RELATIVE STABILITY AND WEAK CONVERGENCE IN NON-DECREASING STOCHASTICALLY MONOTONE MARKOV CHAINS¹

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ABSTRACT

Let $\{\xi_n\}$ be a non-decreasing stochastically monotone Markov chain whose transition probability $Q(\cdot, \cdot)$ has $Q(x, \{x\}) = \beta(x) > 0$ for some function $\beta(\cdot)$ that is non-decreasing with $\beta(x) \uparrow 1$ as $x \rightarrow +\infty$, and each $Q(x, \cdot)$ is non-atomic otherwise. A typical realization of $\{\xi_n\}$ is a Markov renewal process $\{(X_n, T_n)\}$, where $\xi_j = X_n$ for T_n consecutive values of j , T_n geometric on $\{1, 2, \dots\}$ with parameter $\beta(X_n)$. Conditions are given for X_n to be relatively stable and for T_n to be weakly convergent.

Key words: Markov chain, stochastic monotonicity, Markov renewal process, relative stability, weak convergence.

AMS (MOS) subject classifications: 60G, 60K15.

1. INTRODUCTION

In this paper, R is the real line and $\mathcal{B}$ the σ -field of Borel subsets of R . Let $\{\xi_n\}_0^\infty$ be a Markov chain with state space $\{R, \mathcal{B}\}$, an initial distribution π and transition probability Q . The π and Q determine completely and uniquely a probability measure P on the countable product space $\{R^\infty, \mathcal{B}^\infty\}$. When $\pi(\cdot) = \epsilon_x(\cdot)$ (the Dirac measure concentrated at x) we shall write P_x instead of P . The corresponding expectation operator is denoted then by E_x .

Throughout this paper it is assumed that the Q is subject to the following regularity conditions:

- (i) for each $x \in R$ the support of $Q(x, \cdot)$ is in $[x, \infty)$;
- (ii) the chain $\{\xi_n\}_0^\infty$ is stochastically monotone (Daley, [3]); in other words; for any $x_1 \leq x_2$, $Q(x_2, B_y) \leq Q(x_1, B_y)$ where $B_y = (-\infty, y]$; (1.1)

$$(iii) \quad Q(x, \{y\}) = \begin{cases} 0 & x \neq y \\ \beta(x) > 0 & x = y \end{cases}$$

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Concerning the function $\beta(\cdot)$ we assume that $(x_1 \leq x_2)$

$$\beta(x_1) \leq \beta(x_2) \text{ and } \lim_{x \rightarrow +\infty} \beta(x) = 1 \quad (1.2)$$

From (1.1.i) it follows that

$$\xi_0 \leq \xi_1 \leq \dots \quad (1.3)$$

Markov processes of this type are of considerable interest in reliability theory as models of the amount of deterioration of a mechanical device subject to shocks and wear during its service (Barlow and Proschan, [1]; Brown and Changanty, [2]).

Set

$$\begin{aligned} \tau_0 &= \sup\{k; \xi_k = \xi_0\}, \tau_n = \sup\{k; \xi_k = \xi_{\tau_{n-1}+1}\}, T_0 = \tau_0 \\ T_n &= \tau_n - \tau_{n-1}, X_0 = \xi_0, X_n = \xi_{\tau_{n-1}+1}, W_0 = X_0, W_n = X_n - X_{n-1}. \end{aligned} \quad (1.4)$$

From this one readily obtains that

$$P_x\{T_0 \geq i\} = \{\beta(x)\}^i \quad i = 0, 1, \dots \quad (1.5)$$

2. AUXILIARY RESULTS

Here we list some basic properties of the bivariate sequence $\{(X_n, T_n)\}$ needed in the rest of this paper. Some simple calculations show that this sequence is a Markov renewal process with the transition probability

$$P\{X_n \in du, T_n = i \mid X_{n-1}\} = P(X_{n-1}, du)\{1 - \beta(u)\}\{\beta(u)\}^{i-1} \quad (a.s.) \quad (2.1)$$

where $i = 1, 2, \dots$ and

$$P(x, B_y) = \begin{cases} 0 & y < x \\ \frac{Q(x, B_y) - \beta(x)}{1 - \beta(x)} & y \geq x. \end{cases} \quad (2.2)$$

