A NOTE ON CONTROLLABILITY OF NEUTRAL VOLTERRA INTEGRODIFFERENTIAL SYSTEMS¹

K. BALACHANDRAN and P. BALASUBRAMANIAM

*Bharathiar University
Department of Mathematics
Coimbatore 641 046
Tamil Nadu, INDIA*

ABSTRACT

Sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems are established. The results are obtained using the Schauder fixed point theorem.

Key words: Controllability, neutral Volterra systems, fixed point method.

AMS (MOS) subject classifications: 93B05.

1. INTRODUCTION

The theory of functional differential equations has been studied by several authors [6, 9, 11, 15]. The problem of controllability of linear neutral systems has been investigated by Banks and Kent [3], and Jacobs and Langenhop [12]. Motivation for physical control systems and its importance in other fields can be found in [11, 13]. Angell [1] and Chukwu [4] discussed the functional controllability of nonlinear neutral systems, and Underwood and Chukwu [16] studied the null controllability for such systems. Further, Chukwu [5] considered the Euclidian controllability of a neutral system with nonlinear base. Onwuatu [14] discussed the problem for nonlinear systems of neutral functional differential equations with limited controls. Gahl [10] derived a set of sufficient conditions for the controllability of nonlinear neutral systems through the fixed point method developed by Dauer [7]. Recently, Balachandran [2] established sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems and infinite delay neutral Volterra systems. In this paper we shall derive a new set of sufficient conditions for the controllability of nonlinear neutral Volterra integrodifferential systems by suitably adopting the technique of Do [8].

¹Received: November, 1991. Revised: March, 1993.

2. PRELIMINARIES

Let $J = [0, t_1]$, $t_1 > 0$ and let Q be the Banach space of all continuous functions

$$(x, u): J \times J \rightarrow R^n \times R^m$$

with the norm defined by

$$\|(x, u)\| = \|x\| + \|u\|$$

where $\|x\| = \sup_{t \in J} |x(t)|$. Define the norm of a continuous $n \times m$ matrix valued function $D: J \rightarrow R^n \times R^m$ by

$$\|D(t)\| = \max_i \sum_{j=1}^m \max_{t \in J} |d_{ij}(t)|$$

where d_{ij} are the elements of D .

Consider the linear neutral Volterra integrodifferential system of the form

$$\frac{d}{dt}[x(t) - \int_0^t C(t-s)x(s)ds - g(t)] = Ax(t) + \int_0^t G(t-s)x(s)ds + B(t)u(t) \quad (1)$$

and the nonlinear system

$$\begin{aligned} & \frac{d}{dt}[x(t) - \int_0^t C(t-s)x(s)ds - g(t)] \\ &= Ax(t) + \int_0^t G(t-s)x(s)ds + B(t)u(t) + f(t, x(t), u(t)), \end{aligned} \quad (2)$$

where $x(t) \in R^n$, $u(t) \in R^m$, $C(t)$ and $G(t)$ are $n \times n$ continuous matrix valued functions and $B(t)$ is a continuous $n \times m$ matrix valued functions, A a constant $n \times n$ matrix and f and g are, respectively, continuous and absolutely continuous vector functions. We consider the controllability on a bounded interval J of system (2).

Definition 1 [18]: A function $x: J \rightarrow R^n$ is said to be a solution of the initial value problem (1) or (2) through $(0, x(0))$ on J , if

- (i) x is continuous on J ,
- (ii) $[x(t) - \int_0^t C(t-s)x(s)ds - g(t)]$ is absolutely continuous on J ,
- (iii) (1) or (2) holds almost everywhere on J .

