ON THE EXISTENCE OF SOLUTIONS FOR VOLterra
INTEGRAL INCLUSIONS IN BANACH SPACES

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ABSTRACT

In this paper we examine a class of nonlinear integral inclusions defined in a separable Banach space. For this class of inclusions of Volterra type we establish two existence results, one for inclusions with a convex-valued orientor field and the other for inclusions with nonconvex-valued orientor field. We present conditions guaranteeing that the multivalued map that represents the right-hand side of the inclusion is \( \alpha \)-condensing using for the proof of our results a known fixed point theorem for \( \alpha \)-condensing maps.

Key words: Volterra integral inclusions, Aumann selection theorem, radial retraction, \( \alpha \)-condensing map.

AMS (MOS) subject classifications: 35R15, 34G20, 34A60.

1. INTRODUCTION-PRELIMINARIES

In this paper we examine a class of nonlinear integral inclusions defined in a separable Banach space and we establish two existence results. One for inclusions with a convex-valued orientor field and the other for inclusions with a nonconvex valued orientor field. Our work extends existence results of Ragimkhanov [11] and Lyapin [7] and the infinite dimensional results of Chuong [3] and Papageorgiou [10], where the hypotheses on the orientor field \( F(t, x) \) are too restrictive (see theorem 3.1 of Chuong and theorems 3.1-3.3 of Papageorgiou).

Let \((\Omega, \Sigma)\) be a measurable space and \( X \) a separable Banach space. Throughout this work we will be using the following notations:

\[
P_{f(c)} = \{A \subseteq X: \text{nonempty, closed (convex)}\}
\]
and

\[ P_{(w)k(c)}(X) = \{ A \subseteq X: \text{nonempty}, (w) \text{ compact}, \text{(convex)} \}. \]

A multifunction \( F: \Omega \rightarrow P(X) \) is said to be measurable (see Wagner [13]), if for every \( x \in X \), \( \omega \rightarrow d(x, F(\omega)) = \inf \{ \| x - z \| : z \in F(\omega) \} \) is measurable. When there is a \( \sigma \)-field measure \( \mu(\cdot) \) on \( (\Omega, \Sigma) \) and \( \Sigma \) is \( \mu \)-complete, then the above definition of measurability is equivalent to saying that \( GrF = \{ (\omega, x) \in \Omega \times X: x \in F(\omega) \} \in \Sigma \times B(X) \), with \( B(X) \) being the Borel \( \sigma \)-field of \( X \) (graph measurability).

By \( S_F \) we will denote the set of measurable selectors of \( F(\cdot) \) while by \( S^p_F \) \((1 \leq p \leq \infty)\) the set of measurable selectors of \( F(\cdot) \) that belong in the Lebesgue-Bochner space \( L^p(X) \), i.e. \( S^p_F = \{ f \in L^p(X): f(\omega) \in F(\omega)\mu\text{-a.e.} \} \). This set may be empty. It is nonempty if and only if \( \omega \rightarrow \inf \{ \| z \| : z \in F(\omega) \} \in L^+_\mu \).

In particular this is the case if \( \omega \rightarrow \| F(\omega) \| = \sup \{ \| z \| : z \in F(\omega) \} \in L^+_\mu \) in which case we say that \( F(\cdot) \) is \( L^\mu \)-integrably bounded.

If \( Y, Z \) are Hausdorff topological spaces and \( G: Y \rightarrow 2^Z \setminus \{ \emptyset \} \) then we say that \( G(\cdot) \) is lower semicontinuous (l.s.c.), if for all \( U \subseteq Z \) open, the set \( G^{-}(U) = \{ y \in Y: G(y) \cap U \neq \emptyset \} \) is open in \( Y \).

If furthermore \( Y, Z \) are metric spaces, then the above definition is equivalent to saying that for all \( y_n \rightarrow y \) we have \( G(y) \subseteq \lim G(y_n) = \{ z \in Z: z = \lim z_n, z_n \in G(y_n) \} \).

