EXISTENCE OF SOLUTIONS FOR SECOND-ORDER EVOLUTION INCLUSIONS

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ABSTRACT

In this paper we examine second-order nonlinear evolution inclusions and prove two existence theorems; one with a convex-valued orientor field and the other with a nonconvex-valued field. An example of a hyperbolic partial differential inclusion is also presented.

Key words: Evolution Triple, Monotone Operator, Hemicontinuous Operator, Symmetric Operator, Fixed Point, Sobolev Space, Program, Average Turnpike Property, Separation Theorem.

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1. Introduction

In this paper we study the existence of solutions for second order nonlinear evolution inclusions. Our work here complements the existence results of [7], where we considered first order nonlinear evolution inclusions. We present two existence results. One in which the multivalued term (orientor field) is convex valued and the other with a nonconvex valued orientor field. At the end of the paper, we work in detail an example of a hyperbolic partial differential inclusion, illustrating the applicability of our result.

2. Mathematical Preliminaries

Let $T = [0,T]$ and $Y$ a separable Banach space. Throughout this paper we will be using the following notation: $P_{f}(Y) = \{ A \subseteq Y: \text{nonempty, closed (and convex)} \}$. A multifunction (set-valued function), $F: T \to P_{f}(Y)$ is said to be measurable if for all $x \in Y$, the $\mathbb{R}^+$-valued function $t \mapsto d(x,F(t)) = \inf \{ \| x - y \| : y \in F(t) \}$ is measurable. By $S_{\mathbb{R}^+}^{p}(1 \leq p \leq \infty)$, we will denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^{p}(Y)$; i.e. $S_{\mathbb{R}^+}^{p} = \{ f \in L^{p}(Y) : f(t) \in F(t) \text{ a.e.} \}$. It is easy to check using Aumann's selection theorem (see for example Wagner [8], theorem 5.10), that $S_{\mathbb{R}^+}^{p}$ is nonempty if and only if the $\mathbb{R}^+$-valued function $t \mapsto \inf \{ \| x \| : x \in F(t) \}$ belongs to $L^{p}_{\mathbb{R}^+}$.
Let $H$ be a separable Banach space and $X$ a dense subspace of $H$, carrying the structure of a separable, reflexive Banach space, which embeds in $H$ continuously. Identifying $H$ with its dual (pivot space), we have $X \rightarrow H \rightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" (or "Gelfand triple" or "spaces in normal position"). We will also assume that the above embeddings are compact, a condition that is very often satisfied in applications. By $\| \cdot \|$ (resp. $| \cdot |$, $\| \cdot \|_\infty$), we will denote the norm of $X$ (resp. of $H$, $X^*$). Also by $\langle \cdot , \cdot \rangle$ we will denote the duality brackets for the pair $(X, X^*)$ and by $(\cdot , \cdot)$ the inner product of $H$. The two are compatible in the sense that $\langle \cdot , \cdot \rangle X \times H = (\cdot , \cdot)$. To have a concrete example in mind let $Z \subseteq \mathbb{R}^N$ be a bounded domain, $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^* = W^{m,p}(Z)^* = W^{-m,q}(Z)$, $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. From the well-known Sobolev’s embedding theorem we know that $(X, H, X^*)$ is an evolution triple and furthermore all embeddings are compact. Let $W(T) = \{x \in L^2(X): \dot{x} \in L^2(X^*)\}$. The derivative in this definition is taken in the sense of vector valued distributions. Equipped with the norm $\| x \|_{W(T)} = \| x \|_{L^2(X)}^2 + \| \dot{x} \|_{L^2(X^*)}^2)^{1/2}$, $W(T)$ becomes a separable reflexive Banach space. Furthermore if $X$ is a Hilbert space, then $W(T)$ is too, with inner product $(x,y)_{W(T)} = (x,y)_{L^2(X)} + (\dot{x}, \dot{y})_{L^2(X^*)}, x, y \in W(T)$. Note that the elements in $W(T)$ are up to a Lebesgue-null subset of $T$, equal to an $X^*$-valued absolutely continuous function, and, therefore the derivative $\dot{x}(\cdot)$ is also the strong derivative of the function $x:T \rightarrow X^*$. Also, it is well-known that $W(T)$ embeds continuously into $C(T, H)$. Thus, every equivalence class in $W(T)$, has a unique representative in $C(T, H)$. Furthermore, since we have assumed that $X \rightarrow H$ compactly, we have that $W(T)\rightarrow L^2(H)$ compactly. Recently, Nagy [3] proved that if $X$ is a Hilbert space too, then $W(T)\rightarrow C(T, H)$ compactly. For further details on evolution triples and the abstract Sobolev space $W(T)$ we refer to the book of Zeidler [9] and, in particular, chapter 23.

