_i} e^{ML(s)} ds \\
& \leq k_1 B a(t) e^{ML(t)} + k_1 a(t) e^{ML(t)} + A a(t) e^{ML(t)} + a(t)[1 - k_1(1+B) - A] e^{ML(t)} \\
& = a(t) e^{ML(t)},
\end{aligned}$$

which proves that FE is a subset of E .

Now we want to prove that F is continuous on the set E . In order to do this we take $F = F_1 + F_2$, where

$$(F_1 x)(t) = G(x) g(t, x(h_1(t)), \dots, x(h_n(t)))$$

$$\text{and } (F_2 x)(t) = \int_0^t K(t, s, x(H_1(s)), \dots, x(H_m(s))) ds.$$

We shall prove the continuity of F_1 and F_2 separately. Let us fix $\varepsilon > 0$ and take $x, y \in E$ such that $\|x - y\| \leq \varepsilon$. We have

$$\begin{aligned}
& \|F_1 x - F_1 y\| \leq k_1 \sup \{ |g(t, x(h_1(t)), \dots, x(h_n(t))) - g(t, y(h_1(t)), \dots, y(h_n(t)))| \cdot [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} \\
& + |G(x) - G(y)| \sup \{ |g(t, y(h_1(t)), \dots, y(h_n(t)))| \cdot [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} \\
& \leq k_1 \sum_{i=1}^n \sup \{ \alpha_i(t) |x(h_i(t)) - y(h_i(t))| [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} \\
& + |G(x) - G(y)| \sup \{ |g(t, y(h_1(t)), \dots, y(h_n(t))) - g(t, 0, \dots, 0)| \cdot [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} \\
& + |G(x) - G(y)| \sup \{ |g(t, 0, \dots, 0)| \cdot [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} \\
& \leq k_1 \sum_{i=1}^n B_i \sup \{ |x(h_i(t)) - y(h_i(t))| [a(h_i(t))]^{-1} e^{M[L(t)-L(h_i(t))]} e^{-ML(t)-t} : t \geq 0 \} \\
& + |G(x) - G(y)| \sum_{i=1}^n \sup \{ \alpha_i(t) |y(h_i(t))| [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} \\
& + |G(x) - G(y)| \sup \{ e^{-t} : t \geq 0 \} \\
& \leq k_1 \sum_{i=1}^n B_i \sup \{ |x(h_i(t)) - y(h_i(t))| [a(h_i(t)) e^{ML(h_i(t)) + h_i(t)}]^{-1} : t \geq 0 \} \\
& + |G(x) - G(y)| \sum_{i=1}^n \sup \{ a(h_i(t)) e^{ML(t)} [a(t) e^{ML(t)+t}]^{-1} : t \geq 0 \} + |G(x) - G(y)| \\
& \leq k_1 B \|x - y\| + B |G(x) - G(y)| + |G(x) - G(y)|.
\end{aligned}$$

This implies that F_1 is continuous in view of (iv).

Now we prove that F_2 is continuous on the set E . For this let us fix $\varepsilon > 0$ and $x, y \in E$ such that $\|x - y\| \leq \varepsilon$. Further, let us take an arbitrary fixed $T > 0$. In view of (i) and (vi) the function $K(t, s, x_1, \dots, x_m)$ is uniformly continuous on

$$[0, T] \times [0, T] \times [-r(H_1(T)), r(H_2(T))] \times \dots \times [-r(H_m(T)), r(H_m(T))]$$

where $r(T) = \max\{a(s)e^{ML(s)} : s \in [0, T]\}$. Thus, we have for $t \in [0, T]$

$$(3) \quad |(F_2 x)(t) - (F_2 y)(t)| \leq \int_0^t |K(t, s, x(H_1(s)), \dots, x(H_m(s)) - K(t, s, y(H_1(s)), \dots, y(H_m(s)))| ds \leq \beta(\epsilon)$$

where $\beta(\epsilon)$ is some continuous function such that $\lim_{\epsilon \rightarrow 0} \beta(\epsilon) = 0$. Further, let us take

