THE PROBABILISTIC APPROACH TO THE ANALYSIS OF THE LIMITING BEHAVIOR OF AN INTEGRO-DIFFERENTIAL EQUATION DEPENDING ON A SMALL PARAMETER, AND ITS APPLICATION TO STOCHASTIC PROCESSES¹

O.V. BORISENKO

*Kiev Polytechnic Institute
Department of Mathematics N3
Prospect Pobedy 3, Kiev-252056, UKRAINE*

A.D. BORISENKO

*Kiev University
Department of Probability & Mathematical Statistics
Kiev-252017, UKRAINE*

I.G. MALYSHEV

*San Jose State University
Department of Mathematics & Computer Science
San Jose, CA 95192 USA*

ABSTRACT

Using connection between stochastic differential equation with Poisson measure term and its Kolmogorov's equation, we investigate the limiting behavior of the Cauchy problem solution of the integro-differential equation with coefficients depending on a small parameter. We also study the dependence of the limiting equation on the order of the parameter.

Key words: Stochastic process, Kolmogorov's averaging, integro-differential equation, Cauchy problem, limiting behavior, small parameters, white and Poisson noise.

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It is well known that investigation of a nonlinear oscillating systems with a small stochastic white noise at the input, can be accomplished applying the averaging method for Kolmogorov's parabolic equation with coefficients depending on a small parameter [1]. If both white and Poisson types of noise are present, then the corresponding Kolmogorov's equation is integro-differential [2],

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and we shall extend here the averaging principle to such equations.

Let us study behavior, as $\epsilon \rightarrow 0$, of the following equation

$$\begin{aligned} & \frac{\partial}{\partial t} U(t, x) + \epsilon^{k_1} (f(t, x), \nabla U(t, x)) + \frac{\epsilon^{k_2}}{2} \text{Tr}(g(t, x) g^*(t, x) \nabla^2 U(t, x)) \\ & + \int_{R^d} [U(t, x + \epsilon^{k_3} q(t, x, y)) - U(t, x) - \epsilon^{k_3} (q(t, x, y), \nabla U(t, x))] \Pi(dy) = 0, \\ & (t, x) \in [0, T) \times R^d, \end{aligned} \quad (1)$$

where $\epsilon > 0$ is a small parameter and k_1, k_2, k_3 , are some positive numbers, and

$$\nabla U(t, x) = \left\{ \frac{\partial U(t, x)}{\partial x_i}, i = 1, \dots, d \right\}, \quad \nabla^2 U(t, x) = \left\{ \frac{\partial^2 U(t, x)}{\partial x_i \partial x_j}, i, j = 1, \dots, d \right\}.$$

Here Π is a finite measure on Borel sets in R^d , $f(t, x)$, $q(t, x, y)$ are d -dimensional vectors, and $g(t, x)$ is a $d \times d$ square matrix.

Lemma: *If*

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_A^{s+A} b(t, x) dt = \bar{b}(x)$$

uniformly with respect to A for each x , the function $b(x)$ is continuous, and $b(t, x)$ is continuous in x uniformly with respect to (t, x) in arbitrary compact $|x| \leq C$, and stochastic process $\xi(t)$ is continuous, then

$$\lim_{\epsilon \rightarrow 0} \int_0^t b\left(\frac{\tau}{\epsilon}, \xi(\tau)\right) d\tau = \int_0^t \bar{b}(\xi(\tau)) d\tau.$$

The proof is similar to that in [2].