Let $Z$ and $V$ be Hausdorff topological spaces. A multifunction $G:Z \rightarrow 2^V \{\emptyset\}$ is said to be upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for every open set $U \subseteq V$, the set $G^+(U) = \{z \in Z: G(z) \subseteq U\}$ (resp. the set $G^-(U) = \{z \in Z: G(z) \cap U \neq \emptyset\}$) is open in $Z$. Other equivalent definitions and further properties of such multifunctions can be found in the book of Klein-Thompson [2].

3. Existence Theorems

Let $T = [0,r]$ and $(X, H, X^*)$ be an evolution triple of spaces with all embeddings assumed to be compact. We will be considering the following second order nonlinear evolution inclusion:

\[
\begin{cases}
\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) \in F(t, x(t)) \text{ a.e.} \\
x(0) = x_0 \in X, \ \dot{x}(0) = x_1 \in H.
\end{cases}
\]

By a solution of $(\ast)$, we understand a function $x \in C(T, X)$ such that $\dot{x} \in W(T)$ and an $f \in S_F(\cdot, x(\cdot))$ such that $\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t)$ a.e. with $x(0) = x_0$ and $\dot{x}(0) = x_1$. Recall (see Section 2), that $W(T)\rightarrow C(T, H)$ and so the initial condition $\dot{x}(0) = x_1 \in H$ makes sense.

First we prove an existence theorem for $(\ast)$, for the case where the multivalued perturbation term $F(t, x)$ is convex-valued. To this end, we will need the following hypotheses on the data of $(\ast)$.

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**H(A):** $A: T \times X \to X^*$ is a map such that

1. $t \to A(t, v)$ is measurable,
2. $v \to A(t, v)$ is monotone, hemicontinuous (i.e. for all $v, v' \in X$, $(A(t, v) - A(t, v'), v - v') \geq 0$ (monotonicity) and for all vectors $v, y, x \in X$, the map $\lambda \to (A(t, v + \lambda y), x)$ is continuous on $[0, 1]$ (demicontinuity)),
3. $(A(t, v), v) \geq c \| v \|^2$ a.e. with $c > 0$,
4. $\| A(t, v) \| \leq a(t) + b \| v \|$ a.e. with $a(\cdot) \in L^2_+, b > 0$.

**H(B):** $B \in \mathcal{L}(X, X^*)$, $(Bx, y) = \langle x, By \rangle$ for all $x, y \in X$ (i.e. $B$ is symmetric) and $(Bx, x) \geq c' \| x \|^2$ with $c' > 0$.

**H(F):** $F: T \times H \to P_{fc}(H)$ is a multifunction such that

1. $t \to F(t, x)$ is measurable,
2. $x \to F(t, x)$ is u.s.c. from $H$ into $H$,
3. $| F(t, x) | = \sup \{ | v : v \in F(t, x) \} \leq a_1(t) + b_1 | x |$ a.e. with $a_1(\cdot) \in L^2_+, b_1 > 0$.

We will denote the solution set of $(\cdot)$ by $S(x_0, x_1) \subseteq C(T, X)$.

**Theorem 3.1:** If hypotheses $H(A), H(B), H(F)$ hold and $x_0 \in X, x_1 \in H$, then $S(x_0, x_1)$ is a nonempty and compact subset of $C(T, X)$.

**Proof:** First we will derive some *a priori* bounds for the solutions of $(\cdot)$. Let $x(\cdot) \in C(T, X)$ be such a solution. Then, by the definition, for some $f \in S^2_F(\cdot, x(\cdot))$, we have

$$\ddot{x}(t) + A(t, \dot{x}(t)) + B(x(t)) = f(t) \text{ a.e.}$$

It yields

$$\langle \ddot{x}(t), \dot{x}(t) \rangle + \langle A(t, \dot{x}(t)), \dot{x}(t) \rangle + \langle Bx(t), \dot{x}(t) \rangle = \langle f(t), \dot{x}(t) \rangle \text{ a.e.}$$

(1)

Since $\dot{x} \in W(T)$, from proposition 23.23 (iv), p. 422 of Zeidler [9], we know that

$$\langle \ddot{x}(t), \dot{x}(t) \rangle = \frac{1}{2} \frac{d}{dt} | \dot{x}(t) |^2.$$  

(2)

Also because of hypothesis $H(A)$ (3), we have that

$$\langle A(t, \dot{x}(t)), \dot{x}(t) \rangle \geq c \| \dot{x}(t) \|^2 \text{ a.e.}$$

(3)

Using the product rule and the symmetry hypothesis on $B$, we get

$$\frac{d}{dt} \langle Bx(t), x(t) \rangle = \langle B\dot{x}(t), x(t) \rangle + \langle Bx(t), \dot{x}(t) \rangle = 2\langle Bx(t), \dot{x}(t) \rangle.$$ 

