ON A COUNTEREXAMPLE OF GAROFALO - LIN FOR A UNIQUE CONTINUATION OF SCHRÖDINGER EQUATION*

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ABSTRACT

Garofalo and Lin have given a counterexample for which unique continuation fails for the Schrödinger equation

$$-\Delta u + \frac{c}{|x|^{2+\varepsilon}} u = 0, \quad \varepsilon > 0.$$

Their counterexample consists of a Bessel function of the third kind $K_\nu(|x|)$ with the restriction that ν cannot be an integer. In this note we have removed the restriction.

Key Words: Schrödinger Equation, Unique Continuation, Bessel Function of the Third Kind, Inverse Square Potential

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Consider the following Schrödinger equation:

$$(1) \quad -\Delta u + V u = 0.$$

Garofalo and Lin [1] have shown that $V = c/|x|^2$, the inverse square potential, is optimal for unique continuation of (1). Indeed they have given a counterexample for which unique continuation fails once the square inverse potential is replaced by $c/|x|^{2+\varepsilon}$ for any $\varepsilon > 0$ [1: pp. 265-266]. Their counterexample is given by

$$u(x) = |x|^{-(n-2)/2} K_{(n-2)/\varepsilon} \left(\frac{2\sqrt{c}}{\varepsilon} |x|^{-\varepsilon/2} \right)$$

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with the restriction that $(n-2)/\varepsilon$ cannot be an integer for any $\varepsilon > 0$

In this note we show that this restriction is unnecessary. We use the same notations as in [1].

Theorem: *For the following Schrödinger equation,*

$$(2) \quad -\Delta u + \frac{c}{|x|^{2+\varepsilon}} u = 0 \quad \text{in } B_1,$$

where B_1 is a unit ball in $\mathbb{R}^n$, we have a radial solution given by

$$(3) \quad u(x) = |x|^{-(n-2)/2} K_{(n-2)/\varepsilon} \left(-\frac{2\sqrt{c}}{\varepsilon} |x|^{-\varepsilon/2} \right),$$

where $K_{(n-2)/\varepsilon}$ is a Bessel function of the third kind for any real number $(n-2)/\varepsilon$ and $\varepsilon > 0$.

Proof. Since $\Delta_x u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_\theta u$, a radial solution of (2) must satisfy

$$(4) \quad r^2 u''(r) + (n-1) r u'(r) - c r^{-\varepsilon} u(r) = 0, \quad 0 < r < 1.$$

It is known that $I_\nu(z)$ and $L_\nu(z)$, Bessel functions of imaginary argument, satisfy the differential equation [3: p. 77]

$$(5) \quad z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2) u = 0.$$

Notice from (4) and (5) that $I_\nu(z)$ and $L_\nu(z)$ (when ν is an integer, see equation (9) for the definition) cannot be a solution of (4) because of the coefficient $n-1$ of $u'(r)$ in (4). Thus we look for a solution of a form $z^\alpha I_{\pm\nu}(\beta z^\gamma)$, where α , β , and γ are constants to be determined. An easy computation reveals that $z^\alpha I_{\pm\nu}(\beta z^\gamma)$ satisfies:

$$(6) \quad z^2 \frac{d^2}{dz^2} (z^\alpha I_{\pm\nu}(\beta z^\gamma)) + (1 - 2\alpha) z \frac{d}{dz} (z^\alpha I_{\pm\nu}(\beta z^\gamma)) \\ - (\gamma^2 \beta^2 z^{2\gamma} + \gamma^2 \nu^2 - \alpha^2) (z^\alpha I_{\pm\nu}(\beta z^\gamma)) = 0.$$

Comparison of (6) with (4) shows that

$$(7) \quad \alpha = -(n-2)/2, \quad \beta = -2\sqrt{c}/\varepsilon, \quad \gamma = -\varepsilon/2, \quad \text{and } \nu = (n-2)/\varepsilon.$$

Define the third kind of Bessel function $K_\nu(z)$ according to Watson [3: p. 78] by

$$(8) \quad K_\nu(z) = \frac{\pi}{2} \frac{I_\nu(z) - I_{-\nu}(z)}{\sin \nu \pi}, \quad \nu \neq \text{integer}$$

$$(9) \quad K_n(z) = \lim_{\nu \rightarrow n} K_\nu(z), \quad \nu = \text{integer}.$$

Then $K_\nu(z)$ is defined for all real values of ν . Conjunction of equations (4)-(9) yields the solution (3) to Schrödinger equation (2). This completes the proof.

Remark.

Since $K_\nu(r) \sim r^{-1/2} e^{-r}$ for all ν as $r \rightarrow \infty$ [3: p.202], $K_\nu(r)$ vanishes of infinite order as $r \rightarrow \infty$. Consequently unique continuation of Schrödinger equation (2) fails for any $\varepsilon > 0$, which implies that $V = c/|x|^2$, the inverse square potential, is optimal for unique continuation of solutions of Schrödinger equation (1).

References

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