Definition 2: The system (2) is said to be controllable on J if for every $x(0)$, $x_1 \in R^n$ there exists a control function $u(t)$ defined on J such that the solution of (2) satisfies

$$x(t_1) = x_1.$$

The solution of (1) can be written, as in [17], in the form

$$x(t) = Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds + \int_0^t Z(t-s)B(s)u(s)ds,$$

where $\dot{Z}(t-s) = \frac{\partial Z}{\partial t}(t-s)$ and $Z(t)$ is an $n \times n$ continuously differentiable matrix satisfying the equation

$$\frac{d}{dt}[Z(t) - \int_0^t C(t-s)Z(s)ds] = AZ(t) - \int_0^t G(t-s)Z(s)ds$$

with $Z(0) = I$ and the solution of nonlinear system (2) is given by

$$\begin{aligned} x(t) = & Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ & + \int_0^t Z(t,s)[B(s)u(s) + f(s, x(s), u(s))]ds. \end{aligned}$$

Define the matrix W by

$$W(t) = \int_0^t Z(t-s)B(s)B^*(s)Z^*(t-s)ds,$$

where the $*$ denotes the transpose matrix. We know that system (1) is controllable on J if and only if W is nonsingular [2].

It is clear that x_1 can be obtained if there exist continuous functions $x(\cdot)$ and $u(\cdot)$ such that

$$\begin{aligned} u(t) = & B^*(t)Z^*(t_1-t)W^{-1}(t_1)[x_1 - Z(t_1)(x(0) - g(0)) - g(t_1) \\ & - \int_0^{t_1} \dot{Z}(t_1-s)g(s)ds - \int_0^{t_1} Z(t_1-s)f(s, x(s), u(s))ds] \end{aligned} \quad (3)$$

and

$$\begin{aligned} x(t) = & Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t-s)g(s)ds \\ & + \int_0^t Z(t-s)[B(s)u(s) + f(s, x(s), u(s))]ds. \end{aligned} \quad (4)$$

Now, we will find conditions for the existence of such $x(\cdot)$ and $u(\cdot)$. If $\alpha_i \in L^1(J)$ ($i = 1, 2, \dots, q$) then $\|\alpha_i\|$ ($i = 1, 2, \dots, q$) is the L^1 norm of $\alpha_i(s)$ ($i = 1, 2, \dots, q$). That is,

$$\|\alpha_i\| = \int_0^{t_1} |\alpha_i(s)| ds \quad (i = 1, 2, \dots, q).$$

Next, for our convenience, let us introduce the following notations:

$$K = \max\{\|Z(t-s)\| : 0 \leq s \leq t \leq t_1\},$$

$$k = \max\{\|Z(t-s)B(s)\| : t_1, 1\},$$

$$a_1 = 3k\{\|B^*(s)Z^*(t_1-s)\| \|W^{-1}(t_1)\| \|Z(t_1-s)\| \|\alpha_i\|\}, \quad (i = 1, 2, \dots, q)$$

$$b_i = 3K \|\alpha_i\|, \quad (i = 1, 2, \dots, q)$$

$$c_i = \max\{a_i, b_i\}, \quad (i = 1, 2, \dots, q)$$

$$d_1 = 3k \|B^*(s)Z^*(t_1-s)\| \|W^{-1}(t_1)\| [|x_1| + \|Z(t_1)\| |x(0) - g(0)| + |g(t_1)| + |\int_0^{t_1} \dot{Z}(t_1-s)g(s)ds|],$$

$$d_2 = 3 \|Z(t_1)\| |x(0) - g(0)| + |g(t_1)| + |\int_0^{t_1} \dot{Z}(t_1-s)g(s)ds|,$$

$$d = \max\{d_1, d_2\}.$$

3. MAIN RESULTS

Now, we will prove the following main theorem, which is a generalization of Theorem 2 of Balachandran [2].