Also the multifunction \( F: Y \rightarrow 2^Z \setminus \{ \emptyset \} \) is said to be upper semicontinuous (u.s.c.) if and only if for every \( W \subseteq Z \) open, the set \( F^{+}(W) = \{ y \in Y: F(y) \subseteq W \} \) is open in \( Y \).

Finally we say that a multifunction \( G: Y \rightarrow 2^Z \setminus \{ \emptyset \} \) is closed if and only if the set \( GrG = \{ (y, z) \in z \in G(y) \} \) is closed in \( Y \times Z \).

2. EXISTENCE THEOREMS

Let \( T = [0, b], b > 0 \) and let \( X \) be a separable Banach space. The integral inclusion of Volterra type which we will be studying is the following:

\[ x(t) \in p(t) + \int_{0}^{t} K(t, s)F(s, x(s))ds, t \in T \quad (*) \]
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where \( p(\cdot) \in C(T, X) \).

By a solution of \((*)\) we understand a function \( x(\cdot) \in C(T, X) \) such that

\[
x(t) \in p(t) + \int_0^t K(t, s)f(s)ds, \quad t \in T, \quad \text{with } f \in S^b_{p(t, x(\cdot))},
\]

First we prove an existence result for the case where the orientor field \( F(t, x) \) is convex valued. For that purpose we will need the following hypotheses on the data of \((*)\).

**H(F):** \( F: T \times X \rightarrow P_{wsc}(X) \) is a multifunction such that:

1. \( t \rightarrow F(t, x) \) is measurable,
2. \( x \rightarrow F(t, x) \) is u.s.c. from \( X \) into \( X \),
3. \( |F(t, x)| \leq a(t) + b(t)\|x\| \ a.e. \) with \( a(\cdot), b(\cdot) \in L^1_+ \),
4. For any \( \epsilon > 0 \) and \( V \subseteq X \) bounded, there exists \( I_\epsilon \subseteq T \) open such that \( \mu(I_\epsilon) \leq \epsilon \) and \( a(F(J \times V)) \leq \sup_{t \in J} \eta(t)\alpha(V) \) for any \( J \subseteq T \setminus I_\epsilon \) closed and with \( \eta(\cdot) \in L^1_+ \).

**Remark:** We can replace the sublinear growth condition \( H(F)(3) \) by a hypothesis of the form “for every \( B \subseteq X \) bounded there exists \( a_B(\cdot) \in L^1_+ \) s.t. \( \sup_{x \in B} |F(t, x)| \leq a_B(t) \)”. In this case though the existence result is only local.

**H(K):** \( K: T \times T \rightarrow L(X) \) is continuous (we can have \( K \) defined only on \( \Delta \) and set \( K(t, s) = K(t, t), \ t \leq s \).

Now we are ready for our first result:

**Theorem 1:** If hypotheses \( H(F) \) and \( H(K) \) hold and \( M \|\eta\|_1 \leq 1 \) where

\[
\|K(t, s)\|_L \leq M,
\]

then \((*)\) admits a solution.

**Proof:** First we will establish an a priori bound for the solutions of \((*)\). So let \( x(\cdot) \in C(T, X) \) be such a solution. We have:

\[
\|x(t)\| \leq \|p\|_\infty + \int_0^t M |F(s, x(s))| \, ds
\]

for all \( t \in T \) and with \( \|K(t, s)\|_L \leq M \) for all \( (t, s) \in \Delta \) (see hypothesis \( H(K) \)).
Using hypothesis $H(F)(3)$, we get:

$$
\| x(t) \| \leq \| p \|_\infty + \int_0^t (Ma(s) + Mb(s) \| x(s) \| )ds, \quad t \in T.
$$

Invoking Gronwall's inequality, we get $M_1 > 0$ s.t.

$$
\| x(t) \| \leq M_1
$$

for all $t \in T$ and all solutions $x(\cdot) \in C(T, X)$ of $(*)$.

