EXISTENCE THEOREM FOR NONCONVEX
STOCHASTIC INCLUSIONS

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ABSTRACT

An existence theorem for stochastic inclusions \( x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau 
+ \int_s^t G_\tau(x_\tau)d\omega_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) (d\tau,dz) \) with nonanticipative nonconvex-valued right-hand sides is proved.

Key words: Stochastic inclusions, existence solutions, solution set.

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1. Introduction

Existence theorem and weak compactness of the solution set to stochastic inclusion

\[ x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)d\omega_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau) (d\tau,dz), \]

denoted by \( SI(F,G,H) \), with predictable convex-valued right-hand sides have been considered in
the author's paper [4]. These results were obtained by fixed points methods. Applying
the successive approximation method we shall prove here an existence theorem for \( SI(F,G,H) \) with
nonanticipative nonconvex-valued multivalued processes \( F,G \) and \( H \). To begin with, we recall
the basic definitions dealing with set-valued stochastic integrals and stochastic inclusions
presented in [5].

Let a complete filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) be given, where a family \( (\mathcal{F}_t)_{t \geq 0} \)
of \( \sigma \)-algebras \( \mathcal{F}_t \subset \mathcal{F} \) is assumed to be increasing: \( \mathcal{F}_s \subset \mathcal{F}_t \) if \( s \leq t \). Let \( \mathbb{R}_+ = [0, \infty) \) and \( \mathcal{B}_+ \) be
the Borel \( \sigma \)-algebra on \( \mathbb{R}_+ \). We consider set-valued stochastic processes \( \mathcal{F}_t \geq 0 \), \( \mathcal{G}_t \geq 0 \) and
\( \mathcal{H}_t,z \geq 0, z \in \mathbb{R}^n \) taking on values in the space \( Comp(\mathbb{R}^n) \) of all nonempty compact subsets of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). They are assumed to be nonanticipative and such that

\[ \int_0^\infty \| \mathcal{F}_t \|^p dt < \infty, \quad p \geq 1, \quad \int_0^\infty \| \mathcal{G}_t \|^2 dt < \infty \quad \text{and} \quad \int_0^\infty \int_{\mathbb{R}^n} \| \mathcal{H}_{t,z} \|^2 dq(dz) < \infty, \quad \text{a.s.}, \]

where \( q \) is a measure on a Borel \( \sigma \)-algebra \( \mathcal{B}^n \) of \( \mathbb{R}^n \) and \( \| A \| = \sup \{ | a | : a \in A \}, \ A \in Comp(\mathbb{R}^n) \). The space
\( Comp(\mathbb{R}^n) \) is considered with the Hausdorff metric \( h \) defined in the usual way, i.e.,

\[ h(A,B) = \max \{ \overline{h}(A,B), \overline{h}(B,A) \}, \text{ for } A,B \in Comp(\mathbb{R}^N), \] \( \overline{h}(A,B) = \{ \text{dist}(a,B) : a \in A \} \) and
\( \overline{h}(B,A) = \{ \text{dist}(b,A) : b \in B \} \).
2. Basic Definitions and Notations

Throughout the paper, we shall assume that a filtered complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) satisfies the following usual hypotheses:

(i) \(\mathcal{F}_0\) contains all the \(P\)-null sets of \(\mathcal{F}\) and

(ii) \(\mathcal{F}_t = \bigcup_{u > t} \mathcal{F}_u\) for all \(t, 0 \leq t < \infty\); that is, the filtration \((\mathcal{F}_t)_{t \geq 0}\) is right continuous.

As usual, we shall consider a set \(\mathbb{R}_+ \times \Omega\) as a measurable space with the product \(\sigma\)-algebra \(\mathcal{B}_+ \otimes \mathcal{F}\).

An \(n\)-dimensional stochastic process \(z\) is understood as a function \(x: \mathbb{R}_+ \times \Omega \to \mathbb{R}^n\) with measurable sections \(x_t\) for \(t \geq 0\), and it is denoted by \((x_t)_{t \geq 0}\). It is measurable if \(x\) is \(\mathcal{B}_+ \otimes \mathcal{F}\)-measurable. The process \((x_t)_{t \geq 0}\) is \(\mathcal{F}_t\)-adapted or adapted if \(x_t\) is \(\mathcal{F}_t\)-measurable for \(t \geq 0\). Every measurable and adapted process is called nonanticipative. In what follows, the Banach spaces \(L^p(\Omega, \mathcal{F}, P, \mathbb{R}^n)\) and \(L^p(\mathbb{R}_+ \times \Omega, \mathcal{F}, \mathbb{R}^n)\) with the usual norm \(\| \cdot \|\) are denoted by \(L^p(\mathcal{F}_t)\) and \(L^p(\mathcal{F})\), respectively.