$t \geq T$. Then we have

$$|(F_2 x)(t) - (F_2 y)(t)| \leq |(F_2 x)(t)| + |(F_2 y)(t)| \leq 2a(t)e^{ML(t)}$$

$$\text{and } |(F_2 x)(t) - (F_2 y)(t)|/p(t) \leq 2e^{-t}.$$

Hence for sufficiently large T we have

$$(4) \quad |(F_2 x)(t) - (F_2 y)(t)|/p(t) \leq \epsilon.$$

By (3) and (4) we get that F_2 is continuous on the set E . Hence $F = F_1 + F_2$ is continuous on E .

Now we prove that FE is relatively compact. For every $x \in E$ we have $Fx \in E$ which gives $|(Fx)(t)|/p(t) \leq e^{-t}$. Hence $\lim_{T \rightarrow \infty} \sup\{|(Fx)(t)|/p(t) : t \geq T\} = 0$ uniformly with respect to $x \in E$.

Furthermore, let us fix $\epsilon > 0$ and $T > 0$; and let $t, s \in [0, T]$ such that $|t-s| \leq \epsilon$. Then for $x \in E$, we have

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| &\leq |G(x)| |g(t, x(h_1(t)), \dots, x(h_n(t))) - g(s, x(h_1(s)), \dots, x(h_n(s)))| \\ &\quad + \left| \int_0^t K(t, u, x(H_1(u)), \dots, x(H_m(u))) du - \int_0^s K(s, u, y(H_1(u)), \dots, y(H_m(u))) du \right| \end{aligned}$$

$$\begin{aligned}
& \leq k_1 \omega^T(g, \varepsilon) + \left| \int_0^t K(t, u, x(H_1(u)), \dots, x(H_m(u))) du \right. \\
& \quad \left. - \int_0^s K(t, u, x(H_1(u)), \dots, x(H_m(u))) du \right| \\
& \quad + \left| \int_0^s K(t, u, x(H_1(u)), \dots, x(H_m(u))) du \right. \\
& \quad \left. - \int_0^s K(s, u, x(H_1(u)), \dots, x(H_m(u))) du \right| \\
& \leq k_1 \omega^T(g, \varepsilon) + \int_s^t |K(t, u, x(H_1(u)), \dots, x(H_m(u)))| du \\
& \quad + \int_0^s |K(t, u, x(H_1(u)), \dots, x(H_m(u))) \\
& \quad - K(s, u, x(H_1(u)), \dots, x(H_m(u)))| du \\
& \leq k_1 \omega^T(g, \varepsilon + \varepsilon \max\{m(t, u) + a(t)b(u) \sum_{i=1}^m |x(H_i(u))| : 0 \leq u \leq t \leq T\}) \\
& \quad + T \omega^T(K, \varepsilon).
\end{aligned}$$

This tends to zero as $\varepsilon \rightarrow 0$. Thus FE is equicontinuous on $[0, T]$.

Therefore from the lemma FE is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $x \in E$ such that $(Fx)(t) = x(t)$. Hence the theorem holds.

EXAMPLE: Consider the following nonlinear integral equation:

$$(5) \quad x(t) = (1/8) t \sin x(t/2) + \int_0^t [ts + (t^3 + (1/4)) x(s/3)] ds.$$

This is clearly of the form (1) and satisfies all the conditions (i) to (vi) with

$$\begin{aligned}
m &= n = 1, G(x) = 1/8, g(t, x) = t \sin x, \\
K(t, s, x) &= ts + (t^3 + (1/4)) x, \\
m(t, s) &= ts, a(t) = t^3 + (1/4), b(s) = 1, \\
\alpha_1(t) &= t, h_1(t) = t/2, H_1(t) = t/3, \\
M &= 16, A = 1/2, B = 1, k_1 = 1/8, N_1 = 1/16.
\end{aligned}$$

Therefore from our existence theorem the equation (5) has at least one solution x in the space C_p such that

$$|x(t)| \leq (t^3 + 1/4)e^{4(t^4 + t)} \text{ for any } t \geq 0.$$

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