ON A UNIQUE SOLUTION OF A DIFFERENTIAL EQUATION IN VECTOR DISTRIBUTION

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Let us consider a differential equation in vector distribution

\[ \dot{x}(t) = f(x, u, t) + b(x, u, t)\dot{u}(t), \quad x(t_0) = x_0, \]  

where \( x \in \mathbb{R}^n \), functions \( f(x, u, t) \in \mathbb{R}^n \) and \( b(x, u, t) \in \mathbb{R}^{n \times m} \) are piecewise continuous in \( x, u, t \), and \( u(t) = (u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m \) is a bounded variation function which is non-decreasing in the following sense: \( u(t_2) \geq u(t_1) \) as \( t_2 \geq t_1 \), if \( u_i(t_2) \geq u_i(t_1) \) for all \( i = 1, \ldots, m \).

A solution to equation (1) is defined as a vibrosolution [1], i.e., as a unique limit of absolutely continuous solutions to pre-limiting equations (1) corresponding to absolutely continuous non-decreasing approximations of function \( u(t) \). Solutions to pre-limiting equations (1) are regarded as conventional solutions of ordinary differential equations with discontinuous right-hand sides in the sense of Filippov [2]. A vibrosolution is expected to be a function discontinuous at discontinuity points of the function \( u(t) \).

**Theorem 1:** Let the following conditions hold:
1. functions \( f(x, u, t), b(x, u, t) \) are piecewise continuous domains are locally connected;
2. functions \( f(x, u, t), b(x, u, t) \) satisfy the one-side Lipschitz condition in \( x \) [2].

A unique vibrosolution to equation (1) exists if and only if an \( n \times m \)-dimensional system of differential equations in differentials

\[ \frac{d\xi}{du} = b(\xi, u, x), \quad \xi(\omega) = z, \]  

is solvable inside a cone \( K = \{ u \in \mathbb{R}^m: u_i \geq \omega_i, i = 1, \ldots, m \} \) with arbitrary initial values \( \omega \in \mathbb{R}^m \), \( \omega \geq u(t_0), z \in \mathbb{R}^n \), and \( s \geq t_0 \).

Since a vibrosolution is a discontinuous solution to equation (1), it is helpful to design an equation with a measure which enables us to compute jumps of a vibrosolution at discontinuity points of function \( u(t) \). This completely determines the behavior of a vibrosolution.

**Theorem 2:** Let the conditions of Theorem 1 hold. Then a solution of an equivalent equation with a measure

\[ dx(t) = f(x, u, t)dt + b(x, u, t)du(t) + \sum_{t_i} G(x(t_i), u(t_i), \Delta u(t_i), t_i)dx(t - t_i), \quad x(t_0) = x_0, \]

coincides with a vibrosolution of equation (1), if \( G(z, v, u, s) = \xi(z, v, u + v, s) - z \), where
ξ(z,v,u,s) is a solution to system (2); u^c(t) is a continuous component of a nondecreasing function u(t), \( \Delta u(t_i) = u(t_i^+) - u(t_i^-) \) is a jump of a function u(t) at \( t_i \), \( t_i \) are discontinuity points of a function u(t), and \( \chi(t - t_i) \) is the Heaviside function.

References