(4)

Substituting (2), (3) and (4) into (1) above, we finally have

$$\frac{1}{2} \frac{d}{dt} | \dot{x}(t) |^2 + c \| \dot{x}(t) \|^2 + \frac{d}{dt} \langle Bx(t), x(t) \rangle \leq \langle f(t), \dot{x}(t) \rangle \text{ a.e.}$$

Integrating the above inequality, we get that

$$\frac{1}{2} | \dot{x}(t) |^2 - \frac{1}{2} | x_1 |^2 + c \int_0^t \| \dot{x}(s) \|^2 ds + \frac{1}{2} \langle Bx(t), x(t) \rangle - \frac{1}{2} \langle Bx_0, x_0 \rangle \leq \int_0^t \langle f(s), \dot{x}(s) \rangle ds$$

(5)
it yields

\[ |\dot{x}(t)|^2 + 2c \int_0^t \|\dot{x}(s)\|^2 ds + c' \|x(t)\|^2 \leq M + 2 \int_0^t (f(s), \dot{x}(s)) ds \]

(5)

where \( M = |x_0|^2 + \|B\|_F \|x_0\|^2 \).

Applying Cauchy’s inequality with \( \epsilon > 0 \), we get

\[
\int_0^t (f(s), \dot{x}(s)) ds \leq \int_0^t |f(s)| \cdot |\dot{x}(s)| ds
\]

\[
\leq \frac{\epsilon}{2} \int_0^t |f(s)|^2 ds + \frac{1}{2\epsilon} \int_0^t |\dot{x}(s)|^2 ds
\]

\[
\leq \frac{\epsilon}{2} \int_0^t (2a_1(s)^2 + 2b_1^2 |x(s)|^2) ds + \frac{1}{2\epsilon} \int_0^t |\dot{x}(s)|^2 ds
\]

\[
\leq \epsilon \int_0^t (a_1(s)^2 + b_1^2 |x(s)|^2) ds + \frac{1}{2\epsilon} \int_0^t \beta^2 \|\dot{x}(s)\|^2 ds
\]

where \( \beta > 0 \) is such that \( \|\cdot\| \leq \beta \|\cdot\| \). It exists since by hypothesis \( X \to H \) continuously. So, we have

\[
|\dot{x}(t)|^2 + 2c \int_0^t \|\dot{x}(s)\|^2 ds + c' \|x(t)\|^2
\]

\[
\leq M + \epsilon \|a_1\|_2^2 + \epsilon b_1^2 \int_0^t |x(s)|^2 ds + \frac{\beta^2}{2\epsilon} \int_0^t \|\dot{x}(s)\|^2 ds.
\]

Let \( \frac{\beta^2}{2\epsilon} = 2c \) implies that \( \epsilon = \frac{\beta^2}{4c} \). Then we have:

\[
|\dot{x}(t)|^2 + \frac{c'}{\beta^2} |x(t)|^2 \leq M + \frac{\beta^2}{4c} \|a_1\|_2^2 + \frac{\beta^2 b_1^2}{4c} \int_0^t |x(s)|^2 ds.
\]

(\*)

From (\*) by neglecting \( |\dot{x}(t)|^2 \) and using Gronwall’s inequality, we get

\[
|x(t)|^2 \leq \left( \frac{\beta^2}{c'} M + \frac{\beta^4}{4c c'} \|a_1\|_2^2 \right) e^{\frac{\beta^2 b_1^2}{4c c} t} = M_2^2, \ t \in T.
\]

(6)

Using (6) and neglecting \( \frac{c'}{\beta^2} |x(t)|^2 \) in (\*), we obtain
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\[ |\dot{x}(t)|^2 \leq M + \frac{\beta^2}{4c} a_1 \| \dot{x}(t) \|^2 + \frac{\beta^2}{4c} b_1^2 M^2 r = M_{1, t}^1, \quad t \in T. \]  
(7)

Coming back to (5) and using estimates (6) and (7) above, we get

\[ \|\dot{x}\|_{L^2(X)} \leq \frac{1}{2c}(M + 2 \| a_1 \|_{2}^2 + M^2 r + M^2 r) = M_{3}^2. \]  
(8)

Finally, from (5) and (8), we deduce that

\[ \|x(t)\|^2 \leq \frac{1}{c}(M + 2 \| a_1 \|_{2}^2 + 2b_1^2 M^2 r + M^2 r) = M_{4}^2. \]  
(9)