Let $\widehat{F}(t, x) = F(t, p_{M_1}(x))$, with $p_{M_1}(\cdot)$ being the $M_1$-radial retraction.

We will consider the integral inclusion $(*)$ with the orientor field $F(t, x)$ replaced by $\widehat{F}(t, x)$. Note that because of hypothesis $H(F)(1)$ the multifunction $t \mapsto F(t, x)$ is also measurable. Also recalling that $p_{M_1}(\cdot)$ is Lipschitz continuous and using hypothesis $H(F)(2)$, we get from theorem 7.3.11 (ii), p. 87 of Klein-Thompson [5], that $x \mapsto \widehat{F}(t, x)$ is u.s.c. from $X$ into $X_w$. Furthermore $|\widehat{F}(t, x)| \leq a(t) + b(t)M_1 = \phi(t)$ a.e. with $\phi(\cdot) \in L^1_+$.

Finally in hypothesis $H(F)(4)$ we have

$$
\alpha(\widehat{F}(J \times V)) = \alpha(F(J \times p_{M_1}(V))) \leq \sup_{t \in J} \alpha(t) \alpha(p_{M_1}(V)).
$$

But note that $p_{M_1}(V) \subseteq \text{conv}[\{0\} \cup V] \Rightarrow \alpha(p_{M_1}(V)) = \alpha(\{0\} \cup V) \leq \alpha(V)$. So we have $\alpha(\widehat{F}(J \times V)) \leq \sup_{t \in J} \alpha(t) \alpha(V)$ and so we have checked that $\widehat{F}(t, x)$ satisfies hypothesis $H(F)(4)$.

Set

$$
H = \{ y \in C(T, X) : y(t) = p(t) + \int_0^t K(t, s) g(s) ds, \quad t \in T, \| g(t) \| \leq \phi(t) \text{ a.e.} \}.
$$

Next let $R : H \to 2^H$ be defined by

$$
R(x) = \{ y \in C(T, X) : y(t) = p(t) + \int_0^t K(t, s) f(s) ds, t \in T, f \in S^1_{\widehat{F}(\cdot, x(\cdot))} \}.
$$

First we will show that $R(\cdot)$ has nonempty values. Let $\{s_n\}_{n \geq 1}$ be simple functions such that $s_n(t) \to x(t)$ a.e. in $X$.

Then for each $n \geq 1$, $t \to \widehat{F}(t, s_n(t))$ is measurable (since $t \to \widehat{F}(t, x)$ is measurable). So by Aumann's selection theorem (see Wagner [13], theorem 5.10), we get $f_n : T \to X$ measurable such that $f_n(t) \in \widehat{F}(t, s_n(t))$. Clearly $f_n \in L^1(X)$. Note that because $\widehat{F}(t, \cdot)$ is u.s.c. from $X$ into $X_w$,
$U(t) = \bigcup_{n \geq 1} F(t, s_n(t))^{w} \in P_{wk}(X)$ (see Klein-Thompson [5], theorem 7.4.2, p. 90) and $t \rightarrow U(t)$ is measurable. Hence $t \rightarrow \text{conv} U(t) = U_s(t)$ is an integrably bounded, $P_{wk}(X)$-valued multifunction (Krein-Smulian theorem). So from Papageorgiou [8] (see proposition 3.1), we get that $S^1_{L^1_c}$ is $w$-compact in $L^1(X)$.

