Rothe's Method to Semilinear Hyperbolic Integrodifferential Equations

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ABSTRACT

In this paper we consider an application of Rothe's method to abstract semi-linear hyperbolic integrodifferential equations in Hilbert spaces. With the aid of Rothe's method we establish the existence of a unique strong solution.

Key words: Rothe's Method, Positive Definite Operator, V-Elliptic Operator, Lax-Milgram Lemma.

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1. INTRODUCTION

In this paper we are concerned with the application of Rothe's method to the following semi-linear hyperbolic integrodifferential equation

\[\frac{d^2 u(t)}{dt^2} + Au(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t), \ a.e. \ t \in I \]

\[u(0) = U_0 \in \mathcal{V}, \ \frac{du}{dt}(0) = U_1 \in \mathcal{V} \]

where \(u\) is an unknown function from \(I := [0,T], \ 0<T<\infty\), into a real Hilbert space \(\mathcal{H}\), \(A\) is a bounded linear operator from another Hilbert space \(\mathcal{V}\) into its dual space \(\mathcal{V}^*\), \(k\) is a nonlinear mapping from \([0,T] \times \mathcal{V}\) into \(\mathcal{H}\), \(a\) and \(f\), respectively, are real-valued and \(\mathcal{H}\)-valued functions on \([0,T]\).
Earlier, some of the applications of Rothe’s method to the homogeneous and nonhomogeneous linear hyperbolic problems have been considered by Rektorys [6], Putlar [5] and Streiblova [8] (other references are cited in these papers).

Kačur [4] has applied Rothe’s method to a semilinear hyperbolic equation under a global Lipschitz-like condition on nonlinear forcing term. Recently, Bahuguna [1,2] has employed Rothe’s method to a more general case of the problem considered by Kačur [4] and has proved the local existence under local Lipschitz condition on nonlinear forcing terms.

Similar kinds of nonlinear integral perturbations as in (1.1) have been investigated by Bahuguna and Raghavendra [3] (see also [2]) for nonlinear parabolic problems with the aid of Rothe’s method.

2. ASSUMPTIONS AND MAIN RESULT

Let $\mathcal{V}$ and $\mathcal{H}$ be two real Hilbert spaces such that $\mathcal{V}$ is dense in $\mathcal{H}$ and the embedding of $\mathcal{V}$ in $\mathcal{H}$ is compact. We denote by $\|\cdot\|$ and $|\cdot|$ the respective norms of $\mathcal{V}$ and $\mathcal{H}$. Furthermore, the inner product in $\mathcal{H}$ and the usual duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ are denoted by $(u,v)$, $u, v \in \mathcal{H}$; and $\langle f,v \rangle \in \mathcal{V}^*$, $v \in \mathcal{V}$; respectively. Let $I$ denote the interval $[0,T]$ where $0 < T < \infty$ is arbitrary. We introduce the following hypotheses:

1. **(H$_1$)** The bounded linear operator $A: \mathcal{V} \to \mathcal{V}^*$ is symmetric and $\mathcal{V}$-elliptic, i.e.
   \[ \langle Au, v \rangle = \langle Av, u \rangle \quad \text{and} \quad \langle Au, u \rangle \geq \alpha \|u\|^2 \]
   for all $u, v \in \mathcal{V}$ and $\alpha > 0$ is a constant.

2. **(H$_2$)** $k: I \times \mathcal{V} \to \mathcal{H}$ is continuous in both variables and satisfies
   \[ |k(t,u)| \leq C_1 \|u\| + C_2 \]
   for all $t \in I$ and all $u \in \mathcal{V}$, where $C_1$ and $C_2$ are positive constants.

3. **(H$_3$)** The mapping $k$ satisfies
   \[ |k(t,u) - k(t,v)| \leq L(t) \|u - v\| \]
   for $t \in I$ a.e. and all $u, v \in \mathcal{V}$, where $L \in L^1(I)$ is nonnegative.

4. **(H$_4$)** Functions $f: I \to \mathcal{H}$ and $a: I \to \mathbb{R}$ are Lipschitz continuous.