Finally, let \( p \in L^2(X) \) and denote by \( (\cdot, \cdot)_0 \) the duality brackets for the pair \( (L^2(X), L^2(X^*)) \). Also let \( \tilde{A}: L^2(X) \rightarrow L^2(X^*) \) be the Nemitsky operator corresponding to the map \( A(t, x) \); i.e. \( \tilde{A}(x)(t) = A(t, x(t)) \). Then we have:

\[ ((\tilde{A}(x), p))_0 \leq (\tilde{A}(x), p) + (Bx, p) + f, p \]

\[ \leq \| \tilde{A}(x) \|_{L^2(X^*)} + \| Bx \|_{L^2(X^*)} + \| f \|_{L^2(X^*)} \| p \|_{L^2(X)}, \]

\[ \leq \| a \|_{2} + bM_3 + \| B \|_{L^2(X^*}, r^{1/2} + \beta' \| a_1 \|_{2} + \beta' b_1 M_{2}^1 r^{1/2} \| p \|_{L^2(X)}, \]

where \( \beta' > 0 \) is such that \( \| \cdot \|_{*} \leq \beta' \| \cdot \|_{*} \). It exists since \( H \rightarrow X^* \) continuously. Since \( p \in L^2(X) \) was arbitrary, we deduce that there exists \( M_5 > 0 \) such that for all \( x \in S(x_0, x_1) \), we have

\[ \|\dot{x}\|_{L^2(X^*)} \leq M_5. \]  
(10)

From (8) and (10) above, we deduce that the set

\[ S'(x_0, x_0) = \{ \dot{x} \in W(T) : x \in S(x_0, x_1) \} \]

is bounded, hence relatively weakly compact in \( W(T) \).

Now introduce the following modification of the original orientor field \( F(t, x) \):

\[ \tilde{F}(t, x) = \begin{cases} F(t, x) & \text{if } |x| \leq M_2 \\ F(t, \frac{M_2 x}{|x|}) & \text{if } |x| > M_2. \end{cases} \]

Observe that \( \tilde{F}(t, x) = F(t, pM_2(x)) \), where \( pM_2(\cdot) \) is the \( M_2 \)-radial retraction in \( H \). Since \( pM_2(\cdot) \) is Lipschitz continuous, we have, using hypothesis \( H(F)_1 \), that \( t \rightarrow \tilde{F}(t, x) \) is measurable while \( x \rightarrow \tilde{F}(t, x) \) is u.s.c. from \( H \) into \( H_w \). Furthermore, note that \( |\tilde{F}(t, x)| \leq a(t) + bM_2 = \phi(t) \) a.e., with \( \phi(\cdot) \in L^2_. \). Let \( K = \{ h \in L^2(H) : h(t) \leq \phi(t) \forall t \} \). This set, endowed with the relative weak \( L^2(H) \)-topology, is compactly metrizable. In what follows, this will be the topology considered on \( K \). Let \( \gamma: K \rightarrow C(T, X) \) be the map which to each \( h \in K \), assigns the unique solution of the initial value problem \( \ddot{z}(t) + A(t, \dot{z}(t)) + Bz(t) = h(t), \quad x(0) = x_0, \quad \dot{z}(0) = x_1 \) (see Zeidler [9], theorem 33.3.1, p. 224). We claim that \( \gamma(\cdot) \) is continuous. To this end, let \( h_n \rightarrow h \) in \( K \) and let \( x_n = \gamma(h_n) \). Recall that \( \{x_n\}_{n \geq 1} \subseteq W(T) \) is relatively weakly compact. Hence, by passing to a subsequence if necessary, we may assume that \( x_n \overset{w}{\rightarrow} x \) in \( W(T) \). Let \( x = \gamma(h) \). We need to show that \( y = \dot{x} \). We have:
\[
\langle \ddot{x}_n(t) - \dot{x}(t), \dot{x}_n(t) - \dot{x}(t) \rangle + \langle A(t, \dot{x}_n(t)) - A(t, \dot{x}(t)), \dot{x}_n(t) - \dot{x}(t) \rangle \\
+ \langle Bx_n(t) - Bx(t), \dot{x}_n(t) - \dot{x}(t) \rangle \\
= (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \text{ a.e.}
\]

Exploiting the fact that \(A(t, \cdot)\) is monotone and using the integration by parts formula for functions in \(W(T)\) (see Zeidler [9], proposition 23.23, p. 422), we get

\[
\frac{1}{2} \frac{d}{dt} \left| \dot{x}_n(t) - \dot{x}(t) \right|^2 + \frac{1}{2} \frac{d}{dt} \langle B(x_n(t) - x(t)), \dot{x}_n(t) - \dot{x}(t) \rangle \leq (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \text{ a.e.}
\]

But, as before, exploiting the symmetry of the operator \(B\), we have

\[
\langle B(x_n(t) - x(t)), \dot{x}_n(t) - \dot{x}(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle B(x_n(t) - x(t)), x_n(t) - x(t) \rangle.
\]

So we get:

\[
\frac{1}{2} \frac{d}{dt} \left| \dot{x}_n(t) - \dot{x}(t) \right|^2 + \frac{1}{2} \frac{d}{dt} \langle B(x_n(t) - x(t)), x_n(t) - x(t) \rangle \leq (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \text{ a.e.}
\]