Let \(\mathcal{M}^2(\mathcal{F}_t)\) denote the family of all (equivalence classes of) \(n\)-dimensional nonanticipative processes \((f_t)_{t \geq 0}\) such that \(\int_0^\infty |f_t|^2 dt < \infty\), a.s. We shall also consider a subspace \(L^2_n\) of \(\mathcal{M}^2(\mathcal{F}_t)\) defined by \(L^2_n = \{(f_t)_{t \geq 0} \in \mathcal{M}^2(\mathcal{F}_t): E \int_0^\infty |f_t|^2 dt < \infty\}\) with the norm \(\| \cdot \|_{L^2_n}\) defined in the usual way. The Banach spaces \(L^p(\mathbb{R}_+ \times \mathcal{F}_t, dt, \mathbb{R}^n)\), \(p \geq 1\) and \(L^2(\mathcal{B}_+ \otimes \mathcal{F}_t, dt, \mathcal{B}_+ \otimes \mathcal{F}_t)\), with the usual norms \(\| \cdot \|_p\) and \(\| \cdot \|_2\) will be denoted by \(M^p(\mathcal{F}_t)\) and \(M^2(\mathcal{F}_t)\), respectively. Finally, by \(M^p(\mathcal{F}_t)\) we denote a space of all (equivalence classes of) \(n\)-dimensional \(\mathcal{F}_t\)-measurable mappings.

Throughout the paper, by \((w_t)_{t \geq 0}\) we mean a one-dimensional \(\mathcal{F}_t\)-Brownian motion starting at 0, i.e., such that \(P(w_0 = 0) = 1\). By \(\nu(t, A)\) we denote a \(\mathcal{F}_t\)-Poisson measure (see [1]) on \(\mathbb{R}_+ \times \mathbb{R}^n\) and then define a \(\mathcal{F}_t\)-centered Poisson measure \(\tilde{\nu}(t, A)\), \(t \geq 0\), \(A \in \mathcal{B}^n\), by taking \(\tilde{\nu}(t, A) = \nu(t, A) - tq(A), t \geq 0, A \in \mathcal{B}^n\), where \(q\) is a measure on \(\mathbb{R}^n\) such that \(E\nu(t,B) = tq(B)\) and \(q(B) < \infty\) for \(B \in \mathcal{B}^n\).

By \(\mathcal{M}^2(\mathcal{F}_t, q)\), we shall denote the family of all (equivalence classes of) \(\mathcal{B}_+ \otimes \mathcal{F} \otimes \mathbb{R}^n\)-measurable and \(\mathcal{F}_t\)-adapted functions \(h: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n\) such that \(\int_0^\infty \int \|h_{t,x}\|^2 dt dz < \infty\) a.s. Recall that a function \(h: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n\) is said to be \(\mathcal{F}_t\)-adapted or adapted if for every \(x \in \mathbb{R}^n\) and \(t \geq 0\), \(h(t, \cdot, x)\) is \(\mathcal{F}_t\)-measurable. Elements of \(\mathcal{M}^2(\mathcal{F}_t, q)\) will be denoted by \(h = (h_{t,x})_{t \geq 0, x \in \mathbb{R}^n}\). Finally, let \(\mathcal{W}^2_n = \{h \in \mathcal{M}^2(\mathcal{F}_t, q): \|h\|_{\mathcal{W}^2_n}^2 < \infty\}\) where \(\|h\|_{\mathcal{W}^2_n}^2 = E \int_0^\infty \int \|h_{t,x}\|^2 dt dz\).

Given \(g \in \mathcal{M}^2(\mathcal{F}_t)\) and \(h \in \mathcal{M}^2(\mathcal{F}_t, q)\), by \((\int_0^t g_r dw_r)_{t \geq 0}\) and \((\int_0^t h_r \tilde{\nu}(dr, dz))_{t \geq 0}\), we denote their stochastic integrals with respect to an \(\mathcal{F}_t\)-Brownian motion \((w_t)_{t \geq 0}\) and an \(\mathcal{F}_t\)-centered Poisson measure \(\tilde{\nu}(t, A), t \geq 0, A \in \mathcal{B}^n\), respectively. These integrals, understood as \(n\)-dimensional stochastic processes, have quite similar properties (see [1]).

Let us denote by \(D\) the family of all \(n\)-dimensional \(\mathcal{F}_t\)-adapted cádlág (see [6]) processes \((x_t)_{t \geq 0}\) such that \(E\sup_{t \geq 0} |x_t|^2 < \infty\). The space \(D\) is considered as a normed space with the
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Given \(0 < c < \infty\) and \((z_t)_{t > 0} \in D\), let \(z_\alpha^{\beta} = (z_\alpha^{\beta})_{t \geq 0} \geq 0\) be such that \(z_\alpha^{\beta} = z_\alpha\) and \(z_\alpha^{\beta} = z_\beta\) for \(0 \leq t \leq \alpha\) and \(t \geq \beta\), respectively, and \(z_\alpha^{\beta} = z_t\) for \(\alpha \leq t \leq \beta\). It is clear that \(D_\alpha^{\beta} = \{z_\alpha^{\beta} : z_\alpha \in D\}\) is a linear subspace of \(D\), closed in the \(\| \cdot \|_\ell^\infty\) norm topology. Then, \((D_\alpha^{\beta}, \| \cdot \|_\ell^\infty)\) is also a Banach space.