The $P(x, B_y)$ is the transition probability of the Markov chain $\{X_n\}_0^\infty$. It is easy to verify that

$$P(x_2, B_y) \leq P(x_1, B_y) \text{ for all } x_1 \leq x_2. \quad (2.3)$$

From (2.1) we deduce

$$P\{X_n \in du, T_n = i\} = \{1 - \beta(u)\}\{\beta(u)\}^{i-1} P\{X_n \in du\} \quad (2.4)$$

which clearly implies (see (1.5)) that

$$\begin{aligned}
P\{T_n = i \mid X_n\} &= \{1 - \beta(X_n)\}\{\beta(X_n)\}^{i-1} \quad (a.s.) \\
&= P_{X_n}\{T_0 = i - 1\}.
\end{aligned} \tag{2.5}$$

In addition, since

$$P_x\{X_1 \in du, T_1 = i\} = P(x, du)\{1 - \beta(u)\}\{\beta(u)\}^{i-1} \tag{2.6}$$

it follows that

$$P\{X_n \in du, T_n = i \mid X_{n-1}\} = P_{X_{n-1}}\{X_1 \in du, T_1 = j\} \quad (a.s.). \tag{2.7}$$

Denote by $P^n(x, B_y)$ the n -step transition probability of $\{X_n\}_0^\infty$, since

$$X_0 < X_1 < \dots \tag{2.8}$$

we have that $P^{n+1}(x, B_y) \leq P^n(x, B_y)$. On the other hand, the stochastic monotonicity and the Chapman-Kolmogorov equation yield:

$$P^n(x, B_y) \leq (P(x, B_y))^n. \tag{2.9}$$

From this, (2.8) and the Borel-Cantelli lemma it follows that $X_n \rightarrow +\infty$ (a.s.) if $P(x, B_y) < 1$ for all $y < \infty$.

It is clear from (2.5) that T_n is conditionally geometric with parameter $\beta(X_n)$. In addition, since

$$P\{T_n \geq i \mid X_n\} = \{\beta(X_n)\}^{i-1} \quad (a.s.) \tag{2.10}$$

it is apparent that $\{T_n\}$ is stochastically monotone and that $T_n \xrightarrow[n \rightarrow \infty]{d.} \infty$ as $n \rightarrow \infty$. Finally, for each $n = 0, 1, \dots$ we have:

$$P\{T_0 = i_0, \dots, T_n = i_n \mid X_0, \dots, X_n\} = \prod_{j=1}^n P\{T_j = i_j \mid X_j\} \quad (a.s.). \tag{2.11}$$

In other words, conditioned on a realization of $\{X_n\}$ the sequence of sojourn times $\{T_n\}$ becomes a family of independent r.v.'s such that the distribution of T_n depends only on X_n .

Consider

$$\begin{aligned}
P_x\{X_n \in du, T_n = i\} &= \int_x^u P\{X_n \in du, T_n = i \mid X_{n-1} = z\} P^{n-1}(x, dz) \\
&= \{1 - \beta(u)\}\{\beta(u)\}^{i-1} P_x\{X_n \in du\}
\end{aligned}$$

from which we deduce that

$$P_x\{T_n = i \mid X_n\} = \{1 - \beta(X_n)\}\{\beta(X_n)\}^{i-1} \text{ (a.s.)} \quad (2.12)$$

where the right hand side is independent of x .

3. REMARKS ON THE STRUCTURE OF $\{W_n\}$

In this section we investigate asymptotic structure of the sequence $\{W_n\}_0^\infty$ assuming that the following condition holds for all $y \geq 0$:

$$\lim_{x \rightarrow \infty} \frac{P_x\{\xi_1 > x + y\}}{P_x\{\xi_1 > x\}} = 1 - F(y) \quad (3.1)$$

where $F(\cdot)$ is a proper d.f.

Remark 3.1: The condition (3.1) is similar to one introduced by Gnedenko [4].