Integrating and recalling that \(x_n(0) = x(0) = x_0, \dot{x}_n(0) = \dot{x}(0) = x_1\), we have:

\[
\frac{1}{2} \left| \dot{x}_n(t) - \dot{x}(t) \right|^2 + \frac{1}{2} \int_0^t \langle B(x_n(s) - x(s)), x_n(s) - x(s) \rangle ds \leq \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds
\]

which yields

\[
\frac{c'}{2} \| x_n(t) - x(t) \|^2 \leq \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds
\]

which yields

\[
\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds.
\]

Note that \(h_n \overset{w}{\rightarrow} h\) in \(L^2(H)\) and \(\dot{x}_n \overset{w}{\rightarrow} y\) in \(W(T)\). Since \(W(T) \rightarrow L^2(H)\) compactly, we have that \(\dot{x}_n \overset{s}{\rightarrow} y\) in \(L^2(H)\). Thus we have:

\[
\int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds
\]

\[
= \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - y(s)) ds + \int_0^t (h_n(s) - h(s), y(s) - \dot{x}(s)) ds \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

So \(x_n(t) \overset{s}{\rightarrow} x(t)\) in \(X\) yields \(\dot{x} = y \in W(T)\). Now note that

\[
\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} \| h_n - h \|_{L^2(H)} \| \dot{x}_n - \dot{x} \|_{L^2(H)}
\]

Since \(h_n \overset{w}{\rightarrow} h\) in \(K\), we have \(\| h_n - h \|_{L^2(H)} \leq N\) for all \(n \geq 1\) and some \(N > 0\). Thus
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\[ \| x_n(t) - x(t) \| ^2 \leq \frac{2}{c} N \| \dot{x}_n - \dot{x} \| _{L^2(H)} \to 0 \]

which implies that \( \gamma(\cdot) \) is indeed continuous as claimed.

Let \( R: K \to 2^K \) be the multifunction defined by

\[ R(h) = S^2_{\hat{F}(\cdot, \gamma(h)(\cdot))} \]

First we will show that \( R(\cdot) \) has nonempty values. Let \( s_n(\cdot) \) be simple functions such that

\[ s_n(t) \xrightarrow{a.e.} \gamma(h)(t) \text{ a.e. in } H. \]

Then because of hypothesis \( H(F) \), for each \( n \geq 1 \), \( t \to \hat{F}(t, s_n(t)) \) is measurable. Apply Aumann’s selection theorem to get \( f_n: T \to H \) measurable such that \( f_n(t) \in \hat{F}(t, x_n(t)) \) a.e., \( n \geq 1 \). Note that \( |f_n(t)| \leq \phi(t) \) a.e. with \( \phi(\cdot) \in L^2_+ \). Hence by passing to a subsequence if necessary, we may assume that \( f_n \to f \) in \( L^2(H) \). Then theorem 3.1 of [6], tells us that

\[ f(t) \in \text{conv } \text{w-lim} \{f_n(t)\}_{n \geq 1} \]

\[ \subseteq \text{conv } \text{w-lim} \hat{F}(t, s_n(t)) \]

\[ \subseteq \hat{F}(t, \gamma(t)(t)) \text{ a.e.} \]

The last inclusion follows from the fact that \( \hat{F}(t, \cdot) \) is u.s.c. from \( H \) into \( H_w \) and since \( s_n(t) \xrightarrow{a.e.} \gamma(h)(t) \) a.e. in \( H \). Therefore \( f \in S^2_{\hat{F}(\cdot, \gamma(h)(\cdot))} \) and so we have established that the values of the multifunction \( R(\cdot) \) are nonempty. Also since \( F(t, x) \) is \( P_{f\phi}(H) \)-valued, it is clear that for every \( h \in K, \ R(h) \in P_{f\phi}(K) \). Furthermore using theorem 4.2 of [6] and recalling that \( \gamma(\cdot) \) is continuous on \( K \) into \( C(T, X) \), we get that \( R(\cdot) \) is u.s.c. Apply the Kakutani-KyFan fixed point theorem to get \( h \in R(h) \). Then \( z = \gamma(h) \) is a solution of (\( \star \)), with \( F(t, x) \) replaced by \( \hat{F}(t, x) \). But as in the beginning of the proof, with the same \textit{a priori} estimation, we can show that \( |x(t)| \leq M_2 \) for all \( t \in T \) implies that \( \hat{F}(t, x(t)) = F(t, x(t)) \) and this yields that \( x(\cdot) \) solves (\( \star \)).