Given a measure space \((\mathcal{X}, \mathcal{B}, \mu)\), a set-valued function \(\Psi : \mathcal{X} \to \mathcal{C}(\mathbb{R}^n)\) is said to be \(\mathcal{B}\)-measurable if \(\{z \in \mathcal{X} : \Psi(z) \subseteq C\} \in \mathcal{B}\) for every closed set \(C \subseteq \mathbb{R}^n\). For such a multifunction, we define subtrajectory integrals as a set \(\Phi(\Psi) = \{g \in L^p(\mathcal{X}, \mathcal{B}, \mu, \mathbb{R}^n) : g(x) \in \Psi(x) \text{ a.e.}\}\). It is clear that for nonemptiness of \(\Phi(\Psi)\) we must assume more then \(\mathcal{B}\)-measurability of \(\Psi\). In what follows, we shall assume that \(\mathcal{B}\)-measurable set-valued function \(\Phi : \mathcal{X} \to \mathcal{C}(\mathbb{R}^n)\) is \(p\)-integrable bounded, \(p \geq 1\), i.e., that a real-valued mapping: \(\mathcal{X} \ni x \mapsto \|\Psi(x)\| \in \mathbb{R}_+\) belongs to \(L^p(\mathcal{X}, \mathcal{B}, \mu, \mathbb{R}_+)\). It can be verified (see [2], Th. 3.2) that a \(\mathcal{B}\)-measurable set-valued mapping \(\Phi : \mathcal{X} \to \mathcal{C}(\mathbb{R}^n)\) is \(p\)-integrable bounded, \(p \geq 1\), if and only if \(\Phi(\Psi)\) is nonempty and bounded in \(L^p(\mathcal{X}, \mathcal{B}, \mu, \mathbb{R}^n)\). Finally, it is easy to see that \(\Phi(\Psi)\) is decomposable, i.e., such that \(1_{A_f} + 1_{A_f} \in \Phi(\Psi)\) for \(A \in \mathcal{B}\) and \(f_1, f_2 \in \Phi(\Psi)\).

We have the following general result dealing with the properties of subtrajectory integrals (see [2], [3]).

**Proposition 1.** Let \(\Phi : \mathcal{X} \to \mathcal{C}(\mathbb{R}^n)\) be \(\mathcal{B}\)-measurable and \(p\)-integrable bounded, \(p \geq 1\). Then, \(\Phi(\Psi)\) is a nonempty bounded and closed subset of \(L^p(\mathcal{X}, \mathcal{B}, \mu, \mathbb{R}^n)\). Moreover, if \(\Phi\) takes on convex values then \(\Phi(\Psi)\) is convex and weakly compact in \(L^p(\mathcal{X}, \mathcal{B}, \mu, \mathbb{R}^n)\).

Let \(F = (F_t)_{t \geq 0}\) be a set-valued stochastic process with values in \(\mathcal{C}(\mathbb{R}^n)\), i.e., a family of \(\mathcal{B}\)-measurable set-valued mappings \(F_t : \Omega \to \mathcal{C}(\mathbb{R}^n)\), \(t \geq 0\). We call \(F\) measurable if it is \(\mathcal{B}_+ \otimes \mathcal{F}\)-measurable. Similarly, \(F\) is said to be \(\mathcal{F}_t\)-adapted or adapted if \(F_t\) is \(\mathcal{F}_t\)-measurable for each \(t \geq 0\). A measurable and adapted set-valued stochastic process is called nonanticipative.

In what follows, we shall also consider \(\mathcal{B}_+ \otimes \mathcal{F} \otimes \mathbb{R}^n\)-measurable set-valued mappings \(\Phi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to \mathcal{C}(\mathbb{R}^n)\). They will be denoted as families \((\Phi_t, z)_{t \geq 0, z \in \mathbb{R}^n}\) and called measurable set-valued stochastic processes depending on a parameter \(z \in \mathbb{R}^n\). The process \(\Phi = (\Phi_t, z)_{t \geq 0, z \in \mathbb{R}^n}\) is said to be \(\mathcal{F}_t\)-adapted or adapted if \(\Phi_t, z\) is \(\mathcal{F}_t\)-measurable for each \(t \geq 0\) and \(z \in \mathbb{R}^n\). We call it nonanticipative if it is measurable and adapted.

Denote by \(\mathcal{M}_{\mathcal{S}_+}^Z(\mathcal{F}_t)\) and \(\mathcal{M}_{\mathcal{S}_+}^Z(\mathcal{F}_t, q)\) families of all nonanticipative set-valued processes \(\Phi = (\Phi_t)_{t \geq 0}\) and \(\Phi = (\Phi_t, z)_{t \geq 0, z \in \mathbb{R}^n}\), respectively, such that \(\int_0^\infty \|\Phi_t\|^2 dt < \infty\) and \(\int \int \|\Phi_{t, z}\|^2 dt dz < \infty\), a.s. Immediately, from Kuratowski and Ryll-Nardzewski measurable selection theorem (see [3]) it follows that for every \(F, \Phi \in \mathcal{M}_{\mathcal{S}_+}^Z(\mathcal{F}_t)\) and \(\Phi \in \mathcal{M}_{\mathcal{S}_+}^Z(\mathcal{F}_t, q)\) their subtrajectory integrals

\[
\Phi(F) = \{f \in \mathcal{M}_Z(\mathcal{F}_t) : f_t(\omega) \in F_t(\omega), dt \times P - \text{a.e.}\},
\]

\[
\Phi(\Phi) = \{g \in \mathcal{M}_Z(\mathcal{F}_t) : g_t(\omega) \in \Phi_t(\omega), dt \times P - \text{a.e.}\},
\]

