DIFFERENCE EQUATIONS OF VOLTERRA TYPE AND EXTENSION OF LYAPUNOV'S METHOD

M. Zouyousefain

Division of Math, Science, and Technology

Mountain View College, Dallas, Texas 75211

Abstract

In this paper, the stability properties of nonlinear difference equations of Volterra type is discussed. For this purpose some comparison Theorems are developed, then using these results the stability of the nonlinear difference equations of Volterra type is investigated.

Key words: Asymptotic stability, Comparison method, Difference equations of Volterra type, Difference inequalities, Linearization method, Lyapunov's method, Minimal classes of functions, Nonlinear equations, Scalar difference equation, Trivial solution, Uniform stability.

AMS classification index: 39.

1. Introduction.

Consider the nonlinear difference equation of Volterra type

$$\Delta x(n) = f(n, x(n), \sum_{s=n_0}^{n-1} G(n, s, x(s))),$$

(1.1)

$$x(n_0) = n_0, \quad n_0 \in N_{n_0}^{+},$$

Received: January 1990, Revised: May 1990

where f, G : $N_{n_0}^{+} \chi \Re^d \chi \Re^d \to \Re^d$. Stability results of (1.1) are discussed recently by linearization method in [3]. In this paper, we are interested in extending the Lyapunov's method to discuss stability properties of the system (1.1).

When we employ a Lyapunov function, we are faced with two questions. One is to estimate variation of the Lyapunov function relative to the system (1.1) in terms of a function, in which case a basic question is to select a minimal class of functions for which this can be done. Thus by using the theory of difference inequalities and choosing minimal classes of functions suitably, it is possible to establish stability properties of difference equations of Volterra type (1.1) by reducing the study of (1.1) to a simple difference equation. The second approach is to estimate the vairation of Lyapunov function by means of a functional so that the study of (1.1) is reduced to the study of a rlatively simple differenece equation of Volterra type. This approach makes the choice of minimal classes unnecessary but it requires developing the theory of difference inequalities of Volterra type. Therefore, finding the stability properties of even simple diffenence equation of Volterra type is comparatively more difficult than simple difference equations. Both methods offer a unified approach.

Extension of Lyapunov method for integro-differential equations is discussed in [1]. For stability results using Lyapunov method for difference equations see [2].

2. Comparison results.

It is well known that comparison principle is one of the most efficient methods for studying the qualitative behavior of solutions of nonlinear systems. Let us begin by proving the following comparison results relative to the scalar difference equation of Volterra type.

Theorem 2. 1. Assume that $g : N_{n_0}^{+} \chi \Re_{+} \chi \Re_{+} \to \Re$, and $g(n, u, v)$ is nondecreasing in u, v for fixed $n \in N_{n_0}^{+}$. Suppose further that

$$u : N_{n_0}^{+} \to \Re_{+} \quad \text{and} \quad n \geq n_0, \quad u(n_0) = u_0 \geq 0,$$

$$\Delta u(n) \leq g(n, u(n), \sum_{s=n_0}^{n-1} u(s)) - u(n).$$

Then $u(n) \leq r(n)$, $n \geq n_0$, where $r(n) = r(n, n_0, u_0)$ is the solution of the scalar difference equation of Volterra type.

$$\Delta r(n) = r(n+1) - r(n) = g(n, r(n), \sum_{s=n_0}^{n-1} r(s)) - r(n),$$

(2.1)

$$r(n_0) \geq u_0.$$

Proof. Let $u(n_0) \leq r(n_0)$ and suppose that $u(n) > r(n)$, $n \geq n_0$. Then there exists a $k > n_0$ such that

$$u(k) \leq r(k) \text{ and } u(k+1) > r(k+1).$$

This implies that

$$\sum_{s=n_0}^{n-1} u(s) \leq \sum_{s=n_0}^{n-1} r(s).$$

Hence, using the monotone character of g, we get

$$u(k+1) \leq g(k, u(k), \sum_{s=n_0}^{n-1} u(s))$$

$$\leq g(k, r(k), \sum_{s=n_0}^{n-1} r(s)) = r(k+1).$$

This contradiction proves the theorem.

The next comparison result is more general but requires the functional to be nonnegative.

Theorem 2. 2. Assume that

$$g : N_{n_0}^{+} \chi \Re_+ \chi \Re_+ \to \Re_+ , \quad H : N_{n_0}^{+} \chi \Re_+ \chi \Re_+ \to \Re_+$$

and $g(n, u, v)$, $H(n, s, u)$ are nondecreasing in u, v for each $n, s \in N_{n_0}^{+}$.

