

NON-COMPACT RANDOM GENERALIZED GAMES AND RANDOM QUASI-VARIATIONAL INEQUALITIES

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ABSTRACT

In this paper, existence theorems of random maximal elements, random equilibria for the random one-person game and random generalized game with a countable number of players are given as applications of random fixed point theorems. By employing existence theorems of random generalized games, we deduce the existence of solutions for non-compact random quasi-variational inequalities. These in turn are used to establish several existence theorems of non-compact generalized random quasi-variational inequalities which are either stochastic versions of known deterministic inequalities or refinements of corresponding results known in the literature.

Key words: Polish Space, Suslin Space, Measurable Space, Suslin Family, (Random) Fixed Point, (Random) Maximal Element, (Random) Equilibria, (Random) Qualitative Game, (Random) Generalized Game, (Random) Variational Inequality, (Random) Quasi-Variational Inequality, Class L , L -Majorized, Measurable Selection Theorem, Property (K) , Random Operator.

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1. Introduction

Since Spacek [34] and Hans [14] established some existence results of random fixed point theorems in the fifties, random fixed point theory has received much more attention in recent years, e.g., see Bharucha-Reid [5], Bocsan [8], Engl [13], Itoh [16], Kucia and Nowak [21], Lin [23], Liu and Chen [24], Nowak [26], Papageorgiou [27], Rybinski [28], Sarbadhikari and Srivastava [31], Sehgal and Singh [32], Tan and Yuan [37-38] and Xu [45], etc. Recently, we proved a very general random fixed point theorem in [37] (e.g., see Theorem A below). In this paper, as applications of random fixed point theorem in [37], existence theorems of random maximal elements, random equilibria for a random one-person game and random generalized games with a countable number of players are given. By employing existence theorems of random generalized games, we deduce the existence of solutions for non-compact random quasi-variational inequalities which in turn are used to establish several existence theorems of non-compact generalized random quasi-variational inequalities which are either stochastic versions or

improvements of corresponding results in the literature, e.g., Aliprantis et al. [1], Arrow and Debreu [2], Aubin [3], Aubin and Ekeland [4], Bharucha-Reid [5], Borglin and Keiding [6], Border [7], Bocsan [8], Hans [14], Kucia and Nowak [21], Liu and Chen [24], Mas-Colell and Zame [25], Nowak [26], Papageorgiou [27], Rybinski [28], Shih and Tan [33], Spacek [34], Tan [35-36], Tan and Yuan [39], Tarafdar and Mehta [41], Toussaint [42], Tulcea [43], Yannelis and Prabhakar [46], Zhang (Chang) [47], and Zhou and Chen [48].

2. Preliminaries

The set of all real numbers is denoted by \mathbb{R} and the set of natural numbers is denoted by \mathbb{N} . If X is a set, we shall denote by 2^X the family of all subsets of X . Let A be a subset of a topological space X . The set A is said to be compactly open if A is relatively open in each non-empty compact subset of X . We shall denote by $\text{int}_X(A)$ the interior of A in X and by $\text{cl}_X(A)$ the closure of A in X . If A is a subset of a vector space, we shall denote by $\text{co}A$ the convex hull of A . If A is a non-empty subset of a topological vector space E and $S, T: A \rightarrow 2^E$ are correspondences, then $\text{co}T, T \cap S: A \rightarrow 2^E$ are correspondences defined by $(\text{co}T)(x) = \text{co}T(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$. If X and Y are topological spaces and (Ω, Σ) is a measurable space (see definition below), and $T: \Omega \times X \rightarrow 2^Y$ is a correspondence, the Graph of T , denoted by $\text{Graph}T$, is the set $\{(\omega, x, y) \in \Omega \times X \times Y: y \in T(\omega, x)\}$ and the correspondence $\bar{T}: \Omega \times X \rightarrow 2^Y$ is defined by $\bar{T}(\omega, x) = \{y \in Y: (x, y) \in \text{cl}_{X \times Y} \text{Graph}T(\omega, \cdot)\}$, and $\text{cl}T: \Omega \times X \rightarrow 2^Y$ is defined by $\text{cl}T(\omega, x) = \text{cl}_Y(T(\omega, x))$ for each $(\omega, x) \in \Omega \times X$. It is easy to see that $\text{cl}T(\omega, x) \subset \bar{T}(\omega, x)$ for each $(\omega, x) \in \Omega \times X$.

