OPTIMAL CONTROLLABILITY OF IMPULSIVE CONTROL SYSTEMS

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ABSTRACT

The problem of optimal controllability of a nonlinear impulsive control system is studied using the method of vector Lyapunov functions and the generalized comparison principle.

Key words: Impulsive control systems, optimal control, vector Lyapunov functions.

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1. INTRODUCTION

Many evolutionary processes are subject to short term perturbations which act instantaneously in the form of impulses. Thus impulsive differential equations provide a natural description of observed evolutionary processes of several real world processes [1].

Control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems [8]. A central problem in this area is the optimal control problem, that is, the problem of controlling a system in some "best" possible manner by minimizing some function of the trajectories.

In this paper, the problem of optimal controllability of a nonlinear impulsive control system is studied, using the method of vector Lyapunov functions and the generalized comparison principle [3, 4]. An example is provided to illustrate the results.

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2. MAIN RESULTS

We shall consider the following impulsive control system

$$\left\{ \begin{array}{ll} x' = f(t, x, u), & t \neq t_k, \quad k = 1, 2, \dots \\ x(t_k^+) = x(t_k) + I_k(x, u), & k = 1, 2, \dots, \\ x(t_0) = x_0 \end{array} \right. \quad (2.1)$$

where $0 < t_1 < t_2 \dots < t_k < \dots$ and $t_k \rightarrow \infty$ as $k \rightarrow \infty$, $f \in PC[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$, $I_k \in C[\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$ for every k , $k = 1, 2, \dots$, and $u = u(t)$ is a control vector. Let Ω be the control prescribed. Corresponding to any control function $u = u(t)$, we shall denote a solution of (2.1) by $x(t) = x(t; t_0, x_0, u)$, with $x(t_0) = x_0$.

The following result deals with the optimal stabilization of (2.1).

Theorem 2.1: *Assume that*

- (i) $0 < \lambda < A$ are given,
- (ii) $V \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+^N]$, $V(t, x)$ is locally Lipschitzian in x , $Q \in \mathcal{H}[\mathbb{R}_+^N, \mathbb{R}_+]$, $g \in PC[\mathbb{R}_+ \times \mathbb{R}_+^N \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^N]$, $g(t, w, x, u)$ is quasimonotone nondecreasing in w and $\psi_k: \mathbb{R}_+^N \rightarrow \mathbb{R}^N$ is nondecreasing for $k = 1, 2, \dots$,
- (iii) $\Omega \subset \mathbb{R}^m$ is a convex, compact set and for $u^0(t) \in \Omega$, the system (2.1) admits unique solutions for $t \geq t_0$ and for $(t, x) \in \mathbb{R}_+ \times S(A)$,

$$b(\|x\|) \leq Q(V(t, x)) \leq a(\|x\|), \quad a, b \in \mathcal{H}[\mathbb{R}_+, \mathbb{R}_+]$$

and

$$\|x + I_k(x)\| < \rho \text{ whenever } \|x\| < A, \quad \rho > A.$$

- (iv) $B[V, t, x, u^0, g] \equiv V_t(t, x) + V_x^T(t, x)f(t, x, u^0) + g(t, V(t, x), x, u^0) \leq 0$, $t \neq t_k$,
- (v) $C_k[V, t_k, x, \psi_k] \equiv \Delta V + \psi_k(V(t_k, x(t_k))) = 0$, $k = 1, 2, \dots$, where $\Delta V = V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))$,
- (vi) $B[V, t, x, u, g] \geq 0$ for any $u \in \Omega$, $t \neq t_k$,
- (vii) $a(\lambda) < b(A)$ holds,
- (viii) any solution $w(t, t_0, w_0)$ of

$$\left\{ \begin{array}{ll} w' = -g(t, w, x^0(t), u^0(t)), & t \neq t_k \\ \Delta w = \psi_k(w(t_k)) & k = 1, 2, \dots, \\ w(t_0) = w_0 \geq 0 \end{array} \right. \quad (2.2)$$

exists on $[t_0, \infty)$ and satisfies

$$\text{and} \quad Q(w_0) < a(\lambda) \text{ implies } Q(w(t, t_0, w_0)) < b(A), \quad t \geq t_0 \quad (2.3)$$

$$\lim_{t \rightarrow \infty} w(t; t_0, w_0) = 0. \quad (2.4)$$

Then, the control system (2.1) is practically asymptotically stable and the inequality

$$\left\{ \begin{array}{l} \int_{t_0}^{\infty} g(s, V(s, x^0(s)), x^0(s), u^0(s)) ds + \sum_{k=1}^{\infty} \psi_k(V(t_k, x^0(t_k))) \\ = \min_u \int_{t_0}^{\infty} g(s, V(s, x(s)), x(s), u(s)) ds + \sum_{k=1}^{\infty} \psi_k(V(t_k, x(t_k))) \\ \leq V(t_0, x_0) \text{ holds.} \end{array} \right. \quad (2.5)$$

i.e. $u^0 \in \Omega$ assures optimal stabilization.