To apply Rothe’s method to equation (1.1), we proceed as follows. For every positive integer $n$ denote by $\{t^*_j\}$ the partition of the interval $I$ defined by $t^*_j = j \cdot h$, $h = \frac{T}{n}$, $j = 1, ..., n$. Setting

\[
\begin{align*}
(2.1) & \quad u_0^0 = U_0, \quad u_{-1}^0 = U_0 - hU_1 \\
(2.2) & \quad u_{-2}^n = h^2(f(0) - A U_0) - 2hU_1 + U_0, 
\end{align*}
\]

we successively look for a solution $u^n_j \in \mathcal{V}$ of the variational identity.
for all $v \in \mathcal{V}$ and $j=1,2,...,n$. The existence of a unique solution satisfying (2.3) is a consequence of Lax-Milgram Theorem, see Rektorys [7, p. 383]. Denote

$$z_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad s_j^n = \frac{z_j^n - z_{j-1}^n}{h}, \quad j=0,1,...,n$$

and define Rothe's sequences $\{U^n\}$ and $\{Z^n\}$ of Lipschitz continuous functions respectively from $I$ into $\mathcal{V}$ and from $I$ into $\mathcal{H}$ by

\[
\begin{align*}
U^n(t) &= u^n_{j-1} + \frac{1}{h} (t - t^n_{j-1})(u^n_j - u^n_{j-1}) \\
Z^n(t) &= z^n_{j-1} + \frac{1}{h} (t - t^n_{j-1})(z^n_j - z^n_{j-1})
\end{align*}
\]

and sequences $\{u^n\}$, $\{z^n\}$, $\{s^n\}$ of step functions from $(-\frac{h}{2}, \frac{h}{2}]$ into $\mathcal{V}$, by

\[
\begin{align*}
u^n(t) &= u^n_0 & u^n(t) &= u^n_j \\
z^n(t) &= z^n_0 & z^n(t) &= z^n_j \\
s^n(t) &= s^n_0 & s^n(t) &= s^n_j
\end{align*}
\]

After proving some a priori bounds for the sequences of functions $\{U^n\}$, $\{Z^n\}$, $\{u^n\}$, $\{z^n\}$ and $\{s^n\}$ we prove the following main existence result for equation (1.1).

**Theorem 2.1.** Assume that Hypotheses $(H_1)$, $(H_2)$, and $(H_4)$ hold and let $AU_0 \in \mathcal{H}$. Then there exists a function $u$ in $\text{Lip}(I, \mathcal{V})$ with the properties

\[
\frac{du}{dt} \in L_\infty(I, \mathcal{V}) \cap C(I, \mathcal{H}), \quad \frac{d^2u}{dt^2} \in L_\infty(I, \mathcal{H})
\]

$Au \in L_\infty(I, \mathcal{H})$, $u(0) = U_0$, $\frac{du}{dt}(0) = U_1$

and $u$ satisfies the identity

\[
\left( \frac{d^2u}{dt^2}(t), v \right) + \left< Au(t), v \right> = (K(u)(t) + f(t), v)
\]

for $t \in I$ a.e. and for all $v \in \mathcal{V}$, where

\[
K(u)(t) = \int_0^1 a(t-s)k(s, u(s))\,ds
\]

In addition, if $(H_3)$ is also satisfied, then $u$ is unique.

For the notational convenience, we drop the superscript $n$ and denote for $0 \leq i, j \leq n$ by...
\[ a_{ji} = a(t_j - t_i) \]
\[ k_j = k(t_j, u_j) \]
\[ f_j = f(t_j) \]

Henceforth, \( C \) will represent a generic constant independent of \( j, h \) and \( n \). Below we state and prove all lemmas required in the proof of Theorem 2.1 which is proved at the end.