If X and Y are topological spaces, $A \subset X \times Y$, and $F: X \rightarrow 2^Y$, then

- (1) the domain of F , denoted by $\text{Dom}F$, is the set $\{x \in X: F(x) \neq \emptyset\}$;
- (2) the projection of A into X , denoted by $\text{Proj}_X A$, is the set $\{x \in X: \text{there exists some } y \in Y \text{ such that } (x, y) \in A\}$;
- (3) F is said to be lower (respectively, upper) semicontinuous if for each closed (respectively, open) subset C of Y , the set $\{x \in X: F(x) \subset C\}$ is closed (respectively, open) in X ;
- (4) F is said to be compact if for each $x \in X$, there exists a open neighborhood V_x of x in X such that $F(V_x) = \cup_{z \in V_x} F(z)$ is relatively compact in Y ; and
- (5) $x \in X$ is a maximal element of F if $F(x) = \emptyset$.

Note that $\text{Dom}F = \text{Proj}_X \text{Graph}F$.

Let X be a subset of a topological vector space E . The set X is said to have the property (K) (see [43]) if for each compact subset S of X , the convex hull $\text{co}B$ of B is relatively compact in X .

Let X be a topological space, Y a non-empty subset of a vector space E , $\theta: X \rightarrow E$ a (single-valued) mapping and $\phi: X \rightarrow 2^Y$ a mapping. Then

- (1) ϕ is said to be of class L_θ if for every $x \in X$, $\text{co}\phi(x) \subset Y$ and $Q(x) \notin \text{co}\phi(x)$ and for each $y \in Y$, $\phi^{-1}(y) = \{x \in X: y \in \phi(x)\}$ is compactly open in X ;
- (2) a correspondence $\phi_x: X \rightarrow 2^Y$ is said to be an L_θ -majorant of ϕ at $x \in X$ if there exists an open neighborhood N_x of x in X such that
 - (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \text{co}\phi_x(z)$,
 - (b) for each $z \in X$, $\text{co}\phi_x(z) \subset Y$ and
 - (c) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X ;
- (3) ϕ is L_θ -majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an L_θ -majorant of ϕ at $x \in X$.

We shall only deal with either the case (I) $X = Y$ and which is a non-empty convex subset of a

topological vector space and $\theta = I_X$, the identity mapping on X (in this case, the above notions coincide with the corresponding notions introduced in [46]), or the case (II) $X = \prod_{i \in I} X_i$ and $\theta = \pi_j: X \rightarrow X_j$ is the projection of X onto X_j and $X_j = Y$ is a non-empty convex subset of atopolological vector space. In both cases (I) and (II), we shall write L in place of L_θ .

A measurable space (Ω, Σ) is a pair where Ω is a set and Σ is a σ -algebra of subsets of Ω . If X is a set, $A \subset X$, and \mathfrak{D} is a non-empty family of subsets of X , we shall denote by $\mathfrak{D} \cap A$ the family $\{D \cap A: D \in \mathfrak{D}\}$ and by $\sigma_X(\mathfrak{D})$ the smallest σ -algebra on X generated by \mathfrak{D} . If X is a topological space with topology τ_X , we shall use $\mathfrak{B}(X)$ to denote $\sigma_X(\tau_X)$, the Borel σ -algebra on X . If (Ω, Σ) and (Φ, Γ) are two measurable spaces, then $\Sigma \otimes \Gamma$ denotes the smallest σ -algebra on $\Omega \times \Phi$ which contains all the sets $A \times B$, where $A \in \Sigma, B \in \Gamma$, i.e., $\Sigma \otimes \Gamma = \sigma_{\Omega \times \Phi}(\Sigma \times \Gamma)$. We note that the Borel σ -algebra $\mathfrak{B}(X_1 \times X_2)$ contains $\mathfrak{B}(X_1) \otimes \mathfrak{B}(X_2)$ in general. A mapping $f: \Omega \rightarrow \Phi$ is said to be (Σ, Γ) measurable (or simply, measurable) if for each $B \in \Gamma, f^{-1}(B) = \{\omega \in \Omega: f(\omega) \in B\} \in \Sigma$. Let X be a topological space and $F: (\Omega, \Sigma) \rightarrow 2^X$ be a mapping. Then F is said to be measurable (respectively, weakly measurable) if $F^{-1}(B) = \{\omega \in \Omega: F(\omega) \cap B \neq \emptyset\} \in \Sigma$ for each closed (respectively, open) subsets B of X . The map F is said to have a measurable graph if $Graph F := \{\omega, y \in \Omega \times X: y \in F(\omega)\} \in \Sigma \otimes \mathfrak{B}(X)$. A function $f: \Omega \rightarrow X$ is a measurable selection of F if f is a measurable function such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