\[
\Phi_q(\Phi) = \{h \in \mathcal{M}_Z(\mathcal{F}_t, q) : h_t(\omega, z) \in \Phi_t, z(\omega), dt \times P \times q - \text{a.e.}\}
\]

are nonempty. Indeed, let \(\Sigma = \{Z \in \mathcal{B}_+ \otimes \mathcal{F} : Z_t \in \mathcal{F}_t, \text{each } t \geq 0\}\), where \(Z_t\) denotes a section of \(Z\) determined by \(t \geq 0\). It is a \(\sigma\)-algebra on \(\mathbb{R}_+ \times \Omega\) and a function \(f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n\) (a multifunction \(F : \mathbb{R}_+ \times \Omega \to \mathcal{C}(\mathbb{R}^n)\)) is nonanticipative if and only if it is \(\Sigma\)-measurable. Therefore,
by Kuratowski and Ryll-Nardzewski measurable selection theorem every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$ such selector belongs to $\mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$. Similarly, define on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ a $\sigma$-algebra

$$\widetilde{\Sigma} = \{ Z \in \mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n: Z^u_t \in \mathcal{F}_t, \forall t \geq 0, u \in \mathbb{R}^n \},$$

where $Z^u_t = (Z^u)^t_t$ and $Z^u$ is a section of $Z$ determined by $u \in \mathbb{R}^n$.

Given the set-valued processes

$$F_t = (F_t)_{t \geq 0} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t), \quad \mathcal{G}_t = (\mathcal{G}_t)_{t \geq 0} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$$

and

$$\mathcal{B} = (\mathcal{B}_t, z)_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t, \mathcal{G})$$

by their stochastic integrals we mean families

$$(\int_0^t F_r \, d\tau)_t \geq 0, (\int_0^t \mathcal{G}_r \, dw_r)_t \geq 0, \text{ and } (\int_0^t \mathcal{B}_r, z \, (d\tau, dz))_t \geq 0$$

of subsets defined by

$$\int_0^t F_r \, d\tau = \{ \int_0^t f \, \tau: f \in \mathcal{F}(F) \},$$

$$\int_0^t \mathcal{G}_r \, dw_r = \{ \int_0^t g \, dw_r: g \in \mathcal{F}(\mathcal{G}) \}$$

and

$$\{ \int_0^t \mathcal{B}_r, z \, (d\tau, dz) = \{ \int_0^t h \, \tau, z \, (d\tau, dz): h \in \mathcal{F}(\mathcal{B}) \}. \quad \text{on } \mathbb{R}^n$$

Given $0 \leq \alpha < \beta < \infty$ we also define

$$\int_0^t F_r \, ds = \{ \int_0^t f \, ds: f \in \mathcal{F}(F) \},$$

$$\int_0^t \mathcal{G}_r \, dw_r = \{ \int_0^t g \, dw_r: g \in \mathcal{F}(\mathcal{G}) \}$$

and

$$\int_0^t \mathcal{B}_r, z \, (ds, dz) = \{ \int_0^t h \, \tau, z \, (ds, dz): h \in \mathcal{F}(\mathcal{B}) \}. \quad \text{on } \mathbb{R}^n$$

3. Stochastic Inclusions

Let $F = \{(F_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$ and $H = \{(H_t, z(x))_{t \geq 0, z \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$. Assume $F, G$ and $H$ are such that $(F_t(x))_{t \geq 0} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$, $(G_t(x))_{t \geq 0} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$ and $(H_t, z(x))_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t, \mathcal{G})$ each $x \in \mathbb{R}^n$.

By a stochastic inclusion, denoted by $SI(F, G, H)$, corresponding to given above $F, G$ and $H$ we mean a relation

$$x_t - x_s \in \int_s^t F_r \, d\tau + \int_s^t G_r \, dw_r + \int_s^t H_r, z(x_r) \, (d\tau, dz)$$

that is to be satisfied for every $0 \leq s < t < \infty$ by a stochastic process $x = (x_t)_{t \geq 0} \in D$ such that $F \circ x \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$, $G \circ x \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$ and $H \circ x \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t, \mathcal{G})$, where $F \circ x = (F_t(x_t))_{t \geq 0}$, $G \circ x = (G_t(x_t))_{t \geq 0}$ and $H \circ x = (H_t, z(x_t))_{t \geq 0, z \in \mathbb{R}^n}$. Every stochastic process $(x_t)_{t \geq 0} \in D$, satisfying conditions mentioned above is said to be a global solution to $SI(F, G, H)$. 

A stochastic process \((x_t)_{t\geq 0} \in D\) is a local solution to \(SI(F, G, H)\) on \([a, \beta]\) if and only if \(x^{\alpha, \beta}\) is a global solution to \(SI(F^{\alpha, \beta}, G^{\alpha, \beta}, H^{\alpha, \beta})\), where \(F^{\alpha, \beta} = \mathbb{I}_{[a, \beta]} F\), \(G^{\alpha, \beta} = \mathbb{I}_{[a, \beta]} G\) and \(H^{\alpha, \beta} = \mathbb{I}_{[a, \beta]} H\).

