EXISTENCE OF SOLUTION FOR A MIXED NEUTRAL SYSTEM¹

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ABSTRACT

We prove the existence of unique solution of the mixed neutral system

$$x'(t) = f(t, x) + \sum_{j=1}^m A_j(t, x)x'(t + p_j) + g(t, x, x'(t + h))$$

$$x(0) = x_0$$

and also prove the continuous dependence of the solution.

Key words: Functional differential equations, existence and uniqueness.

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1. INTRODUCTION

A differential system in which the expression for $x'(t)$ involves $x'(h(t))$ for some $h(t) \neq t$ is said to be of neutral type. A system of first order functional differential equations in which the present value of $x'(t)$ is expressed in terms of both past and future values of x is said to be of mixed type. So when both of these characteristics are present, the system is of mixed neutral type, or simply a mixed neutral system.

In this paper we consider a mixed neutral system of the form

$$x'(t) = f(t, x) + \sum_{j=1}^m A_j(t, x)x'(t + p_j) + g(t, x, x'(t + h)) \quad (1)$$

where f is an n -vector valued function and each A_j is an $n \times n$ matrix valued function defined on $R \times C(R, R^n) \times C(R, R^n)$ and each p_j and h are constant real numbers.

The literature contains many papers on the problem (1) in the case when $A_j = 0$ and

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$g = 0$ ([1], [3], [4], [6]-[9]). Driver [2] proved the existence and continuous dependence of solutions of a neutral functional differential equation. But little is known when the mixed equation is also of neutral type. In [5], he proved the existence of unique solution and its continuous dependence for the mixed neutral system (1) when $g = 0$.

In this paper we shall prove the existence of unique solution of the mixed neutral system (1), and also prove that the solution of (1) depends continuously on the initial condition x_0 . System (1) is a greatly over-simplified model of a mixed neutral system arising in a two-body problem of classical electrodynamics.

2. BASIC ASSUMPTIONS

First we define the term solution for the system (1). A function $x: R \rightarrow R^n$ which is absolutely continuous locally and satisfies (1) almost everywhere is a solution of the mixed neutral system (1).

Let $|\cdot|$ be a chosen norm in R^n and let $\|\cdot\|$ be the corresponding induced matrix norm. Let $p = \max\{\max |p_j|, h\}$, and assume the existence of positive constants M_f , M_A , M_g , K_f , K_A , K_g , and N_g such that for $t \in R$ and $x, \bar{x}, \bar{x}' \in C(R, R^n)$

- (i) f and each A_j are continuous on $R \times C(R, R^n)$ and g is continuous on $R \times C(R, R^n) \times C(R, R^n)$,
- (ii) $|f| \leq M_f$ and each $\|A_j\| \leq M_A$ on $R \times C(R, R^n)$ and $|g| \leq M_g$ on $R \times C(R, R^n) \times C(R, R^n)$,
- (iii) $|f(t, x) - f(t, \bar{x})| \leq K_f \max_{t-p \leq s \leq t+p} |x(s) - \bar{x}(s)|$,
 $\|A(t, x) - A(t, \bar{x})\| \leq K_A \max_{t-p \leq s \leq t+p} |x(s) - \bar{x}(s)|$ and
 $|g(t, x, x'(t+h)) - g(t, \bar{x}, \bar{x}'(t+h))| \leq K_g \max_{t-p \leq s \leq t+p} |x(s) - \bar{x}(s)|$
 $+ N_g \max_{t-p \leq s \leq t+p} |x'(s+h) - \bar{x}'(s+h)|$
- (iv) the constants p , M_f , M_A , M_g , K_f , K_A , K_g , and N_g are sufficiently small such that

$$e^{ap} \left[\frac{1}{a} (K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}) + mM_A + N_g \right] < 1 \quad (2)$$

for some constant $a > 0$, and

- (v) the solution of (1) satisfies the conditions

$$\int_t^{t+1} |x'(s)| ds \text{ bounded for } t \in R.$$

Remark 1: If x is a solution of (1) satisfying the assumption (v) and if $B = \sup_{t \in R}$