If (Ω, Σ) and (Φ, Γ) are measurable spaces, Y is a topological space, then a mapping $F: \Omega \times \Phi \rightarrow 2^Y$ is called (jointly) measurable (respectively, weakly measurable) if for every closed (respectively, open) subset B of $Y, F^{-1}(B) = \{(\omega, x) \in \Omega \times \Phi: F(\omega, x) \cap B \neq \emptyset\} \in \Sigma \otimes \Gamma$. In the case $\Phi = X$, a topology space, then it is understood that Γ is the Borel σ -algebra $\mathfrak{B}(X)$.

A topological space X is

- (i) a Polish space if X is separable and metrizable by a complete metric;
- (ii) a Suslin space if X is a Hausdorff topological space and the continuous image of a Polish space.

A Suslin subset in a topological space is a subset which is a Suslin space. "Suslin" sets play very important roles in measurable selection theory. We also note that if X_1 and X_2 are Suslin spaces, then $\mathfrak{B}(X_1 \times X_2) = \mathfrak{B}(X_1) \times \mathfrak{B}(X_2)$ (e.g., see [29, p. 113]).

Denote by \mathfrak{J} and \mathfrak{F} the sets of infinite and finite sequences of positive integers respectively. Let \mathfrak{G} be a family of sets and $F: \mathfrak{J} \rightarrow \mathfrak{F}$ be a map. For each $\sigma = (\sigma_i)_{i=1}^\infty \in \mathfrak{J}$ and $n \in \mathbb{N}$, we shall denote $(\sigma_1, \dots, \sigma_n)$ by $\sigma | n$; then $\bigcup_{\sigma \in \mathfrak{J}} \bigcap_{n=1}^\infty F(\sigma | n)$ is said to be obtained from \mathfrak{G} by the Suslin operation. Now, if every set obtained from \mathfrak{G} in this way is also in \mathfrak{G} , then \mathfrak{G} is called a Suslin family (e.g., see [22], [30], [44], etc.).

Let X and Y be topological spaces, (Ω, Σ) a measurable space and $F: \Omega \times X \rightarrow 2^Y$ be a mapping. Then

- (a) F is a random operator if for each fixed $x \in X$, the mapping $F(\cdot, x): \Omega \rightarrow 2^Y$ is a measurable map;
- (b) F is random lower semicontinuous (respectively, random upper semicontinuous, random continuous) if F is a random operator and for each fixed $\omega \in \Omega, F(\omega, \cdot): X \rightarrow 2^Y$ is lower semicontinuous (respectively, upper semicontinuous, continuous); and
- (c) a measurable (single-valued) mapping $\psi: \Omega \rightarrow X$ is said to be a random maximal element of the correspondence F if $F(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$.

Let (Ω, Σ) be a measurable space, X a topological space and $F: \Omega \times X \rightarrow 2^X$ a mapping. The (single-valued) mapping $\varphi: \Omega \rightarrow X$ is said to be

- (i) a deterministic fixed point of F if $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$; and
- (ii) a random fixed point of F if φ is a measurable mapping and $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$.

It should be noted here that some authors define a random fixed point of F to be a measurable mapping φ such that $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for almost every $\omega \in \Omega$, e.g., see [27], [28] and the references therein.

Let I be any set of players and (Ω, Σ) be a measurable space. For each $i \in I$, let its strategy set X_i be a non-empty subset of a topological vector space. Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $P_i: \Omega \times X \rightarrow 2^{X_i}$ be a correspondence. The collection $\Gamma = (\Omega, X_i, P_i)_{i \in I}$ will be called a random qualitative game. A measurable map $\psi: \Omega \rightarrow X$ is said to be a random equilibrium of the random qualitative game Γ if $P_i(\omega, \psi(\omega)) = \emptyset$ for all $i \in I$ and all $\omega \in \Omega$.