Now, replacing t with t/ϵ^k in (1), where $k = \min(k_1, k_2, k_3)$, and denoting $V_\epsilon(t, x) = U(t/\epsilon^k, x)$, we can derive the following equation:

$$\begin{aligned} & \frac{\partial}{\partial t} V_\epsilon(t, x) + \epsilon^{k_1 - k} (f(t/\epsilon^k, x), \nabla V_\epsilon(t, x)) + \frac{\epsilon^{k_2 - k}}{2} \text{Tr}(g(t/\epsilon^k, x) g^*(t/\epsilon^k, x) \nabla^2 V_\epsilon(t, x)) \\ & + \frac{1}{\epsilon^k} \int_{R^d} [V_\epsilon(t, x + \epsilon^{k_3} q(t/\epsilon^k, x, y)) - V_\epsilon(t, x) - \epsilon^{k_3} (q(t/\epsilon^k, x, y), \nabla V_\epsilon(t, x))] \Pi(dy) = 0, \\ & (t, x) \in [0, T) \times R^d. \end{aligned}$$

Theorem: *Let the following conditions hold:*

1) *the functions $f(t, x), g(t, x), q(t, x, y)$ are continuous in (t, x) , bounded and twice continuously differentiable with respect to x , with derivatives also bounded;*

2) *uniformly with respect to A for each $x \in R^d$, $y \in R^d$ there exists the following three limits*

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_A^{s+A} f(t, x) dt = \bar{f}(x), \quad \lim_{s \rightarrow \infty} \frac{1}{s} \int_A^{s+A} g(t, x) g^*(t, x) dt = \bar{G}(x),$$

and

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_A^{s+A} q(t, x, y) q^*(t, x, y) dt = \bar{Q}(x, y).$$

3) *The functions $\bar{f}(x)$, $\bar{G}(x)$, $\bar{Q}(x, y)$ satisfy the Lipschitz condition in x , and the matrix*

$$\bar{B}(x) = \bar{G}(x) + \int_{R^d} Q(x, y) \Pi(dy)$$

is uniformly parabolic.

Then,

a) *if $k_1 = k_2 = 2k_3$ and $V_\epsilon(t, x)$ satisfies (2) and the “Cauchy” condition*

$$\lim_{t \uparrow T} V_\epsilon(t, x) = F(x), \quad F(x) \in C_b^2(R^d), \quad (3)$$

then $\lim_{\epsilon \rightarrow 0} V_\epsilon(t, x) = \bar{V}(t, x)$, where $\bar{V}(t, x)$ is a solution of the problem:

$$\frac{\partial}{\partial t} \bar{V}(t, x) + (\bar{f}(x), \nabla \bar{V}(t, x)) + \frac{1}{2} \text{Tr}(\bar{B}(x) \nabla^2 \bar{V}(t, x)) = 0, \quad (4)$$

$$\lim_{t \uparrow T} \bar{V}(t, x) = F(x). \quad (5)$$

b) *If $k < k_1$, then V satisfies (4)-(5) but in this case there is no term containing $\bar{f}(x)$ in (4); Similarly, if $k < k_2$, then $\bar{B}(x)$ does not depend on $\bar{G}(x)$; and if $k < 2k_3$, then $\bar{B}(x)$ does not contain the term*

$$\int_{R^d} \bar{Q}(x, y) \Pi(dy).$$

Proof: Applying the results of [2-3] to the conditions of the theorem, it follows that the solution of the problem (2)-(3) exists for each ϵ , is unique and can be represented in the form

$$V_\epsilon(t, x) = E[F(\xi_\epsilon(t, x, T))],$$

where $\xi_\epsilon(t, x, T)$ is the solution of the stochastic equation

$$\begin{aligned} \xi_\epsilon(t, x, s) = & x + \epsilon^{k_1 - k} \int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) d\tau \\ & + \epsilon^{\frac{1}{2}(k_2 - k)} \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) dw(\tau) \\ & + \epsilon^{k_3} \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) \tilde{\nu}(d\tau, dy), \end{aligned}$$

where $w(t)$ is a d -dimensional Wiener process, $\nu(t, A)$ is a Poisson measure independent of w , A is a Borel set of R^d , and

$$\tilde{\nu}(t, A) = \nu(t/\epsilon^k, A) - t\Pi(A)/\epsilon^k; \quad E\nu(t, A) = t\Pi(A).$$