Finally to establish the compactness of \( S(x_0, x_1) \) in \( C(T, X) \), note that \( S(x_0, x_1) \subseteq \gamma(K) \) and the latter is compact in \( C(T, X) \) since \( \gamma: K \to C(T, X) \) is continuous. So it suffices to show that \( S(x_0, x_1) \) is closed in \( C(T, X) \). So let \( \{x_n\}_{n \geq 1} \subseteq S(x_0, x_1) \) and assume that \( x_n \to x \) in \( C(T, X) \). Then by definition \( x_n = \gamma(f_n) \) with \( f_n \in S^2_{\hat{F}(\cdot, \gamma(h)(\cdot))} \). Note that because of hypothesis \( H(F) \)

\[ |f_n(t)| \leq a_1(t) + b_1 \hat{N}, \text{ where } \hat{N} = \sup \|x_n\|_{C(T, X)}. \]

So we may assume that \( f_n \to f \) in \( L^2(H) \) implies that \( \gamma(f_n) \to \gamma(f) \) in \( C(T, X) \) which yields \( x = \gamma(f) \) and from theorem 3.1 of [6], we have that \( f(t) \in \text{conv } \text{w-lim} \{f_n(t)\}_{n \geq 1} \subseteq \text{conv } \text{w-lim} F(t, x_n(t)) \subseteq \hat{F}(t, x(t)) \) a.e. which yields \( x \in S(x_0, x_1) \).

Q.E.D.

Now we consider the case where the multivalued perturbation term \( F(t, x) \) is not necessarily convex-valued. We will need the following hypothesis on the orientor field \( F(t, x) \).

\( H(F_2) \): \( F: T \times H \to P_f(H) \) is a multifunction such that

1. \( (t, x) \to F(t, x) \) is graph measurable; i.e. \( \text{Gr} F = \{(t, x, y) \in T \times H \times H: y \in F(t, x)\} \subseteq B(T) \times B(H) \), with \( B(T) \) (resp. \( B(H) \)), being the Borel \( \sigma \)-field of \( T \) (resp. of \( H \)) (recall that measurability of \( F(\cdot, \cdot) \) implies graph measurability).

2. \( x \to F(t, x) \) is l.s.c.

3. \( |F(t, x)| = \sup \{ |y|: y \in F(t, x)\} \leq a_1(t) + b_1 |x| \) a.e. with \( a_1(\cdot) \in L^2_+ \), \( b_1 > 0 \).

**Theorem 3.2:** If hypotheses \( H(A), H(B), H(F)_2 \) hold and \( x_0 \in X, \ x_1 \in H \), then \( S(x_0, x_1) \neq \emptyset \).
Proof: As in the proof of theorem 3.1, let $\tilde{F}(t,x) = F(t,p_{M_2}(x))$ (it is clear that the same a priori estimation is valid in the present situation). Then given that $p_{M_2}(\cdot)$ is Lipschitz continuous, we have that $(t,x) \rightarrow \tilde{F}(t,x)$ is graph measurable, $x \rightarrow \tilde{F}(t,x)$ is $l.s.c.$ and furthermore note that $|\tilde{F}(t,x)| \leq a_1(t) + b_1M_2 = \phi(t)$ a.e. with $\phi(\cdot) \in L^2_+.$

Let $V \subseteq L^1(H)$ be defined by $V = \{ h \in L^1(H); |h(t)| \leq \phi(t) \ a.e. \}.$ From proposition 3.1 of [5], we know that $V,$ equipped with the relative weak $L^1(H)$-topology, is compact metrizable. Consider the multifunction $\Gamma: V \rightarrow P_f(L^1(H))$ defined by $\Gamma(h) = S^1\tilde{F}(\cdot, \gamma(h)(\cdot)).$ It is easy to check using the continuity of $\gamma(\cdot)$ and theorem 4.1 of [6], that $\Gamma(\cdot)$ is l.s.c. (note that if $h_n \rightharpoonup h$ in $V \subseteq L^1(H),$ then $h_n \rightharpoonup h$ in $L^2(H),$ since $\phi(\cdot) \in L^2_+).$ So, we can apply Fryszkowski's continuous selection theorem [1], to get $k: V \rightarrow V$ continuous such that $k(h) \in R(h).$ Applying the Schauder-Tichonov fixed point theorem, we get $h \in V$ such that $h = k(h).$ Then $x = p(h)$ solves $(\ast)$ with $F(t,x)$ replaced by $\tilde{F}(t,x).$ But as before we can check that $|x(t)| \leq M_2$ which implies $\tilde{F}(t,x(t))$ implies that $F(t,x(t))$ which yields $x \in S(x_0,x_1).$ Q.E.D.

4. An Example

In this section we present an example of a nonlinear hyperbolic partial differential inclusion illustrating the applicability of our work.