A stochastic process \((x_t)_{t\geq 0} \in D\) is called a global (local on \([a, \beta]\), resp.) solution to the initial value problem for stochastic inclusion \(SI(F, G, H)\) with an initial condition \(y \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)\) \((y \in \mathcal{F}_0, \mathbb{R}^n)\), resp.) if \((x_t)_{t\geq 0}\) is a global (local on \([a, \beta]\), resp.) solution to \(SI(F, G < h)\) and \(x_0 = y\) \((x_0 = y, \text{resp.})\). An initial-value problem for \(SI(F, G, H)\) mentioned above will be denoted by \(SI_y(F, G, H)\) \((\mathcal{A}^{\alpha, \beta}_y(F, G, H), \text{resp.})\). In what follows, we denote a set of all global (local on \([a, \beta]\) solutions to \(SI_y(F, G, H)\) by \(A_y(F, G, H)\) \((\mathcal{A}^{\alpha, \beta}_y(F, G, H), \text{resp.})\).

Suppose \(F, G, \text{ and } H\) satisfy the following conditions:

(A1) (i) \(F = \{(F_t(x))_{t \geq 0}; x \in \mathbb{R}^n\}\), \(G = \{(G_t(x))_{t \geq 0}; x \in \mathbb{R}^n\}\) and \(H = \{(H_t, z(x))_{t \geq 0}, z \in \mathbb{R}^n; x \in \mathbb{R}^n\}\) are such that mappings \(\mathbb{R}^+ \times \mathbb{R}^n \ni (t, w, x)\) \(\rightarrow F_t(x)(w) \in \text{Comp}(\mathbb{R}^n)\), \(\mathbb{R}^+ \times \mathbb{R}^n \ni (t, w, x)\) \(\rightarrow G_t(x)(w) \in \text{Comp}(\mathbb{R}^n)\) and \(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, w, x, z, x)\) \(\rightarrow H_t, z(x)(w) \in \text{Comp}(\mathbb{R}^n)\) are \(\mathcal{E} \otimes \mathcal{B}^n\) and \(\Sigma \otimes \mathcal{B}^n\)-measurable, respectively, where \(\Sigma \) and \(\mathcal{B}^n\) are \(\sigma\)-algebras on \(\mathbb{R}^+ \times \mathbb{R}^n\) and \(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n\) defined above,

(ii) \((F_t(x))_{t \geq 0}, (G_t(x))_{t \geq 0}\) and \((H_t, z(x))_{t \geq 0}, z \in \mathbb{R}^n; x \in \mathbb{R}^n\) are square integrable bounded for fixed \(x \in \mathbb{R}^n\).

Corollary 1: For every \((x_t)_{t \geq 0} \in D\) and \(F, G, H\) satisfying (A1) one has \(F \circ x, G \circ x \in \mathcal{M}_{s-\nu}^2(\mathcal{F}_t)\) and \(H \circ x \in \mathcal{M}_{s-\nu}^2(\mathcal{F}_t, q)\).

Now, define a linear mapping \(\Phi \) on \(\mathcal{M}^2(\mathcal{F}_t) \times \mathcal{M}^2(\mathcal{F}_t) \times \mathcal{M}^2(\mathcal{F}_t, q)\) by taking \(\Phi(f, g, h) = f^t \chi_t dr + g^t \tilde{\nu}(dr, dz)\) to each \((f, g, h) \in \mathcal{M}^2(\mathcal{F}_t) \times \mathcal{M}^2(\mathcal{F}_t) \times \mathcal{M}^2(\mathcal{F}_t, q)\).

It is clear that \(\Phi\) maps \(\mathcal{L}^2_n \times \mathcal{L}^2_n \times \mathcal{W}^2_n\) into \(D\).

In what follows, we shall deal with \(F = \{(F_t(x))_{t \geq 0}; x \in \mathbb{R}^n\}, G = \{(G_t(x))_{t \geq 0}; x \in \mathbb{R}^n\}\) and \(H = \{(H_t, z(x))_{t \geq 0}; x \in \mathbb{R}^n\}\) satisfying conditions (A1) and any one of the following conditions.

(A2) There are \(k, \ell \in \mathcal{L}^1_1\) and \(m \in \mathcal{W}^2_1\) such that \(\| \int_0^\infty h_t(F \circ x)_t dt \|_{\mathcal{L}^1_1} \leq E \int_0^\infty k |x_t - y_t| dt\), \(\| h_t(G \circ x, G \circ y) \|_{\mathcal{L}^1_1} \leq E \int_0^\infty \ell |x_t - y_t| dt\) and \(\| h_t(H \circ x, H \circ y) \|_{\mathcal{W}^2_1} \leq E \int_0^\infty m |x_t - y_t| d(q(t, dz))\) for all \(x, y \in D\).

(A3) There are \(k, \ell \in \mathcal{L}^2(\mathcal{F}_+^+)\) and \(m \in \mathcal{L}^2(\mathcal{F}_+^+ \times \mathbb{R}^n)\) such that \(h_t(F_t(x))_t(\omega) \leq k(t) |x_t - x_t|_1\), \(h_t(G_t(x))_t(\omega) \leq \ell(t) |x_t - x_t|_2\) and \(h_t(H_t, z(x))_t(\omega) \leq m(t, x) |x_1 - x_2|_1 \) \(a.e., \) each \(t \geq 0\) and \(x_1, x_2 \in \mathbb{R}^n\).