Let

$$\begin{aligned} \zeta_\epsilon(t, x, s) = & \epsilon^{\frac{1}{2}(k_2 - k)} \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) dw(\tau) \\ & + \epsilon^{k_3} \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) \tilde{\nu}_\epsilon(d\tau, dy). \end{aligned}$$

Then we can obtain the following estimates:

$$E |\xi_\epsilon(t, x, s)|^2 \leq C[x^2 + (\epsilon^{2(k_1 - k)} + \epsilon^{k_2 - k} + \epsilon^{2k_3 - k}) |s - t|],$$

$$E |\zeta_\epsilon(t, x, s)|^2 \leq C(\epsilon^{k_2 - k} + \epsilon^{2k_3 - k}) |s - t|,$$

$$E |\xi_\epsilon(t, x, s_2) - \xi_\epsilon(t, x, s_1)|^2 \leq C[\epsilon^{2(k_1 - k)} |s_2 - s_1|^2 + (\epsilon^{k_2 - k} + \epsilon^{2k_3 - k}) |s_2 - s_1|],$$

$$E |\zeta_\epsilon(t, x, s_2) - \zeta_\epsilon(t, x, s_1)|^2 \leq C(\epsilon^{k_2 - k} + \epsilon^{2k_3 - k}) |s_2 - s_1|.$$

From these estimates we infer that the family of processes $(\xi_\epsilon(t, x, s), \zeta_\epsilon(t, x, s))$ satisfies the Skorokhod's compactness conditions [4]. Therefore, for any sequence $\epsilon \rightarrow 0$ there exists a subsequence $\epsilon_m \rightarrow 0$, $m = 1, 2, \dots$, and processes $\bar{\xi}(t, x, s)$, $\bar{\zeta}(t, x, s)$ such that $\xi_{\epsilon_m}(t, x, s) \rightarrow \bar{\xi}(t, x, s)$, $\zeta_{\epsilon_m}(t, x, s) \rightarrow \bar{\zeta}(t, x, s)$ in probability as $\epsilon_m \rightarrow 0$. From (6) we can also find that

$$\xi_{\epsilon_m}(t, x, s) = x + \epsilon_m^{k_1 - k} \int_t^s f(\tau/\epsilon_m^k, \xi_{\epsilon_m}(t, x, \tau)) d\tau + \zeta_{\epsilon_m}(t, x, s). \quad (7)$$

Then, for any fixed $(t, x) \in [0, T]$ we have:

$$\begin{aligned}
E | \xi_\epsilon(t, x, s_2) - \xi_\epsilon(t, x, s_1) |^4 &\leq C[\epsilon^{4(k_1 - k)} |s_2 - s_1|^4 + E | \zeta_\epsilon(t, x, s_2) - \zeta_\epsilon(t, x, s_1) |^4], \\
E | \zeta_\epsilon(t, x, s_2) - \zeta_\epsilon(t, x, s_1) |^4 &\leq C[(\epsilon^{2(k_2 - k)} + \epsilon^{2(2k_3 - k)}) |s_2 - s_1|^2 \\
&\quad + \epsilon^{4k_3 - 3k/2} |s_2 - s_1|^{3/2} + \epsilon^{4k_3 - k} |s_2 - s_1|].
\end{aligned}$$

Therefore,

$$\begin{aligned}
E | \bar{\xi}(t, x, s_2) - \bar{\xi}(t, x, s_1) |^4 &\leq C[|s_2 - s_1|^4 + |s_2 - s_1|^2], \\
E | \bar{\zeta}(t, x, s_2) - \bar{\zeta}(t, x, s_1) |^4 &\leq C |s_2 - s_1|^2,
\end{aligned}$$

and the processes $\bar{\xi}(t, x, s)$, $\bar{\zeta}(t, x, s)$ satisfy the Kolmogorov's continuity condition on s [5].

a) Let us consider the case $k_1 = k_2 = 2k_3$. Then from (7) we obtain:

$$\xi_\epsilon(t, x, s) = x + \int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) d\tau + \zeta_\epsilon(t, x, s). \quad (8)$$