So let $T = [0,r]$ and $Z$ a bounded domain in $\mathbb{R}^N,$ with smooth boundary $\Gamma = \partial Z.$ We will consider the following initial-boundary value problem of hyperbolic type with multivalued terms.

$$
\begin{cases}
\frac{\partial^2 x}{\partial t^2} - \Delta x - \sum_{i=1}^N D_i(k(t, |Dx_i|^2)D_i x_i) \in [f_1(t,z,x(t,z)), f_2(t,z,x(t,z))] \\
x |_{T \times \Gamma} = 0, \quad x(0,z) = x_0(z), \quad x_t(0,z) = x_1(z).
\end{cases}
$$

Here $D_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, N,$ $Dx = (D_1x_1, \ldots, D_Nx) = \text{grad}(x),$ $DxDy = \sum_{i=1}^N D_i x D_i y$ and $|Dx| = \sum_{i=1}^N |D_i x|^2.$

We will need the following hypotheses on the data of $(\ast)$:

$\text{H}(k):$ $k: T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function such that

1. $t \rightarrow k(t, \mu)$ is measurable,
2. $\mu \rightarrow k(t, \mu)$ is continuous,
3. $0 \leq k(t, \lambda^2) \leq L$ for all $(t, \lambda) \in T \times \mathbb{R}_+, \text{ with } L > 0 \text{ and } k(t,0) = 0,$
4. $k(t, \lambda^2) \lambda - k(t, \mu^2) \mu \geq d(\lambda - \mu)$ for all $\lambda, \mu \in \mathbb{R}_+, \lambda \geq \mu \text{ and for some } d > 0.$

$\text{H}(f):$ $f_1, f_2: T \times Z \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions such that $x \rightarrow f_1(t,z,x),$ $- f_2(t,z,x)$ are l.s.c. and $|f_i(t,z,x)| \leq a_1(t,z) + b_1(z)|x|$ a.e. $i = 1, 2$ with $a_1(\cdot, \cdot) \in L^2(T \times Z),$ $b_1(\cdot) \in L^\infty(Z)$ and $f_1 < f_2.$

$A_0:$

$x_0(\cdot) \in H^1_0(Z),$ $x_1(\cdot) \in L^2(Z).$

In this case, $X = H^1_0(Z),$ $H = L^2(Z)$ and $X^* = H^1_0(Z)^* = H^{-1}(Z).$ We know that $(X,H,X^*)$ is an evolution triple with all embeddings being compact (Sobolev embedding theorem). Consider the following Dirichlet forms:

$$
a_1(t,x,y) = \int_Z \sum_{i=1}^N k(t, |Dx_i|^2) D_i x_i D_i y dydz = \int_Z k(t, |Dx|^2) DxDydz
$$

and

$$
a_2(x,y) = \int_Z \sum_{i=1}^N D_i x D_i y dydz = \int_Z DxDydz
$$
for all $x, y \in H^1_0(Z)$.

Using hypothesis $H(k)(\exists)$, we get

$$|a_1(t, x, y)| \leq L \|x\|_{H^1_0(Z)} \|y\|_{H^1_0(Z)}.$$

So there exists a nonlinear operator $A: T \times X \to X^*$ such that

$$\langle A(t, x), y \rangle = a_1(t, x, y).$$

From Fubini’s theorem we have that $t \to a_1(t, x, y)$ is measurable which implies that $t \to A(t, x)$ is weakly measurable. But $H^{-1}(Z)$ is a separable Hilbert space. So the Pettis measurability theorem tells us that $t \to A(t, x)$ is measurable. Also if $x_n \to x$ in $H^1_0(Z)$, then by passing to a subsequence if necessary, we will have that $\|Dx_n(z)\|^2 \to \|Dx(z)\|^2$ a.e. and since by hypothesis $H(k)(\exists)$, $k(t, \cdot)$ is continuous, we have $k(t, |Dx_n(z)|^2) \to k(t, |Dx(z)|^2)$ for all $t \in T$ and almost all $z \in Z$. Also $D_x x_n \to D_x x$ in $L^2(Z)$. Thus

$$\int Z \{k(t, |Dx_n|^2)Dx_n \cdot dydz \to \int Z \{k(t, |Dx|^2)Dx \cdot dydz\}$$

implies that $A(t, x_n) \to A(t, x)$ which yields $A(t, \cdot)$ is demicontinuous, this hemicontinuous. Also we have

$$\langle A(t, x) - A(t, y), x - y \rangle \geq c \|x - y\|_{H^1_0(Z)}, c > 0.$$

which yields that $A(t, \cdot)$ is strongly monotone.

Also since $k(t, 0) = 0$ (by hypothesis $H(k)(\exists)$), we have $A(t, 0) = 0$ yields that $A(t, \cdot)$ is coercive; i.e., $\langle A(t, x), x \rangle \geq c \|x\|^2_{H^1_0(Z)}$. Thus, we satisfied hypothesis $H(A)$.