But observe that $\{f_n\}_{n \geq 1} \subseteq S^1_{U_c}$. So by passing to a subsequence, if necessary, we may assume that $f_n \rightharpoonup f$ in $L^1(X)$. Then from [9] (see theorem 3.1), we get that

$$f(t) \in \text{conv} \lim_{n \rightarrow \infty} \{f_n(t)\} \subseteq \text{conv} \lim_{n \rightarrow \infty} \tilde{F}(t, s_n(t))$$

the last inclusion following from the upper semicontinuity of $\tilde{F}(t, \cdot \cdot)$ from $X$ into $X_w$, the fact that $s_n(t) \rightarrow x(t)$ a.e. in $X$ and the fact that $\tilde{F}(\cdot, \cdot \cdot)$ is $P_{wk}(X)$-valued. So $S^1_{\tilde{F}(\cdot, x(\cdot \cdot))} \neq \emptyset \Rightarrow R(x) \neq \emptyset$ for all $x \in C(T, X)$. Also since $S^1_{\tilde{F}(\cdot, x(\cdot \cdot))} \in P_{wk}(L^1(X))$ (see proposition 3.1 of [8]), we can easily check that $R(\cdot \cdot)$ has closed, convex values in $2^{C(T, X)}\{\emptyset\}$.

Next we will show that $R(\cdot \cdot)$ has a closed graph. To this end let $[x_n, y_n] \in Gr R$ and assume that $[x_n, y_n] \rightharpoonup [x, y]$ in $C(T, X) \times C(T, X)$. Then by definition for every $n \geq 1$ we have

$$y_n(t) = p(t) + \int_{0}^{t} K(t, s)f_n(s)ds, \text{ for } t \in T \text{ and with } f_n \in S^1_{\tilde{F}(\cdot, x_n(\cdot \cdot))}$$

Note that by the Krein-Smulian theorem (see for example Diestel-Uhl [4], theorem II, p. 51), we have that $\bigcup_{n \geq 1} \tilde{F}(t, x_n(t)) \in P_{wk}(X)$ for all $t \in T$. So from proposition 3.1 of [8] and by passing to a subsequence if necessary, we may assume that $f_n \rightharpoonup f$ in $L^1(X)$. Then as above using theorem 3.1 of [9] and the properties of $\tilde{F}(t, x)$, we get

$$f(t) \in \text{conv} \lim_{n \rightarrow \infty} \{f_n(t)\} \subseteq \text{conv} \lim_{n \rightarrow \infty} \tilde{F}(t, x_n(t)) \subseteq \tilde{F}(t, x(t)), \text{ a.e.}$$

Also $\int_{0}^{t} K(t, s)f_n(s)ds \rightharpoonup \int_{0}^{t} K(t, s)f(s)ds$ in $X$. Hence in the limit as $n \rightarrow \infty$ we get:

$$y(t) = p(t) + \int_{0}^{t} K(t, s)f(s)ds, \text{ } t \in T$$
with \( f \in S^1_{\mathcal{F}(x)} \). Therefore \([x, y] \in \text{Gr} R \Rightarrow R(\cdot)\) has a closed graph.

Next by Lusin’s theorem, given \( \epsilon > 0 \) there exists \( I_\epsilon \subseteq T \) open such that \( \lambda(I_\epsilon) < \epsilon/2 \), \( \eta \mid_{T \setminus I_\epsilon} \in C \) and \( \| \phi \chi_{I_\epsilon} \|_1 \leq \epsilon/2M \). Also from hypothesis \( H(F)(4) \) (which as we have already checked earlier, is also valid for the orientor field \( \mathcal{F}(t, x) \)), given \( V \) a nonempty subset of \( H \) we can find \( I_\epsilon^2 \subseteq T \) open with \( \lambda(I_\epsilon^2) < \epsilon/2 \) and

\[
\alpha(F(J \times \hat{V})) \leq \sup_{s \in J} \eta(s)\alpha(\hat{V}) \quad \text{and} \quad \| \phi \chi_{I_\epsilon^2} \|_1 \leq \epsilon/2M
\]

where \( J \subseteq L = T \setminus I_\epsilon \) closed, with \( I_\epsilon = I_\epsilon^1 \cup I_\epsilon^2 \) and \( \hat{V} = \{v(t) \in V, t \in T\} \).