Let $u : N_{n_0}^{+} \to \Re_+$ and for $n \geq n_0$

$$\Delta u(n) \leq g(n, u(n), \sum_{s=n_0}^{n-1} H(n, s, u(s))),$$

$$u(n_0) = u_0 \geq 0.$$

Then $u(n) \leq r(n)$, $n \geq n_0$, where $r(n) = r(n, n_0, u_0)$ is the solution of the scalar difference equation

$$\Delta r(n) = g(n, r(n), \sum_{s=n_0}^{n-1} H(n, s, r(n))),$$

(2.2)

$$r(n_0) \geq u_0.$$

Proof. Set $P(n) = g(n, u(n), \sum_{s=n_0}^{n-1} H(n, s, u(s)))$ so that $u(n) \leq \Delta^{-1}P(n) + W(n)$ where Δ^{-1} is the antidifference operator and $W(n)$ is an arbitrary function of period 1. If $Z(n) = \Delta^{-1}P(n) + W(n)$, then we see that $\Delta Z(n) = P(n) \geq 0$ since $g \geq 0$. Hence $Z(n)$ is nondecreasing and therefore we have

$$u(n) \leq Z(n), \text{ for every } n \geq n_0.$$

Consequently using the monotone character of g and H, we get

$$\Delta Z(n) \leq g(n, Z(n), \sum_{s=n_0}^{n-1} H(n, s, Z(n)))$$

$$= G(n, Z(n)), \; Z(n_0) \geq u_0.$$

Hence by comparison theorem for difference equation, we have

$$Z(n) \leq r(n) , \; n \geq n_0,$$

where $r(n)$ is the solution of (2.2). Since $u(n) \leq Z(n)$, the proof is complete.

Next we shall discuss a comparison result in terms of Lyapunov function. For this purpose, we need to define the variation of a Lyapunov function.

If $V : N_{n_0}^{+} \chi \Re^d \to \Re_+$, then we define the variation of V relative to the system (1.1) by

$$\Delta V(n, x(n)) = V(n+1, x(n+1)) - V(n, x(n)),$$

Also let us define the minimal set Ω given by

$$\Omega = [x(n) : N_{n_0}^{+} \to \Re^d : V(s, x(s)) \leq V(n, x(n)), \ n_0 \leq s \leq n].$$

Then we have the following result.

Theorem 2. 3. Suppose that $g : N_{n_0}^{+} \chi \Re_+ \to \Re_+$, $g(n, u)$ is nondecreasing in u for each $n \in N_{n_0}^{+}$ and

$$\Delta V(n, x(n)) \leq g(n, v(n, x(n))) \tag{2.3}$$

for $x(n) \in \Omega$ and $n \geq n_0$. Then $V(n_0, x(n_0)) \leq u(n_0)$ implies $V(n, x(n)) \leq u(n)$, $n \geq n_0$, where $x(n)$ is the solution of (1.1) and $u(n)$ is the solution of

$$\Delta u(n) = g(n, u(n)), \ u(n_0) = u_0. \tag{2.4}$$

Proof. Suppose the assertion is false. Then there exists a $k > n_0$ such that

$$V(k, x(k)) \leq u(k), \text{ and } V(k+1, x(k+1)) > u(k+1).$$

Since $g \geq 0$, $u(n)$ is nondecreasing sequence and therefore we have for $n_0 \leq s \leq k$,

$$V(s, x(s)) \leq u(s) \leq u(k) \leq u(k+1) \leq V(k+1, x(k+1)).$$

This implies that $x(k+1) \in \Omega$. Consequently, with (2.3) and the monotone character of g,

$$\begin{aligned} V(k, x(k)) + g(k, V(k, x(k)) &\geq V(k+1, x(k+1)) \\ &> u(k+1) = u(k) + g(k, u(k)) \\ &\geq u(k) + g(k, x(k))). \end{aligned}$$

This leads to the contradiction

$$V(k, x(k)) > u(k)$$

and hence the proof is complete.

Another comparison theorem which is sometimes useful, is the following.

Theorem 2. 4. Assume that $g : N_{n_0}^{+} \chi \Re_{+} \to \Re_{+}$, $g(n, u)$ is nondecreasing in u for each $n \in N_{n_0}^{+}$, $A : N_{n_0}^{+} \to \Re_{+}$, $A(n) > 0$ for $n \geq n_0$ and

$$(\Delta V(n, x(n)))A(n+1) + (\Delta A(n))V(n, x(n)) \leq g(n, A(n)V(n, x(n))),$$

where

$$x(n) \in \Omega_A = [x(n) : N_{n_0}^{+} \to \Re^d : A(s)V(s, x(s)) < A(n)V(n, x(n)), \ n_0 \leq s \leq n.$$

Then $A(n_0)\ V(n_0, x(n_0)) \leq u_0$ implies

$$A(n)V(n, x(n)) \leq u(n), \ n \geq n_0,$$

where $u(n)$ is the solution of (2.4).

