QUASILINEARIZATION FOR SOME
NONLOCAL PROBLEMS

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ABSTRACT
The method of generalized quasilinearization [4] is applied to study semilinear parabolic equation $u_t - Lu = f(t, x, u)$ with nonlocal boundary conditions $u(t, x) = \int_{\Omega} \phi(x, y)u(t, y)dy$ in this paper. The convexity of $f$ in $u$ is relaxed by requiring $f(t, x, u) + Mu^2$ to be convex for some $M > 0$. The quadratic convergence of monotone sequence is obtained.

Key words: Quasilinearization, nonlocal problem, semilinear parabolic equation, upper and lower solutions, monotone iterative technique, quadratic convergence.

AMS (MOS) subject classifications: 35K20, 35K60, 35K99.

1. INTRODUCTION
In this paper, we consider the nonlocal boundary value problem (NBVP for short):

\begin{align}
  u_t - Lu &= f(t, x, u) \quad \text{in } D_T = (0, T] \times \Omega, \\
  u(0, x) &= u_0(x) \quad \forall x \in \Omega, \\
  u(t, x) &= \int_{\Omega} \phi(x, y)u(t, y)dy \quad \forall (t, x) \in \Gamma_T,
\end{align}

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\partial \Omega \in C^2$, $\Gamma_T = (0, T) \times \partial \Omega$,

\[ L = \sum_{i, j=1}^{n} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t, x) \frac{\partial}{\partial x_i}, \quad a_{ij}, b_i \in C^{0, \alpha}, 0 < \alpha < 1. \]

$u_0(x) \in C(\overline{\Omega})$, $f \in C(\overline{D_T \times R, R})$ and satisfies Lipschitz condition in $u$. $\phi \in C(\partial \Omega \times \overline{\Omega}, R)$ satisfies $\phi(x, y) \geq 0$, $\int_{\Omega} \phi(x, y)dy \neq 0$, $\forall x \in \partial \Omega$ and $\int_{\Omega} \phi(x, y)dy \leq \rho < 1$, $\forall x \in \partial \Omega$.

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It is shown in [5] that, if \( v, w \) are lower and upper solutions of NBVP such that \( v \leq w \), then there exist monotone sequences which converge uniformly to the unique solution of NBVP. The method of upper and lower solutions and monotone iterative techniques played most important roles in constructing monotone sequences.

Consider the following ordinary differential equation with initial condition:
\[
    u' = g(t, u), \quad u(0) = u_0, \quad t \in J = [0, T]. \tag{1.4}
\]

It is well known [1, 2] that the method of quasilinearization provides not only the monotone sequence but also quadratic convergence of the sequence for (1.4) if requiring more on \( g(t, u) \), namely, \( g(t, u) \) to be convex in \( u \) for \( t \in J \). The result is of computational interest. In [4], this method has been extended to more general cases, namely, without demanding the convexity of \( g(t, u) \), by requiring \( g(t, u) + Mu^2 \) to be convex for some \( M > 0 \) and the same results have been obtained [4].

Our purpose, in this paper, is to apply the method used in [4] to study nonlocal problem (1.1) – (1.3).

2. PRELIMINARIES

Consider NBVP (1.1)-(1.3). A function \( v_0 \in C^{1,2}(D_T) \cap C(\overline{D_T}) \) is called a lower solution of NBVP if
\[
    (v_0)_t - Lv_0 \leq f(t, x, v_0) \quad \text{in } D_T,
\]
\[
    v_0(0, x) \leq u_0(x) \quad \forall x \in \Omega,
\]
\[
    v_0(t, x) \leq \int_{\Omega} \phi(x, y)v_0(t, y)dy \quad \text{on } \Gamma_T.
\]

An upper solution \( w_0 \) is defined analogously by reversing the above inequalities with respect to \( w_0 \). It is shown[5] that \( v_0 \leq w_0 \) on \( \Omega \).