Note that because of hypothesis \( H(K) \) and since by the choice of \( L \eta \mid_L \) continuous, the map \((s, w) \mapsto \| K(t, s) \| L \eta(w) \) is continuous, hence uniformly continuous on \( ([0, t] \cap L) \times L \). Thus given \( \delta > 0 \) we can find \( \theta > 0 \) s.t.

\[
| \| K(t, s) \| L \eta(w) \alpha(\hat{V}) - \| K(t, \tau) \| L \eta(\tau) \alpha(\hat{V}) | \leq \delta
\]

for all \( s, \tau \in [0, t] \cap L \) with \( |s - \tau| \leq \theta \) and all \( w, z \in L \) with \( |w - z| \leq \theta \).

Let \( 0 = t_0 < t_1 < \ldots < t_n = b \) be a subdivision of \( T \) into \((n + 1)\)-parts such that \( t_i - t_{i-1} \leq \theta \) and let \( L_i = [t_{i-1}, t_i] \setminus I_\epsilon \quad i = 1, 2, \ldots, n \).

Also let \( v_i \in L_i \) and \( s_i \in L_i \quad i = 1, 2, \ldots, n \) be such that

\[
\| K(t, v_i) \| L = \sup_{s \in L_i} \| K(t, s) \| L
\]

and \( \eta(s_i) = \sup_{s \in L_i} \eta(s) \). Their existence is guaranteed by hypothesis \( H(K) \) and since \( \eta \mid L \) is continuous. Then we have:

\[
\alpha(\mathcal{F}(L_i \times \hat{V})) \leq \eta(s_i)\alpha(\hat{V}).
\]

Also from the “Mean Value Theorem” for Bochner integrals (see Diestel-Uhl, [4], corollary 8, p. 48), we have:

\[
\{ \int_{L_i} K(t, s)\mathcal{F}(s, x(s))ds : x \in V \} \subseteq \mu(L_i)\overline{\text{conv}}[K(t, s)\mathcal{F}(s, y) : s \in S_i, y \in \hat{V}].
\]

So we have

\[
\{ \int_{L} K(t, s)\mathcal{F}(s, x(s))ds : x \in V \} \subseteq \sum_{i=1}^{n} \mu(L_i)\overline{\text{conv}}[K(t, s)\mathcal{F}(s, y) : s \in S_i, y \in \hat{V}].
\]
Using the subadditivity of the $\alpha(\cdot)$ measure of non-compactness, we get
\[
\alpha\{ \int K(t,s)\widehat{F}(s,x(s))ds: x \in V \} \leq \sum_{i=1}^{n} \mu(L_i) \| K(t,v_i) \|_L \alpha(\widehat{F}(L_i, \widehat{V})) \\
\leq \sum_{i=1}^{n} \mu(L_i) \| K(t,v_i) \|_L \eta(s) \alpha(\widehat{V}).
\]

From (1) above we get
\[
\alpha\{ \int K(t,s)\widehat{F}(s,x(s))ds: x \in V \} \leq \int L K(t,s) \| \eta(s) \alpha(\widehat{V})d\tau + \delta \mu(L).
\]

Also recall from the initial choice of the sets $I_1^e$ and $I_2^e$ that
\[
\int_{I_2^e} \| K(t,s) \|_L \phi(s)ds \leq \varepsilon.
\]

So finally we have:
\[
\alpha(\widehat{V}(t)) \leq \int L K(t,s) \| \eta(s) \alpha(\widehat{V})ds + \delta \mu(S) + \varepsilon
\]
\[
\leq \int_{0}^{b} M \eta(s) \alpha(\widehat{V})ds + \delta \mu(S) + \varepsilon.
\]

Since $\varepsilon, \delta > 0$ were arbitrary, we get
\[
\alpha(\widehat{V}(t)) \leq \int_{0}^{b} M \eta(s) \alpha(\widehat{V})ds = \alpha(\widehat{V})M \| \eta \|_1
\]