Denote by

$$\Phi_n(y \mid x) = P_x\{W_n \leq y\} \quad n = 1, 2, \dots \quad (3.2)$$

then clearly

$$\Phi_1(y \mid x) = P(x, B_{x+y}). \quad (3.3)$$

Some simple calculations yield:

$$\Phi_n(y \mid x) = E_x\{\Phi_1(y \mid X_{n-1})\}. \quad (3.4)$$

Taking into account (2.2) and condition (3.1), we have:

$$\lim_{x \rightarrow \infty} \Phi_1(y \mid x) = F(y). \quad (3.5)$$

This, (3.4) and the Lebesgue bounded convergence theorem imply that

$$\lim_{n \rightarrow \infty} \Phi_n(y \mid x) = F(y). \quad (3.6)$$

In other words, (at least) $W_n \xrightarrow{d} Y$, where Y is a r.v. with the d.f. $F(y)$. The following proposition generalizes this simple observation.

Proposition 3.1: Assume that (3.1) holds and $F(\cdot)$ is continuous, then under P_x , for all $k = 1, 2, \dots$

$$(W_{n+1}, \dots, W_{n+k}) \xrightarrow{d} (Y_1, \dots, Y_k) \text{ as } n \rightarrow \infty \quad (3.7)$$

where $\{Y_i\}_1^\infty$ is an i.i.d. sequence of r.v.'s with common d.f. $F(\cdot)$.

Proof: The method of proof will be amply illustrated by the case $n = 2$. Given $\epsilon > 0$, we obtain

$$\begin{aligned} & P_x\{W_{n+1} \leq y_1, W_{n+2} \leq y_2\} \\ &= \int_x^\infty \int_z^{z+y_1} P(z, du) \Phi_1(y_2 | u) P^n(x, dz) \\ &= E_x\left\{ \int_{X_n}^{X_n+y_1} P(X_n, du) \Phi_1(y_2 | u) \right\} \end{aligned} \quad (3.8)$$

Since by assumption $F(\cdot)$ is continuous the convergence in (3.5) is uniform. Consequently, for any $\epsilon > 0$ there exists x_0 such that

$$|\Phi_1(y_2 | u) - F(y_2)| < \epsilon \text{ for all } u > x_0 \text{ and any } y_2 \in \mathcal{R}.$$

From this and (3.8) we then have:

$$\begin{aligned} & P_x\{W_{n+1} \leq y_1, W_{n+2} \leq y_2\} \leq P^n(x, B_{x_0}) \\ &+ E_x\{I_{\{X_n > x_0\}} \int_{X_n}^{X_n+y_1} \Phi_1(y_2 | u) P(X_n, du)\} \\ &\leq P^n(x, B_{x_0}) + (\epsilon + F(y_2)) E_x\{\Phi_1(y_1 | X_n) I_{\{X_n > x_0\}}\}. \end{aligned}$$

Consequently,

$$\overline{\lim}_{n \rightarrow \infty} P_x\{W_{n+1} \leq y_1, W_{n+2} \leq y_2\} \leq (\epsilon + F(y_2)) F(y_1).$$

In the same fashion, one can show that

$$\underline{\lim}_{n \rightarrow \infty} P_x\{W_{n+1} \leq y_1, W_{n+2} \leq y_2\} \geq F(y_1)(F(y_2) - \epsilon).$$

Since $\epsilon > 0$ is arbitrary, the assertion follows.

Remark 3.2: The last proposition indicates that, roughly speaking, the remote members of $\{W_n\}_0^\infty$ are *i.i.d. r.v.'s*.

Next, we show that the sequence $\{W_n\}_0^\infty$ is endowed with a mixing property, which means, loosely speaking, that its elements far apart are nearly independent. Denote by $\mathfrak{F}_n = \sigma\{W_0, \dots, W_n\}$ and by $\mathfrak{F}^n = \sigma\{W_n, W_{n+1}, \dots\}$ then we have:

Proposition 3.2: For each $n = 1, 2, \dots$ and $k = 1, 2, \dots$

$$\lim_{m \rightarrow \infty} P_x \left(\bigcap_{j=1}^n \{X_j \leq y_j\} \bigcap_{i=1}^k \{W_{n+m+i} \leq z_i\} \right) \quad (3.9)$$

$$= P_x \left(\bigcap_{j=1}^n \{X_j \leq y_j\} \right) \prod_{i=1}^k F(z_i) \quad (x < y_1 < \dots < y_n)$$

Proof: By invoking the Markov property of $\{X_n\}_0^\infty$ and the proposition 3.1, we obtain

$$\begin{aligned} & P_x \left(\bigcap_{j=1}^n \{X_j \leq y_j\} \bigcap_{i=1}^k \{W_{n+m+i} \leq z_i\} \right) \\ &= \int_{(x, y_n]} P_x \left(\bigcap_{j=1}^{n-1} \{X_j \leq y_j\} \mid X_n = s \right) P_s \left(\bigcap_{i=1}^k \{W_{m+i} \leq z_i\} \right) P^n(x, ds) \\ &\rightarrow \int_{(x, y_n]} P_x \left(\bigcap_{j=1}^{n-1} \{X_j \leq y_j\} \mid X_n = s \right) \prod_{i=1}^k F(z_i) P^n(x, ds) \end{aligned}$$

as $m \rightarrow \infty$ which proves the assertion.