Next note that by the Cauchy-Schwartz inequality, we have

$$|a_2(x, y)| \leq \|x\|_{H^1_0(Z)} \|y\|_{H^1_0(Z)}.$$

So there exists a continuous linear operator $B: X \to X^*$ such that

$$\langle Bx, y \rangle = a_2(x, y).$$

Clearly $\langle Bx, y \rangle = \langle x, By \rangle$; i.e. $B$ is symmetric and by Poincaré's inequality, we have $\langle Bx, x \rangle \geq c' \|x\|^2_{H^1_0(Z)} > 0$. Therefore, we satisfied hypothesis $H(B)$.

Next let $F: T \times L^2(Z) \to \text{conv}(L^2(Z))$ be defined by

$$F(t, x) = \{h \in L^2(Z) : f_1(t, z, x) \leq h(z) \leq f_2(t, z, x) \text{ a.e.}\}.$$

Let $\eta: T \times Z \times \mathbb{R} \to \text{conv}(\mathbb{R})$ be defined by $\eta(t, z, x) = [f_1(t, z, x), f_2(t, z, x)]$. Because of hypothesis $H(f)$, we deduce that $\eta(t, z, \cdot)$ is measurable while $\eta(t, \cdot, \cdot)$ is u.s.c. (see Klein-Thompson [2], p. 74). Note that $F(t, x) = S^2_{\eta(t, \cdot, \cdot)}$. So, from theorem 4.2 of [6], we have that $F(t, \cdot)$ is u.s.c. from $H$ into $H_u$, while clearly $t \to F(t, z)$ is measurable. Also, $|F(t, x)| = \sup\{|y|_{L^2(Z)} : y \in F(t, x)\} \leq \hat{a}_1(t) + \hat{b}_1 |x|_{L^2(Z)}$, with $\hat{a}_1(t) = \|a(t, \cdot)\|_{L^2(Z)}$, $\hat{b}_1 = \|b\|_{L^\infty(Z)}$. Thus, we satisfied hypothesis $H(F)_1$. Finally, let $\hat{x}_0 = x_0(\cdot) \in H^1_0(Z), \hat{x}_1 = x_1(\cdot) \in L^2(Z)$. 

"Existence of Solutions for Second-Order Evolution Inclusions"
Rewrite (**) in the following equivalent nonlinear evolution inclusion form:

\[
\begin{align*}
\dot{x}(t) + A(t, \dot{x}(t)) + Bx(t) & \in F(t, x(t)) \\
x(0) &= \hat{x}_0, \dot{x}(0) = \hat{x}_1.
\end{align*}
\]

Theorem 4.1: If hypotheses \( H(k), H(f) \) and \( H_0 \) hold, then (**) has a solution \( x \in C(T, H^1_0(Z)) \) such that \( \frac{\partial x}{\partial t} \in L^2(T, H^1_0(Z)) \cap C(T, L^2(Z)) \) and \( \frac{\partial^2 x}{\partial t^2} \in L^2(T, H^{-1}(Z)) \). Also, the solution set is compact in \( C(T, H^1_0(Z)) \).

Now suppose that (**) corresponds to an optimal control problem; i.e.

\[
f_1(t, z, x) = f(t, z, x)u_1(z)
\]

and

\[
f_2(t, z, x) = f(t, z, x)u_2(z)
\]

with a function \( f: T \times Z \times \mathbb{R} \to \mathbb{R}_+ \) such that \( (t, z) \mapsto f(t, z, x) \) is measurable, \( x \mapsto f(t, z, x) \) is continuous and \( |f(t, z, x)| \leq a_1(t, z) + b_1(z)|x| \) a.e., with \( a_1(\cdot, \cdot) \in L^2(T \times Z), b_1(\cdot) \in L^\infty(Z) \). The control constraint set is defined as

\[
U(t, z) = \{ v \in \mathbb{R}: u_1(z) \leq v \leq u_2(z) \}
\]

with \( 0 < u_1(z) < u_2(z) \leq M \) a.e.

We are also given a cost functional \( J(x) = \int_0^b \int_Z L(t, z, x(t, z))dzdt \) to be minimized over all admissible trajectories. Assume that \( L: T \times Z \times \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \) is a measurable integrand such that \( L(t, z, \cdot) \) is l.s.c. and \( \phi(t, z) - M(z)|x| \leq L(t, z, x) \) a.e. with \( \phi(\cdot, \cdot) \in L^1(T \times Z), M(\cdot) \in L^\infty_+ (Z) \). Then, \( J(\cdot) \) is l.s.c. on \( C(T, H^1_0(Z)) \), and so, using theorem 4.1 above, we deduce that this distributed parameter optimal control problem has a solution. Analogous results for parabolic systems can be found in [4].

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References


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