Lemma 1: Let \(\varphi \in \mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)\). Suppose \(F, G, \text{ and } H\) satisfy (A1) and (A2) or (A3). Let \(x^n = \varphi + \Phi(f^{n-1}, g^{n-1}, h^{n-1})\), each \(n = 1, 2, \ldots, \) with \((f^0, g^0, h^0) \in \mathcal{F}(F \circ 0) \times \mathcal{F}(G \circ 0) \times \mathcal{F}(H \circ 0)\) and \(\| f^{n-1}_t(\omega) - f^n_t(\omega) \| = \text{dist}(f^{n-1}_t(\omega), (F \circ x^n)_t(\omega))\) and \(\| g^{n-1}_t(\omega) - g^n_t(\omega) \| = \text{dist}(g^{n-1}_t(\omega), (G \circ x^n)_t(\omega))\) and \(\| h^{n-1}_t, z_t(\omega) - h^n_t, z_t(\omega) \| = \text{dist}(h^{n-1}_t, z_t(\omega), (H \circ x^n)_t, z_t(\omega))\) on \(\mathbb{R}^+ \times \mathbb{R}^n\) and \(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n\), respectively. If \(L = \| \int_0^\infty k |dt| \|_{\mathcal{L}^1_1} + 2 \| \int_0^\infty \ell_1 |dt| \|_{\mathcal{L}^1_1} + 2 \| \int_0^\infty \mathbb{R}^n m_t, x_d q(t, dz) \|_{\mathcal{L}^1_1} < 1\) or \(L' = \| k |1 + 2 |\ell|_2 + 2 |m|_2 < 1\), respectively then \((x^n)_{n=1}^\infty\) is a Cauchy sequence of \((D, \| \cdot \|_{\ell})\).
Proof: Let \( (x^n)_{n=1}^\infty \) be such as above. By \((A_2)\) it follows
\[
E \left[ \sup_{t \geq 0} \left| \int_0^t (f^n_\tau - f^{n-1}_\tau) d\tau \right| \right]^2 \leq E \left( \int_0^\infty |f^n_\tau - f^{n-1}_\tau| d\tau \right)^2
\]
\[
\leq E \left( \int_0^\infty \tilde{h}((F \circ x^n)_\tau,(F \circ x^{n-1})_\tau) d\tau \right)^2 \leq \left( E \int_0^\infty k_\tau |x^n_\tau - x^{n-1}_\tau| d\tau \right)^2
\]
\[
\leq E \left( \sup_{t \geq 0} |x^n_t - x^{n-1}_t| \cdot \int_0^\infty k_\tau d\tau \right)^2 \leq E \left( \int_0^\infty k_\tau d\tau \right)^2 \cdot \|x^n - x^{n-1}\|_\ell^2.
\]

Similarly, by Doob's inequality, we obtain
\[
E \left[ \sup_{t \geq 0} \left| \int_0^t (g^n_\tau - g^{n-1}_\tau) d\omega_\tau \right| \right]^2 \leq 4E \int_0^\infty |g^n_\tau - g^{n-1}_\tau|^2 d\tau
\]
\[
\leq 4E \int_0^\infty \tilde{h}((G \circ x^n)_\tau,(G \circ x^{n-1})_\tau)^2 d\tau \leq 4 \left( E \int_0^\infty \ell_\tau |x^n_\tau - x^{n-1}_\tau| d\tau \right)^2
\]
\[
\leq 4 \left( \sup_{t \geq 0} |x^n_t - x^{n-1}_t| \cdot \int_0^\infty \ell_\tau d\tau \right)^2 \leq 4E \left( \int_0^\infty \ell_\tau d\tau \right)^2 \cdot \|x^n - x^{n-1}\|_\ell^2.
\]

Quite similarly we also get
\[
E \left[ \sup_{t \geq 0} \left| \int_0^t \int_{\mathbb{R}^n} (h^n_\tau - h^{n-1}_\tau) d\mu(\tau, dz) \right| \right]^2 \leq 4E \left( \int_0^\infty \int_{\mathbb{R}^n} m_\tau dz d\tau \right)^2 \cdot \|x^n - x^{n-1}\|_\ell^2.
\]

Therefore, \( \|x^{n+1} - x^n\|_\ell \leq L^n \|x^1\|_\ell \), where \( L \) is such as above. This implies that
\[
\|x^m - x^n\|_\ell \leq \frac{L^n \cdot \|x^1\|_\ell}{1 - L},
\]
each \( m > n \geq 1 \). Using conditions \((A_3)\) instead of \((A_2)\) we also get
\[
\|x^m - x^n\|_\ell \leq \frac{(L')^n \cdot \|x^1\|_\ell}{1 - L'},
\]
for \( m > n \geq 1 \). Therefore, \( \|x^m - x^n\|_\ell \to 0 \) as \( n \to \infty \). \( \square \)
Lemma 2: Let $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. Suppose $F, G$ and $H$ satisfy (A$_1$) and (A$_3$). IF $L = |k|_1 + 2 |\ell|_2 + 2 \|m\|_2 < 1$, then $\Lambda_\varphi(F, G, H) \neq \emptyset$.