**Lemma 2.1.** Assume that hypotheses \( (H_1) \), \( (H_2) \) and \( (H_4) \) hold. Then there exists a positive integer \( N \) such that
\[ |z_j|^2 + \|u_j\|^2 \leq C, \quad j = 1, 2, \ldots, n, \quad n > N. \]

**Proof.** Using the notations of \( (2.4) \) and \( (2.9) \) in \( (2.3) \), for all \( v \in \mathbb{V} \) and \( j = 1, 2, \ldots, n \), we have
\[ (z_j - z_{j-1}, v) + h < A u_j, v > = h^2 \left( \sum_{i=0}^{j-1} a_{ji} k_i, v \right) + h f_j, v). \]

Putting \( v = z_j \) in \( (2.10) \), using \( (H_2) \) and the identities
\[ 2(z_j - z_{j-1}, z_j) = |z_j|^2 + |z_j - z_{j-1}|^2 - |z_{j-1}|^2, \]
\[ 2<A u_j, u_j - u_{j-1}> = \|u_j\|^2 + \|u_j - u_{j-1}\|^2 - \|u_{j-1}\|^2, \]
we obtain
\[ |z_j|^2 - |z_{j-1}|^2 + \|u_j\|^2 - \|u_{j-1}\|^2 \leq Ch |z_j|^2 + Ch^2 \sum_{i=0}^{j-2} \|u_i\|^2 + Ch. \]

Choose a positive integer \( N \) such that \( CT/N < 1 \). Then for \( n > N \), inequality \( (2.11) \) implies that
\[ (1 - Ch) |z_j|^2 + \|u_j\|^2 \leq (1 + Ch^2) |z_{j-1}|^2 + \|u_{j-1}\|^2 + Ch^2 \sum_{i=0}^{j-1} \|u_i\|^2 + Ch. \]

Applying inequality \( (2.12) \) recursively, we obtain
\[ (1 - Ch)^j |z_j|^2 + \|u_j\|^2 \leq (1 + j Ch^2)^j |z_0|^2 + \|u_0\|^2 + j Ch. \]

Inequality \( (2.13) \) implies
\[ |z_j|^2 + \|u_j\|^2 \leq C \]
which together with the \( \mathbb{V} \)-ellipticity of \( A \) proves the assertion of the lemma.

**Lemma 2.2.** Assume the hypotheses of Lemma 2.1 and let \( A u_0 \in \mathbb{K} \). Then there exists a positive integer \( N \) such that
\[ \|z_j\|^2 + |z_j|^2 \leq C, \quad j = 1, 2, \ldots, n, \quad n > N. \]

**Proof.** We rewrite \( (2.10) \) as
\[ (s_j, v) + < A u_j, v > = h \sum_{i=0}^{j-1} (a_{ji} k_i, v) + (f_j, v). \]

Thus we have
\[ (s_j, v) + < A u_j - A u_{j-1}, v > = (s_{j-1}, v) + h (a_{j, j-1} k_{j-1}, v) \]
\[ + h \sum_{i=0}^{j-1} ([a_{ji} - a_{j-1, i}] k_i, v) \]
Putting $v = s_j$ in (2.15) using (H2) and (H3) and (H4) we obtain

$$|s_j|^2 - |s_{j-1}|^2 + \|z_j\|^2 - \|z_{j-1}\|^2$$

$$\leq C h |s_j|^2 + C h^2 \sum_{i=0}^{j-1} \|z_i\|^2 + C h.$$  

We assume that $N$ is large enough such that $N > 1$. For $n > N$, inequality (2.16) then implies that

$$(1 - C h) [ |s_j|^2 + \|z_j\|^2 ]$$

$$\leq (1 + C h^2) [ |s_{j-1}|^2 + \|z_{j-1}\|^2 ]$$

$$+ C h^2 \sum_{i=0}^{j-2} \|z_i\|^2 + C h.$$  

Proceeding similarly as in Lemma 1.1 we obtain the required result of the lemma.

**Remark 2.1.** Lemmas 2.1 and 2.2 imply the estimates

$$\|u^n(t)\| + \|U^n(t)\| + \|z^n(t)\| + \|Z^n(t)\| + |s^n(t)| \leq C,$$

$$\|U^n(t) - u^n(t)\| + |Z^n(t) - z^n(t)| \leq \frac{C}{h},$$

$$\|U^n(t) - U^n(s)\| + \|Z^n(t) - Z^n(s)\| \leq C |t - s|$$

for all $t, s \in I$ and $n > N$.