From this point we shall omit the subindex m in ϵ_m for simplicity. Then for each fixed $(t, x) \in [0, T]$ the process

$$\zeta_\epsilon(t, x, s) = \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) dw(\tau) + \epsilon^{k/2} \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) \tilde{v}(d\tau, dy)$$

is a vector-valued martingale with matrix characteristic

$$\begin{aligned}
\langle \zeta_\epsilon(t, x, s), \zeta_\epsilon(t, x, s) \rangle &= \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) g^*(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) d\tau \\
&\quad + \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) q^*(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) \Pi(dy) d\tau.
\end{aligned}$$

Using the above lemma, it is easy to show that

$$P - \lim_{\epsilon \rightarrow 0} \int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) d\tau = \int_t^s \bar{f}(\bar{\zeta}(t, x, \tau)) d\tau, \quad (9)$$

and

$$P - \lim_{\epsilon \rightarrow 0} \langle \zeta_\epsilon(t, x, s), \zeta_\epsilon(t, x, s) \rangle = \int_t^s \bar{B}(\bar{\zeta}(t, x, \tau)) d\tau. \quad (10)$$

Hence, from (8), (9), and (10) we obtain a continuous square integrable vector-valued martingale

$$\bar{\zeta}(t, x, s) = x + \int_t^s \bar{f}(\bar{\zeta}(t, x, \tau)) d\tau + \bar{\zeta}(t, x, s),$$

with matrix characteristic

$$\langle \bar{\zeta}(t, x, s), \bar{\zeta}(t, x, s) \rangle = \int_t^s \bar{B}(\bar{\zeta}(t, x, \tau)) d\tau.$$

It follows from [6] that there exists a d -dimensional Wiener process $w(t)$ such that

$$\bar{\zeta}(t, x, s) = \int_t^s \bar{\sigma}(\bar{\zeta}(t, x, \tau)) d\bar{w}(\tau),$$

where

$$\bar{\sigma}(x)\bar{\sigma}^*(x) = \bar{B}(x).$$

Consequently, the process $\bar{\xi}(t, x, s)$ satisfies the equation which, according [2], has a unique solution:

$$\bar{\xi}(t, x, s) = x + \int_t^s \bar{f}(\bar{\xi}(t, x, \tau)) d\tau + \int_t^s \bar{\sigma}(\bar{\xi}(t, x, \tau)) d\bar{w}(\tau). \quad (11)$$

The matrix $\bar{B}(x)$ is positive definite for all $x \in R^d$, satisfies Lipschitz conditions, and therefore matrix $\bar{\sigma}(x)$ satisfies Lipschitz condition as well. Then, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\epsilon_m \rightarrow 0} V_{\epsilon_m}(t, x) = \bar{V}(t, x) = E[F(\bar{\xi}(t, x, T))]$$

for any sequence $\epsilon_m \rightarrow 0$. But as it follows from [7] the function $\bar{V}(t, x)$ is a unique solution of the problem (4)-(5), which completes the proof of the part *a*) of the theorem.

b) When $k < k_1$, the boundedness of $f(t, x)$ implies that

$$E \left| \int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) d\tau \right| \leq C$$

and therefore the second term in the right side of (6) converges to 0 with $\epsilon \rightarrow 0$ in probability. The matrix characteristic of the martingale $\zeta_\epsilon(t, x, s)$ in (7) has the form

$$\begin{aligned} \langle \zeta_\epsilon, \zeta_\epsilon \rangle &= \epsilon^{k_2 - k} \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) g^*(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau)) d\tau \\ &+ \epsilon^{2k_3 - k} \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) q^*(\tau/\epsilon^k, \xi_\epsilon(t, x, \tau), y) \Pi(dy) d\tau. \end{aligned} \quad (12)$$

From the boundedness of g , q , similarly to the inference made above, we obtain

that either first or second term in the right side of (12) converges to 0 (respectively to the $k < k_2$ or $k < 2k_3$ case) as $\epsilon \rightarrow 0$, which allows to complete the proof of the theorem as in part a).

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