Proof. Setting $L(n, x(n)) = A(n)V(n, x(n))$, it is easy to compute that

$$\Delta L(n, x(n)) \leq g(n, L(n, x(n))),$$

for $x(n) \in \Omega = [x(n) : L(s, x(s)) < L(n, x(n)), \ n_0 \leq s \leq n]$. It then follows from Theorem 2. 3 the stated result and the proof is complete.

3. Stability Results.

Having the necessary comparison results, it is now easy to investigate stability properties of solutions of the system (1.1). For this purpose, we shall assume that $f(n, 0, 0) = 0$, and $G(n, s, 0) = 0$, so that we have the trivial solution $x(n) = 0$ for the system (1.1).

Let us recall that a function $\phi(u)$ is said to be of class κ if it is continuous in $[0, \rho)$, strictly increasing in u and $\phi(0) = 0$. Using the comparison Theorem 2. 3, we can now prove stability porperties of the null solution of (1.1) in a unified way.

Theorem 3. 1. Suppose that there exists two functions $V(n, x)$ and $g(n, u)$ satisfying the conditions:

(i) $g : N_{n_0}^{+} \chi \Re_{+} \to \Re$, $g(n, 0) = 0$, $g(n, u)$ is nondecreasing in u for each $n \in N_{n_0}^{+}$,

(ii) $V : N_{n_0}^{+} \chi S(\rho) \to \Re_{+}$, $V(n, x)$ is continuous in x and the variation of V relative to (1.1) satisfies the estimate

(*) $\Delta V(n, x(n)) \leq g(n, v(n, x(n)))$ whenever $x(n) \in \Omega$, $n \geq n_0$, where $S(\rho) = [x \in \Re^d : ||x|| < \rho]$.

(iii) $b(||x||) \leq V(n, x) \leq a(||x||)$, where $a, b \in \kappa$.

Then the stability properties of the trivial solution of (2.4) imply the corresponding stability properties of the Volterra system (1.1).

Proof. Let $0 < \epsilon < \rho$ and $n_0 \geq 0$ be given. Assume that the trivial solution of (2.4) is stable. Then, given $b(\epsilon) > 0$ and $n_0 \geq 0$, there exists a $\delta_1 = \delta_1(n_0, \epsilon) > 0$ such that

$$u(n_0) < \delta_1 \text{ implies } u(n) < \epsilon, \quad n \geq n_0. \tag{3.1}$$

Choose $\delta = \delta(n_0, \epsilon) > 0$ such that $a(\delta) < \delta_1$. Then we claim that the null solution of (1.1) is stable with this δ. If this is false, then there would exist a solution $x(n)$ of (1.1) such that $\| x(n_0) \| < \delta$ and an $n_1 > n_0$ with

$$\| x(n_1) \| = \epsilon \text{ and } \| x(s) \| \leq \epsilon < \rho, \quad n_0 < s < n_1. \tag{3.2}$$

This shows by Theorem 2. 3 that

$$V(n, x(n)) \leq u(n), \quad n_0 \leq n \leq n_1, \tag{3.3}$$

where $u(n) = u(n, n_0, u(n_0))$ is the solution of (2.4). We choose $V(n_0, x(n_0)) = u(n_0)$, so that when $||x(n_0)|| < \delta$, we have $u(n_0) \leq a(\delta) < \delta_1$. Now the relation (3.1), (3.2), (3.3) and condition (iii) lead to the contradiction

$$b(\epsilon) = b(||x(n_1)||) \leq V(n_1, x(n_1)) \leq u(n_1) = u(n_1, n_0, \delta_1) < b(\epsilon).$$

Hence the trivial solution of (1.1) is stable.

If we suppose that the trivial solution of (2.4) is uniformly stable, then it is clear from the above proof that δ is independent of n_0 and hence we get the uniform stability of the trivial solution of (1.1).

If we suppose that the trivial solution of (2.4) is asymptotically stable, we then have

$$V(n, x(n)) \leq u(n), \text{ for all } n \geq n_0,$$

in view of stability. Consequently, condition (iii) implies the asymptotic stability of the trivial solution of (1.1). The proof of the theorem is complete.