Proof: To prove this theorem, we have to show two things:

- (1) the control $u^0(t) \in \Omega$ assures practical asymptotic stability,
- (2) the relation (2.5) holds.

Let $x^0(t) = x(t; t_0, x_0, u^0)$ be the solution of (2.1) corresponding to the control $u^0(t) \in \Omega$. Then, setting $m(t) = V(t, x^0(t))$, $w_0 = V(t_0, x_0)$ and using assumptions (i)-(v), (vii) and (viii), we can prove that the system (2.1) is practically stable following the standard arguments of [1, 5, 7]. Then, we also have

$$V(t, x^0(t)) \leq w(t; t_0, w_0), \quad t \geq t_0. \quad (2.6)$$

Consequently, (2.4) implies that $\lim_{t \rightarrow \infty} x^0(t) = 0$, which proves practical asymptotic stability.

Now, to prove (2.5), let us suppose that another control $u^*(t) \in \Omega$ also assures practical asymptotic stability of (2.1). Then, the corresponding solution $x^*(t)$ also satisfies $\|x^*(t)\| < A$, $t \geq t_0$, provided $\|x_0\| < \lambda$, and $\lim_{t \rightarrow \infty} x^*(t) = 0$. This implies that

$$\lim_{t \rightarrow \infty} V(t, x^*(t)) = 0 \quad (2.7)$$

and we also have from (2.6)

$$\lim_{t \rightarrow \infty} V(t, x^0(t)) = 0. \quad (2.8)$$

Then, by (iv), we get

$$\int_{t_0}^{\infty} g(s, V(s, x^0(s)), x^0(s), u^0(s)) ds + \sum_{k=1}^{\infty} \psi_k(V(t_k, x^0(t_k))) \leq V(t_0, x_0). \quad (2.9)$$

But by (2.7) and (vi), we get

$$\int_{t_0}^{\infty} g(s, V(s, x^*(s)), x^*(s), u^*(s)) ds + \sum_{k=1}^{\infty} \psi_k(V(t_k, x^*(t_k))) \geq V(t_0, x_0). \quad (2.10)$$

The inequalities (2.9) and (2.10) prove the desired relation (2.5) and the proof is complete.

Q.E.D.

The following simple example illustrates this result.

Example 2.1: Consider the following impulsive control system

$$\left\{ \begin{array}{ll} x' = F(t, x) + R(t, x)u & t \neq t_k, \\ x(t_k^+) = b_k x_k & k = 1, 2, \dots, \\ x(t_0) = x_0 & \end{array} \right. \quad (2.11)$$

where $F \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $R(t, x)$ is an $n \times m$ matrix and u is a control.

We shall base the solution of the problem on the consideration of the function $V(t, x)$ given by

$$V(t, x) = \sum_{i=1}^N a_i V_i(t, x), \quad a_i = \text{const} > 0$$

where $V_i(t, v)$ are the components of Lyapunov's vector function.

Suppose we have

$$V_t(t, x) + V_x^T(t, x)F(t, x) \equiv p(t, x) \leq \lambda'(t)V(t, x) \quad (2.12)$$

where $\lambda'(t) \geq 0$, $t \geq t_0$ and $\lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$.

Define, for $t \neq t_k$,

$$B[V, t, x, u] = p(t, x) + V_x^T R(t, x)u + w(t, x) + u^T D u \quad (2.13)$$

where D is an $m \times m$ non-singular matrix.

We shall find the control $u^0 = u^0(t) \in \Omega$ from the condition of the minimum of B :

$$B[V, t, x, u] = 0 \text{ at } u = u_0 \quad (2.14)$$

$$\frac{\partial B}{\partial u}[V, t, x, u] = 0 \text{ at } u = u_0. \quad (2.15)$$

Thus we obtain

$$R^T(t, x)V_x(t, x) + 2Du^0 = 0 \quad (2.16)$$

and it then follows that

$$u^0(t) = -\frac{1}{2}D^{-1}R^T(t,x)V_x(t,x). \quad (2.17)$$

To discuss the problem of minimization of $\int_0^\infty g(s, V(s, x(s)), x(s), u(s, x(s)))ds$, we obtain from (2.13), (2.14) and (2.16) the relation

$$w(t, v) + p(t, x) - u^{0T}Du^0 = 0$$

which yields

$$w(t, x) = -p(t, x) + u^{0T}Du^0.$$

Thus

$$\begin{aligned} -g(t, v, x, u) &= p(t, x) - u^{0T}Du^0 - u^T Du \\ &\leq \lambda'(t)V - u^{0T}Bu^0 - u^T Du, \quad t \neq t_k. \end{aligned} \quad (2.18)$$

For $t = t_k$, we want

$$V(t_k^+, x(t_k^+)) \leq d_k V(t_k, x(t_k)) \quad (2.19)$$

where $\alpha d_k \leq e^{\lambda(t_k) - \lambda(t_{k+1})}$, $\alpha > 1$. Thus, $u^0 = -\frac{1}{2}D^{-1}R^T(t, x)V_x(t, x)$ assures optimal stabilization of (2.1).

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