Proof: Let $(x^n)^\infty_{n=1}$ be such as in Lemma 1 and let $x = \lim_{n \to \infty} x^n$. The existence of such a sequence follows immediately from the measurable selection theorem given in [3] (see Th. II, 3.13). We shall now show that $(f^n)^\infty_{n=1}$, $(g^n)^\infty_{n=1}$ and $(h^n)^\infty_{n=1}$ are Cauchy sequences of $L^n_2$ and $W^n_2$, respectively. Indeed, one obtains

$$\|f^n - f^m\|_{L^n_2} = \sum_{j=n+1}^m \|f^j - f^{j-1}\|_{L^n_2}^{1/2} \leq \sum_{j=n+1}^m \left[ E \int_0^\infty h^2((F \circ x^j)_r, (F \circ x^{j-1})_r)d\tau \right]^{1/2}$$

Therefore, $(f^n)^\infty_{n=1}$ is a Cauchy sequence of $L^n_2$. Quite similarly, it also follows that $(g^n)^\infty_{n=1}$ and $(h^n)^\infty_{n=1}$ are Cauchy sequences of $L^n_2$ and $W^n_2$, respectively. Let $f, g \in L^n_2$ and $h \in W^n_2$ be such that $\|f^n - f\|_{L^n_2} \to 0$, $\|g^n - g\|_{L^n_2} \to 0$ and $\|h^n - h\|_{W^n_2} \to 0$ as $n \to \infty$. One gets $\|x^n - \varphi - \Phi(f, g, h)\|_\ell \to 0$ as $n \to \infty$. Therefore, $x = \varphi + \Phi(f, g, h)$. To prove that $x_t - x_s \in \int_s^t (F \circ x)_r d\tau + \int_s^t (G \circ x)_r dw_r + \int_s^t \int (H \circ x)_{t, \omega} \tilde{\nu} (d\tau, dz)$ for every $0 \leq s < t < \infty$ it suffices only to verify that $(f, g, h) \in \mathcal{F}(F \circ x) \times \mathcal{F}(G \circ x) \times \mathcal{F}(H \circ x)$. For this aim, denote by $\text{Dist}(a, B)$ and $\bar{H}$ the distance of $a \in L^n_2$ to a nonempty set $B \subset L^n_2$ and the Hausdorff subdistance, respectively induced by the norm of $L^n_2$. Now let $v$ be a fixed element of $\mathcal{F}(F \circ x^n)$. Select $u \in \mathcal{F}(F \circ x)$ such that $|v_r(\omega) - u_r(\omega)| = \text{dist}(v_r(\omega), (F \circ x)_r(\omega))$ for $(\tau, \omega) \in \mathbb{R}_+ \times \Omega$. Then

$$\text{Dist}(v, \mathcal{F}(F \circ x)) \leq \|v - u\|_{L^n_2}$$

$$\leq \left( E \int_0^\infty h^2((F \circ x^n)_r(\omega), (F \circ x)_r(\omega))d\tau \right)^{1/2} \leq |k|_2 \|x^n - x\|_\ell,$$

which implies $\bar{H}(\mathcal{F}(F \circ x^n), \mathcal{F}(F \circ x)) \leq |k|_2 \|x^n - x\|_\ell$, each $n = 1, 2, \ldots$. Thus $\bar{H}(\mathcal{F}(F \circ x^n), \mathcal{F}(F \circ x)) \to 0$ as $n \to \infty$. In a similar way we also get $\bar{H}(\mathcal{F}(G \circ x^n), \mathcal{F}(G \circ x)) \to 0$ and $\bar{H}(\mathcal{F}(H \circ x^n), \mathcal{F}(H \circ x)) \to 0$ as $n \to \infty$. Now we get

$$\text{Dist}(f, \mathcal{F}(F \circ x)) \leq \|f - f_n\|_{L^n_2} + \text{Dist}(f_n, \mathcal{F}(F \circ x^{n-1}))$$

$$+ \bar{H}(\mathcal{F}(F \circ x^{n-1}), \mathcal{F}(F \circ x))$$

for $n = 1, 2, \ldots$, which implies that $\text{Dist}(f, \mathcal{F}(F \circ x)) = 0$. But, $\mathcal{F}(F \circ x)$ is a nonempty closed subset of $L^n_2$. Therefore, $f \in \mathcal{F}(F \circ x)$. In a similar way we can also verify that $g \in \mathcal{F}(G \circ x)$ and $h \in \mathcal{F}(H \circ x)$. \qed
Lemma 3: Let $0 < \alpha < \beta < \infty$ and $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. Suppose $F$, $G$, and $H$ satisfy ($A_1$) and ($A_3$). If $\Lambda^\alpha_{[\alpha, \beta]} = 1_{[\alpha, \beta]}$, then $\Lambda^\beta_{[\alpha, \beta]}(F, G, H) = \emptyset$.

Proof: The proof follows immediately from Lemma 2 applied to $F^{\alpha \beta} = 1_{[\alpha, \beta]} F$, $G^{\alpha \beta} = 1_{[\alpha, \beta]} G$ and $H^{\alpha \beta} = 1_{[\alpha, \beta]} H$.