$\int_{t-r}^{t+r} |x'(s)| ds$ for some choice of $r > 0$ then for each t ,

$$\begin{aligned} \int_{t-r}^{t+r} |x'(s)| ds &\leq \int_{t-r}^{t+r} |f(s, x)| ds + \sum_{j=1}^m \int_{t-r}^{t+r} \|A_j(s, x)\| |x'(s + p_j)| ds \\ &\quad + \int_{t-r}^{t+r} |g(s, x, x'(s + h))| ds \\ &\leq 2rM_f + mM_A B + 2rM_g. \end{aligned}$$

So if $mM_A < 1$,

$$B \leq \frac{2r(M_f + M_g)}{1 - mM_A}.$$

3. EXISTENCE AND UNIQUENESS

Theorem 1: Under the assumptions (i) – (v) there exists a unique solution for the mixed neutral system (1).

Proof: Let $mM_A < 1$ and choose $r > 0$. Define the set

$$S = \{\omega \in L^1_{loc}(R, R^n) : \int_{t-r}^{t+r} |\omega(s)| ds \leq 2r(M_f + M_g)/(1 - mM_A) \text{ for all } t\}$$

Then S is the space of allowable derivatives of solutions of (1).

For any constant $a > 0$, define the metric

$$d(\omega, \bar{\omega}) = \sup_{t \in R} [e^{-a|t|} \int_{t-r}^{t+r} |\omega(s) - \bar{\omega}(s)| ds] \text{ for } \omega, \bar{\omega} \in S.$$

Clearly (S, d) is a complete metric space. For $\omega \in S$, define $x(t) = x_0 + \int_0^t \omega(s) ds$ for all t and then

$$(T\omega)(t) = f(t, x) + \sum_{j=1}^m A_j(t, x)\omega(t + p_j) + g(t, x, \omega(t + h))$$

for all t . Now

$$\begin{aligned} \int_{t-r}^{t+r} |(T\omega)(s)| ds &\leq 2rM_f + mM_A \frac{2r(M_f + M_g)}{1 - mM_A} + 2rM_g \\ &\leq \frac{2r(M_f + M_g)}{1 - mM_A}. \end{aligned}$$

Therefore T maps S into S . Now let $\omega, \bar{\omega} \in S$. Then for $t \geq 0$

$$\begin{aligned}
\int_0^t |\omega(s) - \bar{\omega}(s)| ds &= \int_0^{2r} + \int_{2r}^{4r} + \dots + \int_{2r[t/2r]}^t |\omega(s) - \bar{\omega}(s)| ds \\
&\leq d(\omega, \bar{\omega})(e^{ar} + e^{3ar} + \dots + e^{2r + 2ar[t/2r]}), \\
&\leq d(\omega, \bar{\omega}) \frac{e^{3ar + at}}{e^{2ar} - 1}.
\end{aligned}$$

Similarly, for $t \leq 0$

$$\int_t^0 |\omega(s) - \bar{\omega}(s)| ds \leq d(\omega, \bar{\omega}) \frac{e^{3ar + a|t|}}{e^{2ar} - 1}.$$

Using this,

$$\begin{aligned}
|(T\omega)(t) - (T\bar{\omega})(t)| &\leq |f(t, x) - f(t, \bar{x})| + \sum_{j=1}^m \|A_j(t, x) - A_j(t, \bar{x})\| |\omega(t + p_j)| \\
&\quad + \sum_{j=1}^m \|A_j(t, \bar{x})\| |\omega(t + p_j) - \bar{\omega}(t + p_j)| \\
&\quad + |g(t, x, \omega(t + h)) - g(t, \bar{x}, \bar{\omega}(t + h))| \\
&\leq \max_{t-p \leq s \leq t+p} \left\{ \int_{t-p}^0 |\omega(s) - \bar{\omega}(s)| ds, \int_0^{t+p} |\omega(s) - \bar{\omega}(s)| ds \right\} \\
&\quad \times [K_f + K_g + K_A \sum_{j=1}^m |\omega(t + p_j)|] + \sum_{j=1}^m M_A |\omega(t + p_j) - \bar{\omega}(t + p_j)| \\
&\quad + N_g \max_{t-p \leq s \leq t+p} |\omega(s + h) - \bar{\omega}(s + h)| \\
&\leq d(\omega, \bar{\omega}) \frac{e^{3ar + ap + a|t|}}{e^{2ar} - 1} [K_f + K_g + K_A \sum_{j=1}^m |\omega(s + p_j)|] \\
&\quad + \sum_{j=1}^m M_A |\omega(s + p_j) - \bar{\omega}(s + p_j)| + N_g \max_{t-p \leq s \leq t+p} |w(s + h) - \bar{\omega}(s + h)|.
\end{aligned}$$