We first state two lemmas which we need in the proof of the main result. Define \( Q_T = \{(t, x, u): v_0 \leq u \leq w_0, (t, x) \in D_T \} \). If the existence of upper and lower solutions of NBVP is known, then we can prove the existence of solutions of NBVP in the closed set \( Q_T \) which is the statement of Lemma 2.1. For proof, see [5].

**Lemma 2.1:** Consider
\[
    u_t - Lu = G(t, x, \eta(t, x), u(t, x)) \quad \text{in } D_T, \tag{2.1}
\]
with (1.2) and (1.3), where η(t, x) is a given function on \( \bar{D}_T \). Suppose that

(i) \( v_0, w_0 \) are lower solutions of (2.1), (1.2) and (1.3),

(ii) \( G(t, x, \eta, u) \) satisfies Lipschitz condition in \( u \), i.e., \( \exists N > 0 \) such that

\[
-N(u_1 - u_2) \leq G(t, x, \eta, u_1) - G(t, x, \eta, u_2) \leq N(u_1 - u_2),
\]

whenever \( u_1 \geq u_2 \), for \( (t, x, u_1), (t, x, u_2) \in Q_T \). Then there exists a unique solution \( u \) of NBVP such that \( u \in C^{1,2}(D_T) \cap C(\bar{D}_T) \) and \( v_0 \leq u \leq w_0 \) on \( D_T \).

**Lemma 2.2:** Assume that \( p \in C^{1,2}(D_T) \cap C(\bar{D}_T) \) satisfies

\[
p_t(Lp + Kp + C) \leq 0 \text{ in } D_T,
p(0, x) \leq 0 \quad \forall x \in \Omega,
p(t, x) \leq \int_{\Omega} \phi(x, y)p(t, y)dy \quad \forall (t, x) \in \Gamma_T,
\]

where \( K > 0, C > 0 \) are constants. Then, \( p(t, x) \leq Ce^{(K+1)t} \) on \( \bar{D}_T \).

**Proof:** Let \( \Psi = Ce^{(K+1)t} \), then \( \Psi \) verifies

\[
\Psi_t - (L\Psi + K\Psi + C) = C(e^{(K+1)t} - 1) > 0 \text{ in } D_T,
\]

\[
\Psi(0, x) > 0 \quad \forall x \in \Omega,
\]

\[
\Psi(t, x) > \int_{\Omega} \phi(x, y)\Psi(t, y)dy \quad \text{on } \Gamma_T.
\]

By Theorem 2.1 of [5], \( p(t, x) \leq \Psi \) on \( \bar{D}_T \). This completes the proof of the lemma.

**3. MAIN RESULTS**

In this section, we shall apply the method of quasilinearization generalized in [3] to NBVP in order to obtain the monotone sequence and quadratic convergence of the sequence. Suppose now, \( f \in C^2 \) in \( u \) and \( f + Mu^2 \) is uniformly convex for \( (t, x, u) \in Q_T \). Define \( F(t, x, u) \equiv f(t, x, u) + Mu^2 \), then

\[
F_{uu} = f_{uu} + 2M > 0.
\]

If \( u_1 \geq u_2 \), it follows

\[
F_u(t, x, u_1) = f_u(t, x, u_1) + 2Mu_1 \geq F_u(t, x, u_2) + 2Mu_2;
\]

\[
F(t, x, u_1) - F_u(t, x, u_2)u_1 \geq F(t, x, u_2) - F_u(t, x, u_2)u_2.
\]
We also assume that there exist constants \( K_1 > 0, C_1 > 0 \) such that

\[
|F_u| \leq K_1 \quad \text{and} \quad F_{uu} \leq C_1, \text{ for } (t, x, u) \in Q_T.
\]

Now, we are in the position to prove the main results.

**Theorem 3.1:** Suppose that

(i) \( v_0, w_0 \) are lower and upper solutions of NBVP such that \( v_0 < w_0 \) on \( \bar{D}_T \);

(ii) \( F(t, x, u) \equiv f(t, x, u) + Mu^2 \) is uniformly convex in \( Q_T \).