A random generalized game (abstract economy) is a collection $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ where I is a (finite or infinite) set of players (agents) such that for each $i \in I$, X_i is a non-empty subset of a topological vector space and $A_i, B_i: \Omega \times X \rightarrow 2^{X_i}$ are random constraint correspondences where $X = \prod_{i \in I} X_i$, and $P_i: \Omega \times X \rightarrow 2^{X_i}$ is a preference correspondence (which are interpreted as for each player (or agent) $i \in I$, the associated constraint and preferences A_i , B_i and P_i have stochastic actions). A random equilibrium of Γ is a (single-valued) measurable mapping $\Psi: \Omega \rightarrow X$ such that for each $i \in I$, $\pi_i(\psi(\omega)) \in \overline{B_i}(\omega, \psi(\omega))$ and $A_i(\omega, \psi(\omega)) \cap P_i(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$. Here, π_i is the projection from X onto X_i . If $x \in X$, we shall also write x_i in place of $\pi_i(x)$ if there is no ambiguity. We remark that if A_i , B_i and P_i of the random generalized game $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ are independent of the variable $\omega \in \Omega$, i.e., $A_i(\omega, \cdot) = A_i(\cdot)$, $B_i(\omega, \cdot) = B_i(\cdot)$ and $P_i(\omega, \cdot) = P_i(\cdot)$ for all $\omega \in \Omega$, when $\overline{B_i}(\hat{x}) = cl_{X_i} B_i(\hat{x})$ for each $\hat{x} \in X$ (which is the case when B_i has a closed graph in $X \times X_i$; in particular, when $cl B_i$ is upper semicontinuous with closed values), our definition of an equilibrium point coincides with that of Ding et al. [12] in the deterministic case; and if in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point coincides with the standard definition of the deterministic case, e.g., in Borglin and Keiding [7], Tulcea [43], and Yannelis and Prabhakar [46].

We shall now list some results which will be needed in this paper. The following very general random fixed point theorem is Theorem 2.2 of Tan and Yuan in [37].

Theorem A. *Let (Ω, Σ) be a measurable space, Σ a Suslin family and X a Suslin space. Suppose $F: \Omega \times X \rightarrow 2^X \setminus \{\emptyset\}$ is such that $\text{Graph} F \in \Sigma \otimes \mathfrak{B}(X \times X)$. Then F has a random fixed point if and only if F has a deterministic fixed point in X , i.e., for each $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in X .*

For a non-self mapping generalization of the above result, we refer the reader to [38, Theorem 2.3]. The following measurable selection theorem is due to Leese [22, Corollary, p. 408-409].

Theorem B. *Let (Ω, Σ) be a measurable space, Σ a Suslin family and X a Suslin space. Suppose $F: \Omega \rightarrow 2^X$ has non-empty values such that $\text{Graph} F \in \Sigma \otimes \mathfrak{B}(X)$. Then there exists a sequence $\{g_n\}_{n=1}^\infty$ of measurable selections of F such that for each $\omega \in \Omega$, the set $\{g_n(\omega): n \in \mathbb{N}\}$ is dense in $F(\omega)$.*

The following lemma is Theorem 3.3 of Tan and Yuan in [39].

Lemma 1. *Let $\mathfrak{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

- (a) *for each $i \in I$, X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i ;*
- (b) *for each $i \in I$, $A_i: X \rightarrow 2^{X_i}$ is lower semicontinuous such that for each $x \in X$, $A_i(x)$ is non-empty and $co A_i(x) \subseteq B_i(x)$;*
- (c) *for each $i \in I$, $A_i \cap P_i$ is L_C -majorized;*
- (d) *for each $i \in I$, the set $E^i = \{x \in X: (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X ;*
- (e) *there exist a non-empty compact convex subset X_0 of X and a non-empty compact*

subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then \mathfrak{G} has an equilibrium point in K , i.e., there exists a point $\hat{x} = (\hat{x}_i)_{i \in I} \in K$ such that for each $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

The following result is Theorem 5.3 of Tan and Yuan in [40].

Lemma 2. Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:

- (a) for each $i \in I$, X_i is a non-empty closed convex subset of a locally convex Hausdorff topological vector space E_i and X_i has the property (K);
- (b) for each $i \in I$, B_i is compact and upper semicontinuous with non-empty compact convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;
- (c) for each $i \in I$, P_i is lower semicontinuous and L_C -majorized;
- (d) for each $i \in I$, $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X ;
- (e) there exist a nonempty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in K$ such that for each $i \in I$, $\bar{x}_i \in \overline{B}_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

We also need the following result (e.g., see Theorem 1 of Ding and Tan [11]).

Lemma 3. Let X be a non-empty paracompact convex subset of a Hausdorff topological vector space and $P: X \rightarrow 2^X$ be L -majorized (i.e., L_{LX} -majorized). Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there exists $x \in \text{co}(X_0 \cup \{y\})$ with $x \in \text{co}P(y)$. Then there exists an $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$.