**Lemma 2.3.** Assume the hypotheses of Lemma 2.2. Then there exists $u \in Lip(I, \mathcal{Y})$ with the properties

$$\frac{du}{dt}, \frac{d^2u}{dt^2} \in L_\infty(I, \mathcal{Y}) \cap \mathcal{C}(I, \mathcal{X}), \frac{d^2u}{dt^2} \in L_\infty(I, \mathcal{X})$$

such that

$$U^n \to u \text{ in } \mathcal{C}(I, \mathcal{Y}) \text{ and } Z^n \to \frac{du}{dt} \text{ in } \mathcal{C}(I, \mathcal{X}).$$

**Proof.** Since $\{u^n\}$ and $\{z^n\}$ are uniformly bounded in $\mathcal{Y}$, and $\mathcal{Y}$ is compactly embedded in $\mathcal{X}$, there exists a subsequence $\{n_k\}$ of the indices $\{n\}$ such that

$$u^{n_k}(t) \to u(t) \text{ and } z^{n_k}(t) \to z(t) \text{ in } \mathcal{X} \text{ as } k \to \infty$$

for some functions $u$ and $z$ from $I$ into $\mathcal{X}$. Remark 2.1 implies that

$$U^{n_k}(t) \to u(t) \text{ and } Z^{n_k}(t) \to z(t) \text{ as } k \to \infty.$$  

We notice that the families $\{U^{n_k}\}$ and $\{Z^{n_k}\}$ are equicontinuous in $\mathcal{C}(I, \mathcal{X})$. Also, $\{U^{n_k}(t)\}$ and
\{Z^n(t)\} are relatively compact in \(\mathcal{H}\) for every \(t \in I\). Therefore

\[ U^n \to u \text{ and } Z^n \to z \text{ in } C(I, \mathcal{H}) \text{ as } k \to \infty. \]

Now we show that \(U^n \to u\) in \(C(I, \Psi)\) as \(k \to \infty\). We denote by

\[ K^n(0):= h a_{10}, \quad K^n(0):= \sum_{i=0}^{j-1} a_{ji}, \]

for \(t \in (t_{j-1}, t_j]\).

\[ f^n(0):= f(0), \quad f^n(t):= f(t_j) \]

Clearly, \(\{K^n(t)\}\) and \(\{f^n(t)\}\) are uniformly bounded and \(f^n(t) \to f(t)\) uniformly on \(I\) as \(n \to \infty\).

From (2.14) for positive integers \(p, q > N, t \in (0, T]\) and all \(v \in \Psi\), we get

\[
(s^p(t) - s^q(t), v) + A\, u^p(t) - A\, u^q(t), v) = (K(t) - K(t) + f(t) - f(t), v).
\]

(2.18)

Putting \(v = u^p(t) - u^q(t)\) in (2.18) and rearranging the terms, we obtain

\[
||u^p(t) - u^q(t)||^2_A \leq |s^p(t) - s^q(t)| + |K^p(t) - K^q(t)| + |f^p(t) - f^q(t)| + C \cdot |u^p(t) - u^q(t)|.
\]

(2.19)

Since \(\{u^n\}\) converges in \(C(I, \mathcal{H})\), inequality (2.19) implies that \(\{u^n\}\) is a Cauchy sequence in \(C(I, \Psi)\). From Remark 2.1 it follows that \(u : I \to \Psi\) and \(u : I \to \mathcal{H}\) are Lipschitz continuous hence

\[ \frac{du}{dt} \in L_{\infty}(I, \Psi) \text{ and } \frac{dz}{dt} \in L_{\infty}(I, \mathcal{H}). \]

Now for all \(v \in \Psi\),

\[
(U^n(t), v) = \int_0^t (\frac{dU^n}{dt}(s), v) \, ds + (U_0, v)
\]

(2.20)

We pass through the limit as \(k \to \infty\) in (2.20) to obtain

\[
(u(t), v) = \int_0^t (z(s), v) \, ds + (U_0, v).
\]

Therefore \(\frac{du}{dt}(t) = z(t)\) a.e. on \(I\) and hence \(\frac{d^2u}{dt^2}(t) \in L_{\infty}(I, \mathcal{H})\). The proof of the lemma is complete.