Corollary 3.1: $\{W_n\}_0^\infty$ and $\{Y_j\}_1^\infty$ are independent families. Set

$$\mathcal{T} = \bigcap_{n=0}^{\infty} \mathcal{F}^n.$$

It follows from the last proposition that $\mathcal{F}_n$ and $\mathcal{T}$ are independent σ -algebras for all $n = 0, 1, \dots$. Therefore $\mathcal{F}_n \cap \mathcal{T}$ is a trivial σ -algebra (its elements are either sure or null events). By letting $n \rightarrow \infty$ we have that $\mathcal{F}_\infty \supset \mathcal{T}$ and that their intersection is a trivial σ -algebra. This clearly implies that the tail σ -algebra $\mathcal{T}$ is a trivial one.

4. RELATIVE STABILITY OF $\{X_n\}$

The sequence $\{X_n\}$ is said to be relatively stable if there exist constants $\{a_n\}$ such that $X_n/a_n \rightarrow 1$ in probability (Gnedenko and Kolmogorov, [5]). If the convergence is (a.s.) the sequence is called (a.s.) relatively stable (Resnik, [6]). The following proposition gives a sufficient condition for (a.s.) relative stability of $\{X_n\}$.

Proposition 4.1: Assume that

$$\sup_x E_x \{W_1^2\} < \infty \quad (4.1)$$

then $X_n/n \rightarrow \alpha_1$ (a.s.) where $\alpha_1 = E\{Y_1\}$.

Proof: From (3.4), (4.1) and Fubini's theorem we deduce that

$$\begin{aligned}
E_x\{W_k^2\} &= \int_0^\infty y[1 - \Phi_k(y|x)]dy \\
&= E_x\left\{\int_0^\infty y[1 - \Phi_1(y|X_{k-1})]dy\right\} \\
&= E_x(E_{X_{k-1}}\{W_1^2\}) \leq \sup_x E_x\{W_1^2\} < \infty.
\end{aligned} \tag{4.2}$$

Consequently,

$$\sup_{x,k} E_x\{W_k^2\} < \infty.$$

Next, since

$$\left\{\frac{1}{n}\sum_{k=1}^n W_k \text{ converges}\right\} \in \mathcal{T}$$

and $\mathcal{T}$ is a trivial σ -algebra, to prove the proposition it suffices to show that

$$P_x\left\{\left|\frac{1}{n}X_n - \alpha_1\right| > \epsilon\right\} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{4.3}$$

Consider

$$Var\left\{\frac{1}{n}X_n\right\} = \frac{1}{n^2} \left(\sum_{k=1}^n Var\{W_k\} + \sum_{i=1}^{n-1} \sum_{j=i+1}^n Cov(W_i, W_j) \right).$$

It is clear from (4.1) and (4.2) that under P_x

$$\frac{1}{n^2} \sum_{k=1}^n Var\{W_k\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, due to propositions 3.1 and 3.2

$$\lim_{k \rightarrow \infty} Cov(W_k, W_{k+n}) = 0 \quad \lim_{k \rightarrow \infty} Cov(W_n, W_{n+k}) = 0 \tag{4.4}$$

for each $n = 0, 1, \dots$. Thus, given $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that

$$|Cov(W_i, W_j)| < \epsilon \quad |Cov(W_n, W_{n+k})| < \epsilon \tag{4.5}$$

if $\min\{i, j\} > n_0$ and $k > n_0$. Now, take $n > 2n_0$, then

$$\begin{aligned}
&\left| \sum_{i=1}^{n-1} \sum_{j=i+1}^n Cov(W_i, W_j) \right| \leq \left| \sum_{i=1}^{n_0} \sum_{j=i+1}^{2n_0} Cov(W_i, W_j) \right| \\
&+ \left| \sum_{i=1}^{n_0} \sum_{j=2n_0+1}^n Cov(W_i, W_j) \right| + \left| \sum_{i=n_0+1}^{n-1} \sum_{j=i+1}^n Cov(W_i, W_j) \right|
\end{aligned}$$

$$\leq \left| \sum_{i=1}^{n_0} \sum_{j=i+1}^{2n_0} \text{Cov}(W_i, W_j) \right| + \epsilon(n - 2n_0) + \epsilon(n - n_0 - 1)(n - n_0)/2.$$