3. Random Equilibria of Random Games

As an application of our random fixed point theorem, namely, Theorem A above, we shall first prove the following existence theorem of random maximal elements:

Theorem 1. Let (Ω, Σ) be a measurable space, Σ Suslin family, X a non-empty paracompact convex and Suslin subset of a Hausdorff topological vector space E and $Q: \Omega \times X \rightarrow 2^X$ such that for each given $\omega \in \Omega$, $Q(\omega, \cdot)$ is L_{LX} -majorized and $\text{Dom}Q \in \Sigma \otimes \mathfrak{B}(X)$. Suppose that for each fixed $\omega \in \Omega$, there exists a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there is an $x \in \text{co}(X_0(\omega) \cup \{y\})$ with $x \in \text{co}Q(\omega, y)$. Then Q has a random maximal element, i.e., there exists a measurable mapping $\psi: \Omega \rightarrow X$ such that $Q(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$.

Proof. By Lemma 3, for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that $Q(\omega, x_\omega) = \emptyset$. Define $F: \Omega \times X \rightarrow 2^X$ by $F(\omega, x) = \{y \in X : Q(\omega, y) = \emptyset\}$ for each $(\omega, x) \in \Omega \times X$. Then for each fixed $\omega \in \Omega$, x_ω is a fixed point of $F(\omega, \cdot)$. In order to prove that $\text{Graph}F \in \Sigma \otimes \mathfrak{B}(X \times X)$, we define a mapping $C: \Omega \times X \times X \rightarrow \Omega \times X \times X$ by

$$C(\omega, x, y) = (\omega, y, x)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then C is measurable. By hypothesis, $\text{Dom}Q \in \Sigma \otimes \mathfrak{B}(X)$. Since

$$\text{Graph}F = \{(\omega, x, y) \in \Omega \times X \times X : Q(\omega, y) = \emptyset\}$$

$$= C^{-1}[(\Omega \times X \setminus \text{Dom}Q) \otimes X] \in \Sigma \otimes \mathfrak{B}(X \times X),$$

then, by Theorem A, F has a random fixed point ψ , i.e., there exists $\psi: \Omega \rightarrow X$ is measurable such that $\psi(\omega) \in F(\omega, \psi(\omega))$ for all $\omega \in \Omega$ which implies that $Q(\omega, \psi(\omega)) = \emptyset$ for each $\omega \in \Omega$. \square

As an application of Theorem A again, we have the following existence theorem of random equilibria for random one-person games:

Theorem 2. *Let (Ω, Σ) be a measurable space, Σ a Suslin family and X a non-empty precompact convex and Suslin subset of a Hausdorff topological vector space. Let $A, B, P: \Omega \times X \rightarrow 2^X$ be such that*

- (i) *for each $\omega \in \Omega$, $A(\omega, \cdot) \cap P(\omega, \cdot)$ is L -majorized;*
- (ii) *$A(\omega, x)$ is non-empty and $\text{co}A(\omega, x) \subset B(\omega, x)$ for each $(\omega, x) \in \Omega \times X$;*
- (iii) *$(A(\omega, \cdot))^{-1}(y) = \{x \in X: y \in A(\omega, x)\}$ is open in X for each $(\omega, y) \in \Omega \times X$;*
- (iv) *$\text{Dom}(A \cap P)$ and $\text{Proj}_{\Omega \times X}[(\text{Graph}\bar{B}) \cap (\Omega \times \Delta)] \in \Sigma \otimes \mathfrak{B}(X)$ where $\Delta = \{(x, x): x \in X\}$;*
- (v) *for each fixed $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there is an $x \in \text{co}(X_0(\omega) \cap \{y\})$ with $x \in \text{co}(P(\omega, y) \cap A(\omega, y))$.*

Then the random one-person game $(\Omega; X; A, B; P)$ has a random equilibrium, i.e., there exists a measurable mapping $\psi: \Omega \rightarrow X$ such that $\psi(\omega) \in \bar{B}(\omega, \psi(\omega))$ and $A(\omega, \psi(\omega)) \cap P(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$.