Lemma 2.4. Assume the hypotheses of Lemma 2.3 and let \(u(t)\) be defined as in Lemma 2.3. Then

\[ K^n(t) \to K(u)(t) \text{ as } k \to \infty \text{ in } \mathcal{H} \text{ uniformly on } I. \]

The proof of Lemma 2.4 is same as the proof of Lemma 2.4 in [3] (also, see [2, Chapter IV]).
Proof of Theorem 1.1. For $n_k$, we write (2.3) as

\begin{equation}
\left( \frac{d}{dt} Z(t), v \right) + \langle Au^n(t), v \rangle = (K^n(t) + f^n(t), v)
\end{equation}

for all $v \in \mathcal{V}$ and all $t \in (0, T]$. Integrating (2.21) over $(0, t)$, we get

\begin{equation}
\left( Z^n(t), v \right) - \left( U_1, v \right) + \int_0^t \left( \frac{d}{dt} Z^n(s), v \right) ds = \int_0^t (K^n(s) + f^n(s), v) ds.
\end{equation}

Passing through the limit as $k \to \infty$, using Lemma 2.4 and bounded convergence theorem, we have

\begin{equation}
\left( Z(t), v \right) - \left( U_1, v \right) + \int_0^t \langle Au(s), v \rangle ds = \int_0^t (K(u)(s) + f(s), v) ds.
\end{equation}

Differentiating (2.23) with respect to $t$, we get

\begin{equation}
\left( \frac{d}{dt} Z(t), v \right) + \langle Au(t), v \rangle = (K(u)(t) + f(t), v)
\end{equation}

for all $v \in \mathcal{V}$ and a.e. $t \in I$ which implies identity (2.7). Now we prove the uniqueness under hypothesis (H_3). Let $u_1$ and $u_2$ be two functions satisfying the assertions of Theorem 2.1. Let

\begin{equation}
W := \frac{a_0}{(\alpha)^{1/2}} \int_0^T w(t) dt, \text{ where } a_0 = \max_I |a(t)|.
\end{equation}

We divide the interval $I$ into a finite number of subintervals of equal lengths $p$ such that

\begin{equation}
Wp^2 < \frac{1}{2}.
\end{equation}

Let $t_1, t_2 \in [0, p]$ be such that

\begin{align}
&\left| \frac{d}{dt} u(t_1) \right| = \max_{[0, p]} \left| \frac{d}{dt} u(t) \right|, \\
&\| u(t_2) \|_A = \max_{[0, p]} \| u(t) \|_A.
\end{align}

Then we have

\begin{equation}
\int_0^{t_1} \left| \frac{d}{dt} Z(t) \right|^2 dt + \int_0^{t_2} \left| \frac{d}{dt} u(t) \right|^2 dt \\
\leq \int_0^p \left[ \frac{d}{dt} \left| \frac{d}{dt} u(t) \right|^2 + \left| \frac{d}{dt} u(t) \right|^2 \right] dt.
\end{equation}

Now from identity (2.7) for $v = \frac{d}{dt} u(t)$, we have

\begin{equation}
\frac{d}{dt} \left| \frac{d}{dt} u(t) \right|^2 + \left| \frac{d}{dt} u(t) \right|^2 \leq 2 \left( K(u_1)(t) - K(u_2)(t), \frac{d}{dt} u(t) \right).
\end{equation}

Therefore
\[
\int_0^t \left[ \frac{d}{dt} \left( \frac{du}{dt} (t) \right)^2 + \frac{d}{dt} \| u(t) \|_A^2 \right] dt \leq 2 \frac{\alpha_0}{(\alpha)^{1/2}} \int_0^t \left( \int_0^{t_1} u(s) \| u(s) \|_A \right) ds \| \frac{du}{dt} (t) \| dt
\]

\[ (2.31) \]

From inequalities (2.29), (2.26) and (2.31) we have

\[ \frac{du}{dt} (t) \equiv 0, \quad u(t) \equiv 0 \text{ on } [0, p]. \]

Repeating the above arguments for \([ip, (i + 1) p], i = 1, 2, \ldots\), we have that \(u(t) \equiv 0\) on \(I\).

Therefore \(u_1 \equiv u_2\). The proof of Theorem 2.1 is thus complete.

REFERENCES