Consequently, for any $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \text{Var}\left\{\frac{1}{n}X_n\right\} < \epsilon/2.$$

Finally, since $E_x\{W_n\} \rightarrow \alpha_1$ as $n \rightarrow \infty$ it follows that

$$\sum_{k=1}^n E_x\{W_k\}/n \rightarrow \alpha_1 \text{ as } n \rightarrow \infty.$$

Therefore, for n sufficiently large

$$P_x\left\{\left|\frac{1}{n}X_n - \alpha_1\right| > \epsilon\right\} \leq P_x\left\{\frac{1}{n} \left| \sum_{i=1}^n [W_i - E_x\{W_i\}] \right| > \epsilon\right\}.$$

This and (4.1) prove the assertion.

Corollary 4.1: From proposition 4.1 we readily deduce that for each $\epsilon > 0$

$$P_x\{X_n \leq (\alpha_1 + \epsilon)n\} \rightarrow 1 \text{ and } P_x\{X_n \leq (\alpha_1 - \epsilon)n\} \rightarrow 0 \quad (4.6)$$

as $n \rightarrow \infty$. Consequently

$$\lim_{n \rightarrow \infty} P^n(x, B_{ny}) = \begin{cases} 1 & \text{if } y \geq \alpha_1 \\ 0 & \text{if } y < \alpha_1. \end{cases} \quad (4.7)$$

Denote by

$$T(y) = \inf\{k; X_k > y\} \quad (4.8)$$

then for any $x < y$

$$\begin{aligned} P_x\{T(y) \leq n\} &= P_x\{X_n > y\} \\ &= 1 - P^n(x, B). \end{aligned} \quad (4.9)$$

Proposition 4.2: Under P_x

$$\frac{1}{y}T(y) \rightarrow \alpha_1^{-1} \text{ in probability as } y \rightarrow +\infty.$$

Proof: Assume $\alpha_1 > 0$, then we have to show

$$P_x\left\{\left|\frac{1}{y}T(y) - \alpha_1^{-1}\right| \leq \epsilon\right\} \rightarrow 1 \text{ as } y \rightarrow +\infty$$

for any $\epsilon > 0$. Choose $\epsilon \in (0, \alpha_1^{-1})$ then from (4.9) we deduce

$$P_x\left\{\left|\frac{1}{y}T(y) - \alpha_1^{-1}\right| \leq \epsilon\right\} = P_x\{X_{[(\alpha_1^{-1} - \epsilon)y]} \leq y\} - P_x\{X_{[(\alpha_1^{-1} + \epsilon)y]} \leq y\}$$

$$= P^{[(\alpha_1^{-1} - \epsilon)y]}(x, B_y) - P^{[(\alpha_1^{-1} + \epsilon)y]}(x, B_y)$$

where, as usual, $[x]$ stands for the integer part of x . Since

$$\frac{y}{[(\alpha_1^{-1} - \epsilon)]} \geq \frac{\alpha_1}{1 - \epsilon\alpha_1} > \alpha_1$$

it follows from (4.7) that

$$\lim_{y \rightarrow \infty} P^{[(\alpha_1^{-1} - \epsilon)y]}(x, B_y) = 1.$$

Similarly, when $y \rightarrow \infty$

$$\frac{y}{[(\alpha_1^{-1} + \epsilon)y]} < \frac{y}{(\alpha_1^{-1} + \epsilon)y - 1} \sim \frac{\alpha_1}{1 + \epsilon\alpha_1} < \alpha_1.$$

This and (4.7) imply

$$\lim_{y \rightarrow \infty} P^{[(\alpha_1^{-1} + \epsilon)y]}(x, B_y) = 0$$

which proves the proposition.