Then, there exists a monotone sequence \( \{v_n\} \) which converges uniformly to the unique solution of NBVP and moreover, the convergence is quadratic.

**Proof:** Consider the modified nonlocal problem \((P^*)\):

\[
\begin{align*}
u_t - Lu &= f(t, x, \eta) + (f_u(t, x, \eta) + 2M\eta)(u - \eta) - M(u^2 - \eta^2) \\
&\quad \equiv F(t, x, \eta) + F_u(t, x, \eta)(u - \eta) - Mu^2,
\end{align*}
\]

(3.3)

with (1.2) and (1.3), where \( \eta \in C^1,2(D_T) \cap (\bar{D}_T) \) is such that

\[
v_0(t, x) \leq \eta(t, x) \leq w_0(t, x) \text{ on } D_T.
\]

Define

\[
G(t, x, \eta, u) \equiv F(t, x, \eta) + F_u(t, x, \eta)(u - \eta) - Mu^2.
\]

(3.4)

Since \( v_0 \) is a lower solution of NBVP and if \( \eta = v_0 \), because of (3.4), we have

\[
(v_0)_t - Lv_0 \leq f(t, x, v_0) \equiv G(t, x, v_0, v_0) \text{ in } D_T,
\]

(3.5)

From (3.3) and (3.4),

\[
u_t - Lu = G(t, x, v_0, u) \text{ in } D_T.
\]

(3.6)

Similarly,

\[
(w_0)_t - Lw_0 \geq f(t, x, w_0) \geq F(t, x, v_0) + F_u(t, x, v_0)(w_0 - v_0) - Mw_0^2 \equiv G(t, x, v_0, w_0) \text{ in } D_T.
\]

(3.7)

From (3.5), (3.6) and (3.7) together with the initial and boundary conditions, we conclude that \( v_0(t, x) \) and \( w_0(t, x) \) are lower and upper solutions of \((P^*)\) with \( \eta = v_0 \). It is easy to see that \( G(t, x, v_0, u) \) satisfies Lipschitz condition (2.2) for \( (t, x, u) \in Q_T \) since \( F(t, x, u) \) is convex in \( u \) and \( v_0(t, x) \) is a given function. Therefore, by Lemma 2.1, there exists a unique solution \( u \) of
(\(P^*\)) with \(\eta = v_0\) such that \(u \in C^{1,2}(D_T) \cap C(\bar{D}_T)\) and \(v_0 \leq u \leq w_0\) on \(\bar{D}_T\).

Now, we construct the sequence \(\{v_n\}\) by

\[
(v_{n+1})_t - Lv_{n+1} = f(t, x, v_n) + (fu(t, x, v_n) + 2Mv_n)(v_{n+1} - v_n) - M(v_{n+1}^2 - v_n^2)
\]

\[
= F(t, x, v_n) + F_u(t, x, v_n)(v_{n+1} - v_n) - Mv_{n+1}^2 \quad \text{in } D_T, \tag{3.8}
\]

\[
v_{n+1}(0, x) = u_0(x) \quad \forall x \in \Omega,
\]

\[
v_{n+1}(t, x) = \int_{\Omega} \phi(x, y)v_{n+1}(t, y)dy \quad \text{on } \Gamma_T,
\]

for \(n = 0, 1, 2, \ldots\).

Clearly, from the above discussion, we obtain that \(P^*\) has a unique solution \(v_1\) with \(\eta = v_0\) and \(v_0 \leq v_1 \leq w_0\) on \(\bar{D}_T\).