Corollary 3. 1. The functions

(i) $g(n, u) = 0$

(ii) $g(n, u) = a_n u, \ a_n \geq 0$ with $\prod_{i=n_0}^{n-1} a_i < r(n_0)$,

are admissable to yield uniform stability of (1.1) in Theorem 3. 1. Furthermore, if in (ii),

$$\prod_{i=n_0}^{n-1} a_i < M\eta^{n-n_0}$$

with $0 < \eta < 1$, then exponential stability of the system (1.1) follows.

Theorem 3. 2. Let the assumptions (i) and (iii) of Theorem 3. 1 hold. Suppose further that

$$A : N_{n_0}^{+} \to [1, \infty), \ A(n) \to \infty \text{ as } n \to \infty,$$

$$V : N_{n_0}^{+} \chi \ S(\rho) \to \Re_{+}, \ V(n, x) \text{ is continuous in } x \text{ and}$$

$$(\Delta V(n, x(n)))A(n+1) + (\Delta A(n))V(n, x(n)) \leq g(n, A(n)V(n, x(n)))$$

for $x(n) \in \Omega_A$. Let the trivial solution of (2.4) be stable. Then the trivial solution of

(1.1) is asymptotically stable.

Proof. Proceeding as in the proof of Theorem 3. 1, we obtain the stability of the trivial solution of (1.1) since $A(n) \geq 1$ for $n \geq n_0$. Then it is easy to get the estimate

$$A(n)V(n, x(n)) \leq u(n), \quad n \geq n_0, \tag{3.4}$$

provided $||x(n_0)|| < \delta_0$ where $\delta_0 = \delta(n_0, \rho)$ corresponding to $\epsilon = \rho$. It then follows that from (2.4), in view of the assumptions on A(n), that $\lim_{n \to \infty} ||x(n)|| = 0$ which proves the asymptotic stability of the trivial solution of (1.1). The proof is complete.

Let us next employ the comparison Theorem 2. 1 so that we do not need minimal class of functions.

Theorem 3. 3. Let there exist two function V(n, x) and g(n, u, v) satisfying the conditions:

(i) $g : N_{n_0}^{+} \chi \Re_+ \chi \Re_+ \to \Re$, $g(n, 0, 0) = 0$, $g(n, u, r)$ is nondecreasing in u, r for each n,

(ii) $V : N_{n_0}^{+} \chi S(\rho) \to \Re_+$, V(n, x) is continuous in x and

$$\Delta V(n, x(n)) \leq g(n, V(n, x(n)), \sum_{s=n_0}^{n-1} V(s, x(s))), \text{ for } n \geq n_0.$$

(iii) $b(||x||) \leq V(n, x) \leq a(||x||)$, where $a, b \in \kappa$.

Then the stability properties of the trivial solution of the scalar difference equation of Volterra type (2.1) imply the corresponding stability properties of the trivial solution of the system (1.1).

Proof. If x(n) is any solution of (1.1), we obtain, using Theorem 2. 1 the estimate

$$V(n, x(n)) \leq r(n), \quad n \geq n_0,$$

where r(n) is the solution of the scalar equation (2.1). Hence using an argument similar

to Theorem 3. 1 with suitable modifications, we can construct the proof of the theorem. We omit the details.

Theorem 3. 4. Let the assumptions of Theorem 3. 3 hold except that the equation (3.5) is now replaced by

$$\Delta V(n, x(n)) \leq g(n, V(n, x(n)), \sum_{s=n_0}^{n-1} H(n, s, V(s, x(s)))),$$

where $H : N_{n_0}^{+} \chi N_{n_0}^{+} \chi \Re_{+} \rightarrow \Re$ and $g \geq 0$. Then the stability properties of the null solution of (2.2) imply the corresponding stabilty properties of the null solution of (1.1).

Proof. Based on the comparison Theorem 2. 2 and the proof of Theorem 3. 1, it is not difficult to construct the proof. We omit the details to avoid monotony.

Acknowledgment. The author would like to express his gratitude to Professor V. Lakshmikantham for his helpful comments and suggestions.

References:

[1] Laksmikantham, V. and Rao, M., Integro-differential equations and extension of Lyapunov's method, J. M. A. A. , 30: 435-447 (1970).

[2] Laksmikantham, V. and Trigiante, D., Theory of Difference Equations: Numerical Methods and Applications, Academic Press, New York, 1988.

[3] Zouyousefain, M. and Leela, S. Stability results for difference equations of Volterra type, Applied Mathematics and Computations, (To appear).