Lemma 4: Let $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$ and let $(\tau_n)_{n=1}^\infty$ be a sequence of positive numbers increasing to $+\infty$. Suppose $F, G$ and $H$ satisfy ($A_1$) and ($A_3$). If $x^1 \in \Lambda^\tau_{[\tau_n]}(F, G, H)$ and $x^{n+1} = \Lambda^\tau_{[\tau_n]}(F, G, H)$ for $n = 1, 2, \ldots$, then $x = \sum_{n=1}^\infty \Lambda^\tau_{[\tau_{n-1}, \tau_n]}(x^n)$ belongs to $\Lambda^\tau_{[\tau, \infty]}(F, G, H)$, where $\tau = 0$.

Proof: It is clear that $x_0 = \varphi$ because $x_0 = x^1_0 = \varphi$. Let $0 \leq s < t < \infty$ be fixed and suppose $s \in [\tau_{k-1}, \tau_k)$, and $t \in [\tau_{m-1}, \tau_m)$, for $1 \leq k < m$. One obtains

$$x_t - x_s = (x_t^m - x_{t-1}^m) + \sum_{j=k+1}^m (x_{t-1}^{j-1} - x_{t-1}^j) + (x_{t-1}^k - x_{t-1}^k).$$

Let $(f^j, g^j, h^j) \in S(F \circ x^j) \times S(G \circ x^j) \times S_q(H \circ x^j)$, each $j = k, k+1, \ldots, m$ be such that

$$x_t^m - x_{t-1}^m = \int_{\tau_{m-1}}^t f^m d\tau + \int_{\tau_m}^t g^m dw + \int_{\tau_{m-1}}^t h^m d\nu(d\tau, dz),$$

$$x_{t-1}^j - x_{t-2}^{j-1} = \int_{\tau_{j-2}}^t f^j_j d\tau + \int_{\tau_{j-1}}^t g^j dw + \int_{\tau_{j-2}}^t h^j d\nu(dt, dz),$$

each $j = k+1, \ldots, m-1$, and

$$x_k^k - x_s^k = \int_{\tau_{k-1}}^t f^k d\tau + \int_{\tau_k}^t g^k dw + \int_{\tau_{k-1}}^t h^k d\nu(d\tau, dz).$$

Let $f = 1_{[\tau_k, \infty)} f^k + \sum_{j=k+1}^m 1_{[\tau_{j-1}, \tau_j)} f^j + 1_{[\tau_m, \infty)} f^m$, $g = 1_{[\tau_k, \infty)} g^k + \sum_{j=k}^m 1_{[\tau_{j-1}, \tau_j)} g^j + 1_{[\tau_m, \infty)} g^m$, and $h = 1_{[\tau_k, \infty)} h^k + \sum_{j=k+1}^m 1_{[\tau_{j-1}, \tau_j)} h^j + 1_{[\tau_m, \infty)} h^m$. It is clear that $(f, g, h) \in S(F \circ x) \times S(G \circ x) \times S_q(H \circ x)$ and $x_t - x_s = \int_{\tau_{k-1}}^t f d\tau + \int_{\tau_k}^t g dw + \int_{\tau_{k-1}}^t h d\nu(d\tau, dz)$. Therefore

$$x_t - x_s = \int_{\tau_{k-1}}^t (F \circ x) d\tau + \int_{\tau_k}^t (G \circ x) dw + \int_{\tau_{k-1}}^t (H \circ x) d\nu(d\tau, dz).$$

We can prove now the main result of this paper.

Theorem 5: Let $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. Suppose $F$, $G$ and $H$ satisfy ($A_1$) and ($A_3$). Then $\Lambda^\tau_{[\tau, \infty]}(F, G, H) \neq \emptyset$.

Proof: Let $(\tau_n)_{n=1}^\infty$ be a sequence of positive numbers increasing to $\infty$. Select a positive number $\sigma$ such that $L_{k_\sigma} < 1$ for $k = 0, 1, \ldots, \infty$, where $L_{k_\sigma}$ is such as in Lemma 3. Suppose a positive integer $n_1$ is such that $n_\sigma < \tau_1 \leq (n_1 + 1) \sigma$. By virtue of Lemma 3, there
is $z^1 \in \Lambda_{\phi}^{0,\sigma}(F,G,H)$. By the same argument, there is $z^2 \in \Lambda_{\tau}^{1,2\sigma}(F,G,H)$. Continuing the above procedure we can finally find a $z^{n_1+1} \in \Lambda_{\tau}^{n_1,\sigma,\tau_1}(F,G,H)$. Put

$$
z^1 = \sum_{k=0}^{n_1-1} [k,(k+1)\sigma]^z_{k+1} + ([n_1,\tau_1]z_{n_1+1} + ([\tau_1,\infty) z_\tau_{n_1+1}.
$$

Similarly as in the proof of Lemma 4, we can easily verify that $x^1 \in \Lambda_{\phi}^{0,\tau}(F,G,H)$. Repeating the above procedure to the interval $[\tau_1,\tau_2]$, we can find $x^2 \in \Lambda_{\tau_1}^{r_1,\tau_2}(F,G,H)$.

Continuing this process, we can define a sequence $(x^n)$ of $D$ satisfying conditions of Lemma 4. Therefore $\Lambda_{\phi}(F,G,H) \neq \emptyset$.

REFERENCES