Assume that for some \(n\), \((P^*)\) has a unique solution \(v_n\) with \(\eta = v_{n-1}\) and \(v_{n-1} \leq v_n \leq w_0\) on \(\bar{D}_T\), by (3.2) and (3.8)

\[
(v_n)_t - Lv_n = F(t, x, v_{n-1}) + F_u(t, x, v_{n-1})(v_n - v_{n-1}) - Mv_n^2
\]

\[
\leq F(t, x, v_n) - Mv_n^2 = f(t, x, v_n) \quad \text{in } D_T. \tag{3.9}
\]

On the other hand,

\[
(v_{n+1})_t - Lv_{n+1} = F(t, x, v_n) + F_u(t, x, v_n)(v_{n+1} - v_n) - Mv_{n+1}^2
\]

\[
\equiv G(t, x, v_n, v_{n+1}) \quad \text{in } D_T. \tag{3.10}
\]

Now, (3.9) becomes

\[
(v_n)_t - Lv_n \leq f(t, x, v_n) \equiv G(t, x, v_n, v_n) \quad \text{in } D_T. \tag{3.11}
\]

Similarly,

\[
w_0 \leq v_0 \leq \cdots \leq v_n \leq \cdots \quad \text{on } \bar{D}_T,
\]

and \(\{v_n\}\) is bounded above by \(w_0\) on \(\bar{D}_T\). Therefore, we obtain a sequence \(\{v_n\}\) which is monotone increasing and uniformly bounded in \(C^{1,2}(D_T) \cap C(\bar{D}_T)\). By standard arguments [3, 5], \(\{v_n\}\) converges uniformly to the unique solution of NBVP.
It remains to show that convergence of \( \{v_n\} \) to the solution \( u \) of NBVP is quadratic. Observe that, for all \( n \geq 0 \), \( v_n \leq u \). Set \( p_n = u(t, x) - v_n(t, x) \) on \( \bar{D}_T \). Define
\[
\|p_n\| = \sup_{(t, x) \in \bar{D}_T} |u(t, x) - v_n(t, x)|.
\]
Then,
\[
(p_n)_t - Lp_n = f(t, x, u) - [F(t, x, v_{n-1}) + F_u(t, x, v_{n-1})(v_n - v_{n-1}) - \alpha v_n^2]
\]
\[
= F_u(t, x, \theta)(u - v_{n-1}) - F_u(t, x, v_{n-1})(u - v_{n-1}) + F_u(t, x, v_{n-1})(u - v_n) + M(u^2 - u^2)
\]
\[
= F_{uu}(t, x, \xi)(\theta - v_{n-1}) + F_u(t, x, v_{n-1})(u - v_n) + M(v_n^2 - u^2) \quad \text{in } D_T,
\]
where \( v_{n-1} \leq \xi \leq \theta \leq u \) for \( (t, x) \in D_T \).

Since \( 0 < F_{uu} \leq C_1 \), \( |F_u| \leq K_1 \) on \( Q_T \) and \( v_0 \leq v_n, u \leq w_0 \), we have
\[
(p_n)_t - Lp_n \leq C_1(u - v_{n-1})^2 + Kp_n - M(v_n + u)p_n
\]
\[
\leq (K_1 - 2Mv_0)p_n + C_1 \|p_{n-1}\|^2 \leq Kp_n + C \text{ in } D_T,
\]
where \( K = K_1 + 2MV \) and \( |v_0| \leq V \) on \( \bar{D}_T \), \( C = C_1 \|p_{n-1}\|^2 \). Therefore, by Lemma 2.2, we get
\[
p_n(t, x) \leq Ce^{(K + 1)t} \leq C_1e^{(K + 1)T} \|p_{n-1}\|^2 = C_1e^{(K + 1)T} \sup_{(t, x) \in \bar{D}_T} |u - v_{n-1}|^2 \text{ on } \bar{D}_T,
\]
which yields
\[
\sup_{(t, x) \in \bar{D}_T} |u - v_n| \leq \alpha \sup_{(t, x) \in \bar{D}_T} |u - v_{n-1}|^2, \quad \alpha = C_1e^{(K + 1)T}.
\]

The proof of Theorem 3.1 is therefore complete.

REFERENCES