Theorem: Let measurable functions $\phi_i: R^n \times R^m \rightarrow R^+$ ($i = 1, 2, \dots, q$) and L^1 functions $\alpha_i: J \rightarrow R^+$ ($i = 1, 2, \dots, q$) be such that

$$|f(t, x, u)| \leq \sum_{i=1}^q \alpha_i(t) \phi_i(x, u) \quad \text{for every } (t, x, u) \in J \times R^n \times R^m.$$

Then the controllability of (1) implies the controllability of (2) if

$$\lim_{r \rightarrow \infty} \sup(r - \sum_{i=1}^q c_i \sup\{\phi_i(x, u) : \|(x, u)\| \leq r\}) = \infty. \quad (5)$$

Proof: Define $T: Q \rightarrow Q$ by

$$T(x, u) = (y, v),$$

where

$$v(t) = B^*(t)Z^*(t_1-t)W^{-1}(t_1)[x_1 - Z(t_1)(x(0) - g(0)) - g(t_1)]$$

$$- \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds - \int_0^{t_1} Z(t_1 - s)f(s, x(s), u(s))ds] \quad (6)$$

and

$$\begin{aligned} y(t) &= Z(t)[x(0) - g(0)] + g(t) + \int_0^t \dot{Z}(t - s)g(s)ds \\ &+ \int_0^t Z(t - s)[B(s)v(s) + f(s, x(s), u(s))]ds. \end{aligned} \quad (7)$$

Based on our assumptions, T is continuous. Clearly the solutions $u(\cdot)$ and $x(\cdot)$ to (3) and (4) are fixed points of T . We will prove the existence of such fixed points by using the Schauder fixed point theorem.

Let $\psi_i(r) = \sup\{\phi_i(x, u): \|(x, u)\| \leq r\}$. Since (5) holds, there exists $r_0 > 0$ such that

$$\sum_{i=1}^q c_i \psi_i(r_0) + d \leq r_0.$$

Now, let

$$Q_{r_0} = \{(x, u) \in Q: \|(x, u)\| \leq r_0\}.$$

If $(x, u) \in Q_{r_0}$, from (6) and (7), we have

$$\begin{aligned} \|v\| &\leq \|B^*(t)Z^*(t_1 - t)\| \|W^{-1}(t_1)\| [\|x_1\| + \|Z(t_1)\| \|x(0) - g(0)\| \\ &+ \|g(t_1)\| + \left\| \int_0^{t_1} \dot{Z}(t_1 - s)g(s)ds \right\| + \left\| \int_0^{t_1} Z(t_1 - s) \left\| \sum_{i=1}^q \alpha_i(s) \phi_i(x(s), u(s)) \right\| ds \right\| \\ &\leq (d_1/3k) + (1/3k) \sum_{i=1}^q a_i \psi_i(r_0) \\ &\leq (1/3k)(d + \sum_{i=1}^q c_i \psi_i(r_0)) \\ &\leq (r_0/3k) \leq (r_0/3) \end{aligned}$$

and

$$\begin{aligned} \|y\| &\leq \|Z(t)\| \|x(0) - g(0)\| + \|g(t)\| + \left\| \int_0^t \dot{Z}(t - s)g(s)ds \right\| \\ &+ \left\| \int_0^t Z(t - s)B(s) \|v\| ds \right\| + \left\| \int_0^t Z(t, s) \left\| \sum_{i=1}^q \alpha_i(s) \phi_i(x(s), u(s)) \right\| ds \right\| \\ &\leq (d/3) + k \|v\| + K \sum_{i=1}^q \|\alpha_i\| \psi_i(r_0) \end{aligned}$$

$$\begin{aligned}
&\leq (d/3) + k \|v\| + (1/3) \sum_{i=1}^q c_i \psi_i(r_0) \\
&\leq (1/3)(d + \sum_{i=1}^q c_i \psi_i(r_0)) + k \|v\| \\
&\leq (r_0/3) + (r_0/3) = 2(r_0/3).
\end{aligned}$$

Hence T maps Q_{r_0} into itself. Further, it is easy to see that $T(Q_r)$ is equicontinuous for all $r > 0$ [8]. By the Ascoli-Arzelà theorem, $\overline{T(Q_{r_0})}$ is compact in Q . Since Q_{r_0} is nonempty, closed, bounded and convex, by the Schauder fixed point theorem, solutions of (3) and (4) exist. Hence the proof is complete.