5. WEAK CONVERGENCE OF $\{T_n\}$

In this section we show that a sequence of scale factors $\{d_n\}$ exists such that under P_x

$$d_n T_n \xrightarrow{d} Z \quad (5.1)$$

where the r.v. Z has an exponential distribution independent of x . But first, we need the following auxiliary result.

Proposition 5.1: Assume that condition (4.1) holds, then

$$\frac{1 - \beta(X_n)}{1 - \beta(n\alpha_1)} \xrightarrow{P_x} 1 \quad \text{as } n \rightarrow \infty \quad (5.2)$$

where $\alpha_1 = E\{Y_1\}$.

Proof: Set

$$\beta^{-1}(y) = \inf\{x; \beta(x) > y\} \quad (5.3)$$

then for any $\epsilon \in (0, 1)$

$$P_x\left\{\left|\frac{1 - \beta(X_n)}{1 - \beta(n\alpha_1)} - 1\right| \leq \epsilon\right\} = P_x\left\{\frac{X_n}{n\alpha_1} \leq \frac{\beta^{-1}(\beta(n\alpha_1)(1 - \epsilon) + \epsilon)}{n\alpha_1}\right\}$$

$$-P_x\left\{\frac{\beta^{-1}(\beta(n\alpha_1)(1+\epsilon)-\epsilon)}{n\alpha_1}\right\}.$$

Since

$$\beta^{-1}(\beta(n\alpha_1)(1-\epsilon)+\epsilon) > n\alpha_1$$

$$\beta^{-1}(\beta(n\alpha_1)(1+\epsilon)-\epsilon) < n\alpha_1$$

for all $n = 1, 2, \dots$, the assertion now follows from proposition 4.1.

Proposition 5.2: Suppose that (4.1) holds, then

$$\lim_{n \rightarrow \infty} P_x\{[1 - \beta(n\alpha_1)]T_n > u\} = e^{-u}. \quad (5.4)$$

Proof: Denote by

$$U_n = [1 - \beta(X_n)]T_n. \quad (5.5)$$

Then taking into account (2.10), we have:

$$\begin{aligned} P_x\{U_n > u\} &= E_x(P_x\{T_n > \frac{u}{1 - \beta(X_n)} \mid X_n\}) \\ &= E_x(\{\beta(X_n)\}^{\lceil \frac{u}{1 - \beta(X_n)} \rceil}). \end{aligned} \quad (5.6)$$

Set

$$R_n(u, \omega) = \{\beta(X_n(\omega))\}^{\lceil \frac{u}{1 - \beta(X_n(\omega))} \rceil}.$$

Since the function

$$h(y) = \exp\left\{\frac{1}{1 - \beta(y)} n \beta(y)\right\}$$

is non-decreasing on R , it follows that

$$R_n(u, \cdot) \leq R_{n+1}(u, \cdot)$$

at least (a.s.). From this we readily obtain

$$\lim_{n \rightarrow \infty} R_n(u, \omega) = e^{-u} \quad (5.7)$$

at least (a.s.) P_x . Invoking now the monotone convergence theorem, we deduce from (5.6) that

$$U_n \xrightarrow{d} Z. \quad (5.8)$$

Finally, write

$$[1 - \beta(n\alpha_1)]T_n = \frac{1 - \beta(n\alpha_1)}{1 - \beta(X_n)} U_n$$

then the proof of (5.4) follows (5.8), proposition 4.2 and a Slutsky's theorem.

Remark 5.1: One can easily show that the sequence of r.v.'s $\{U_n\}_0^\infty$ has the following properties:

$$E_x\{U_n\} = 1 \quad \text{Var}\{U_n\} = E_x\{\beta(X_n)\}$$

$$E_x\{U_{n+1} \mid U_1, \dots, U_n\} = 1.$$

Remark 5.2: The result of the last proposition can be easily extended as follows:
Set

$$V_n = [1 - \beta(n\alpha_1)]T_n$$

then after some straight forward calculations (see Todorovic and Gani, [7]) one can show that for each $k = 1, 2, \dots$

$$(V_{n+1}, \dots, V_{n+k}) \xrightarrow{d} (Z_1, \dots, Z_k)$$

under P_x , where $\{Z_k\}_1$ is an *i.i.d.* sequence of r.v.'s with common non-negative exponential distribution of x . The sequence also possesses a mixing property.

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