Thus for each t

$$\begin{aligned}
\int_{t-r}^{t+r} |(T\omega)(s) - (T\bar{\omega})(s)| ds &\leq d(\omega, \bar{\omega}) \frac{e^{4ar + ap + a|t|}}{e^{2ar} - 1} [2r(K_f + K_g) + \frac{mK_A(M_f + M_g)}{1 - mM_A}] \\
&\quad + mM_A d(\omega, \bar{\omega}) e^{a|t| + ap} + N_g d(\omega, \bar{\omega}) e^{a|t| + ap}.
\end{aligned}$$

Therefore $d(T\omega, T\bar{\omega}) \leq \rho d(\omega, \bar{\omega})$ where

$$\rho = e^{ap} \left[\frac{2re^{4ar}}{e^{2ar} - 1} (K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}) + mM_A + N_g \right]. \quad (3)$$

By the contraction mapping theorem, T will have a unique fixed point in S if $\rho < 1$.

Letting $r \rightarrow 0$ the sufficient condition becomes

$$e^{ap} \left[\frac{1}{a} (K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}) + mM_A + N_g \right] < 1.$$

If the above condition holds, then T will be a contraction mapping for some choice of sufficiently small $r > 0$. This yields the desired unique solution of (1) and the theorem is proved.

Remark 2: To make the sufficient condition (2) more specific, we can set $a = \frac{1}{p}$. By setting $a = \frac{1}{p}$ in (2) we get

$$p(K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A}) + mM_A + N_g < \frac{1}{e}.$$

4. CONTINUOUS DEPENDENCE

We complete this paper by giving a continuous dependence result. That is, we shall obtain an estimate for the change in the solution of the mixed neutral system (1) due to a change in the initial condition x_0 . Assuming we actually measured $\bar{x}_0$ instead of x_0 , the question is this: If $\bar{x}_0$ is close to x_0 , will the corresponding solution $\bar{x}$ be close to the solution x ? The following theorem asserts that the answer is 'yes' if the conditions (i)–(v) are satisfied.

Theorem 2: *Under the same hypotheses as in Theorem 1, the solution depends continuously on x_0 . More precisely, assume the smallness condition (2) for some $a > 0$, and let $\bar{x}$ be the unique solution of (1) with $\bar{x}(0) = \bar{x}_0$ and with $\int_t^{t+1} |x'(s)| ds$ bounded. Then for any $T > 0$,*

$$\sup_{|t| < T} [|x(t) - \bar{x}(t)| + \int_t^{t+1} |x'(s) - \bar{x}'(s)| ds] \rightarrow 0 \text{ as } |x_0 - \bar{x}_0| \rightarrow 0.$$

Proof: Choose a sufficiently small $r > 0$ so that $\rho < 1$ as in equation (3) of Theorem 1.

Then a computation quite analogous to that in the Theorem 1 yields

$$d(x', \bar{x}') \leq |x_0 - \bar{x}_0| e^{-a|t|} 2r \left[K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A} \right] + \rho d(x', \bar{x}').$$

But since $\rho < 1$, we have

$$d(x', \bar{x}') \leq |x_0 - \bar{x}_0| 2r \left[K_f + K_g + \frac{mK_A(M_f + M_g)}{1 - mM_A} \right].$$

So $d(x', \bar{x}') \rightarrow 0$ as $|x_0 - \bar{x}_0| \rightarrow 0$.

The assertion of the theorem is a straightforward consequence of this.

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