To apply the above theorem we have to construct α_i 's and ϕ_i 's such that (5) is satisfied. These constructions are different for different situations. However, an obvious construction of α_i 's and ϕ_i 's is easily achieved by taking $q = 1$, $\alpha_1 = \alpha = 1$ and

$$\phi_1(x, u) = \phi(x, u) = \sup\{|f(t, x, u)| : t \in J\}.$$

In this case (5) holds if

$$\lim_{r \rightarrow \infty} \inf(1/r) \sup\{\phi(x, u) : \|(x, u)\| \leq r\} < 1/c_1.$$

Now, we will state a corollary which is a particular case of the above theorem and it was proved by Balachandran [2].

Corollary: *If the continuous function f satisfies the condition*

$$\lim_{\|(x, u)\| \rightarrow \infty} \frac{|f(t, x, u)|}{\|(x, u)\|} = 0$$

uniformly in $t \in J$ and if system (1) is controllable on J , then system (2) is controllable on J .

REFERENCES

- [1] Angell, T.S., On controllability for nonlinear hereditary systems: A fixed point approach, *Nonlinear Analysis: Theory, Methods and Applications* **4** (1980), 529-548.
- [2] Balachandran, K., Controllability of neutral Volterra integrodifferential systems, *J. of Australian Math. Soc., Ser. B*, **34** (1992), 18-25.
- [3] Banks, H.T. and G.A. Kent, Control of functional differential equations of retarded and neutral type to target sets in function space. *SIAM J. on Control* **10**, (1972), 562-593.

- [4] Chukwu, E.N., Functional inclusion and controllability of nonlinear neutral functional differential systems, *J. of Optim. Theory and Appl.* **29** (1979), 291-300.
- [5] Chukwu, E.N., On the Euclidian controllability of a neutral system with nonlinear base, *Nonlinear Analysis: Theory, Methods and Applications* **11** (1987), 115-123.
- [6] Chukwu, E.N. and H.C. Simpson, Perturbations on nonlinear systems of neutral type, *J. of Diff. Equations* **82** (1989), 28-59.
- [7] Dauer, J.P., Nonlinear perturbations of quasilinear control systems, *J. of Math. Anal. and Appl.* **54** (1976), 717-725.
- [8] Do, V.N., Controllability of semilinear systems, *J. of Optim. Theory and Appl.* **65** (1990), 41-52.
- [9] Fuchs, E., The degeneracy property in linear autonomous functional differential equations of neutral type, *J. of Math. Anal. and Appl.* **90** (1983), 527-549.
- [10] Gahl, R.D., Controllability of nonlinear systems of neutral type, *J. of Math. Anal. and Appl.* **63** (1978), 33-42.
- [11] Hale, J.K., *Theory of Functional Differential Equations*, Springer Verlag, New York 1977.
- [12] Jacobs, M.Q. and C.E. Langenhop, Criteria for function space controllability of linear neutral systems, *SIAM J. on Control and Optim.* **14** (1976), 1009-1048.
- [13] O'Conner, D.A., *State Controllability and Observability for Linear Neutral Systems*, Washington University, D.Sc. Thesis 1978.
- [14] Onwuatu, J.U., On the null controllability in function space of nonlinear systems of neutral functional differential equations with limited controls, *J. of Optim. Theory and Appl.* **42** (1984), 397-420.
- [15] Tadmor, G., Functional differential equations of retarded and neutral type: analytic solutions and piecewise continuous controls, *J. of Diff. Equations* **51** (1984), 151-181.
- [16] Underwood, R.G. and E.N. Chukwu, Null controllability of nonlinear neutral differential equations, *J. of Math. Anal. and Appl.* **129** (1988), 326-345.
- [17] Wu, J., Globally stable periodic solutions of linear neutral Volterra integrodifferential equations, *J. of Math. Anal. and Appl.* **130** (1988), 474-483.
- [18] Wu, J., Unified treatment of local theory of NFDES with infinite delay, *Tamkang J. of Math.* **22** (1991), 51-72.