Since $H$ is bounded and equicontinuous, from Ambrosetti’s theorem (see theorem 1.4.2 p. 20 of Lakshmikantham-Leela [6]) we have that
\[
\alpha(\widehat{V}) \leq \widehat{\alpha}(V)
\]
and $\sup_{t \in T} \alpha(\widehat{V}(t)) = \widehat{\alpha}(R(V))$. Thus we get
\[
\widehat{\alpha}(R(V)) \leq M \| \eta \|_1 \widehat{\alpha}(V).
\]

Since by hypothesis $M \| \eta \|_1 < 1$, we get that $R(\cdot)$ is $\widehat{\alpha}(\cdot)$-condensing.

Apply theorem 4.1 of Tarafdar-Vyborny [12], to get $x \in R(x)$. Then $x \in C(T,X)$ solves $(\ast)$ with the orientor field $\widehat{F}(t,x)$. Using the definition of $\widehat{F}(t,x)$ and same estimation as in the beginning of the proof, we get that
\[
\| x(t) \| \leq M_1 \Rightarrow \widehat{F}(t,x(t)) = F(t,x(t)) \Rightarrow x(\cdot) \in C(T,X) \text{ solves (\ast)}.
\]

Q.E.D.
We can have a variant of Theorem 1, where the orientor field is not convex-valued. For this we will need the following hypothesis.

\( H(F)' \): \( F: T \times X \to P_k(X) \) is a multifunction such that

1. \((t, x) \to F(t, x)\) is graph measurable,
2. \( x \to F(t, x) \) is l.s.c.,

and the hypotheses \( H(F)(3) \) and \( (4) \) also hold.

**Theorem 2:** If hypotheses \( H(F)' \) and \( H(K) \) hold and \( M \| \eta \|_1 < 1 \), then \((*)\) admits a solution.

**Proof:** As in the proof of Theorem 1, we can show that for every solution \( x(\cdot) \in C(T, X) \) of \((*)\), we have \( \| x \|_{C(T, X)} \leq M_1 \). Then define \( \hat{F}(t, x) = F(t, p_{M_1}(x)) \). This has the same measurability and continuity properties as \( F(t, x) \) satisfies \( H(F)(4) \), (see the proof of Theorem 1) and \( |\hat{F}(t, x)| \leq \phi(t) \) a.e. with \( \phi(\cdot) \in L^1_+ \).

Let \( \Gamma: C(T, X) \to P_f(L^1(X)) \) be defined by

\[
\Gamma(x) = \left\{ \begin{array}{ll}
F(t, x) & t \in T, \\
S^1_{\hat{F}(t, x)} & \end{array} \right.
\]

Then from Papageorgiou [9] (see theorem 4.1) we get that \( \Gamma(\cdot) \) is l.s.c. Apply theorem 3 of Bressan-Colombo [2] to get a continuous map \( \gamma: C(T, X) \to L^1(X) \) such that \( \gamma(x) \in \Gamma(x) \) for all \( x \in C(T, X) \).

As in the proof of Theorem 1, let

\[
H = \{ y \in C(T, X): y(t) = p(t) + \int_0^t K(t, s)g(s)ds, t \in T, \| g(t) \| \leq \phi(t) \ \text{a.e.} \}.
\]

This is bounded and equicontinuous. Let \( R: H \to H \) be defined by

\[
R(x)(t) = p(t) + \int_0^t K(t, s)\gamma(x)(s)ds.
\]

Since \( \gamma(\cdot) \) is continuous, we can easily check that \( R(\cdot) \) is continuous too. From the proof of Theorem 1, we know that it is \( \tilde{a} \)-condensing. So there exists \( x = R(x) \). This is the desired solution of \((*)\).

Q.E.D.
Acknowledgement: The author wishes to thank the referee for his/her corrections and constructive criticism.

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