Proof. Define $\Psi: \Omega \times X \rightarrow 2^X$ by

$$\Psi(\omega, x) = \{y \in X: A(\omega, y) \cap P(\omega, y) = \emptyset \text{ and } y \in \bar{B}(\omega, y)\}$$

for each $(\omega, x) \in \Omega \times X$. Then by Theorem 2 of Ding and Tan [11], for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that $x_\omega \in \Psi(\omega, x)$ for all $x \in X$. It follows that $\Psi: \Omega \times X \rightarrow 2^X \setminus \{\emptyset\}$ and $x_\omega \in \Psi(\omega, x_\omega)$ for all $\omega \in \Omega$ so that Ψ has a deterministic fixed point in X . Now define a mapping $C: \Omega \times X \times X \rightarrow \Omega \times X \times X$, by $C(\omega, x, y) = (\omega, y, x)$ for each $(\omega, x, y) \in \Omega \times X \times X$. Then C is measurable. Note that

$$\begin{aligned} \text{Graph}\Psi &= C^{-1}([\Omega \times X] \setminus \text{Dom}(A \cap P)) \times X \\ &\cap C^{-1}(\text{Proj}_{\Omega \times X}[(\text{Graph}\bar{B}) \cap (\Omega \times \Delta)]) \times X \\ &\in \Sigma \otimes \mathfrak{B}(X \times X), \end{aligned}$$

so that $\text{Graph}\Psi \in \Sigma \otimes \mathfrak{B}(X \times X)$. By Theorem A, Ψ has a random fixed point ψ , i.e., $\psi: \Omega \times X$ is measurable such that $A(\omega, \psi(\omega)) \cap P(\omega, \psi(\omega)) = \emptyset$ and $\psi(\omega) \in \bar{B}(\omega, \psi(\omega))$ for all $\omega \in \Omega$. \square

As another application of Theorem A, we have the following:

Theorem 3. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ a random generalized game such that I is countable and $X = \prod_{i \in I} X_i$ is paracompact. For each $i \in I$, suppose that the following conditions are satisfied:*

- (I) *X_i is a non-empty convex and Suslin subset of a locally convex Hausdorff topological vector space;*
- (II) *$\text{Dom}(A_i \cap P_i)$, $\text{Proj}_{\Omega \times X}[(\text{Graph}\bar{B}_i) \cap (\Omega \times \Delta_i)] \in \Sigma \otimes \mathfrak{B}(X)$ where $\Delta_i = \{(x, \pi_i(x)): x \in X\}$.*
- (III) *for each $\omega \in \Omega$, $E_i(\omega) = \{x \in X: A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X ;*
- (IV) *for each fixed $\omega \in \Omega$, either*
 - (i) (a) *$A_i(\omega, \cdot): X \rightarrow 2^{X_i}$ is lower semicontinuous such that for each $x \in X$, $A_i(\omega, x)$ is non-empty and $\text{co}A_i(\omega, x) \subset B_i(\omega, x)$, and*

- (b) $A_i(\omega, \cdot) \cap P_i(\omega, \cdot)$ is L -majorized;
- or
- (ii) (a) $B_i(\omega, \cdot)$ is upper semicontinuous with non-empty compact and convex values such that for each $x \in X$, $A_i(\omega, x) \subset B_i(\omega, x)$, and
- (b) $P_i(\omega, \cdot)$ is lower semicontinuous and L -majorized, and X_i is closed and has the property (K) ;
- (V) for each fixed $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there is an $x \in \text{co}(X_0(\omega) \cup \{y\})$ with $x_i \in \text{co}(A_i(\omega, y) \cap P_i(\omega, y))$ for all $i \in I$.

Then Γ as a random equilibrium.

Proof. First we note that as each X_i is a Suslin space and I is countable, X is also a Suslin space. For each $i \in I$, define $\Psi_i: \Omega \times X \rightarrow 2^X$ by

$$\Psi_i(\omega, x) = \{y \in X: A_i(\omega, y) \cap P_i(\omega, y) = \emptyset \text{ and } \pi_i(y) \in \overline{B_i}(\omega, y)\}$$

for each $(\omega, x) \in \Omega \times X$. Define $\Psi: \Omega \times X \rightarrow 2^X$ by $\Psi(\omega, x) = \bigcap_{i \in I} \Psi_i(\omega, x)$ for each $(\omega, x) \in \Omega \times X$. Then by Lemma 1 or Lemma 2, for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that $x_\omega \in \Psi_i(\omega, x)$ for all $x \in X$ and for all $i \in I$ so that $x_\omega \in \Psi(\omega, x)$ for all $x \in X$. It follows that $\Psi: \Omega \times X \rightarrow 2^X \setminus \{\emptyset\}$ and $x_\omega \in \Psi(\omega, x_\omega)$ for all $\omega \in \Omega$ so that Ψ has a deterministic fixed point in X . Now define a mapping $C: \Omega \times X \times X \rightarrow \Omega \times X \times X$ by

$$C(\omega, x, y) = (\omega, y, x)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then C is measurable. Note that

$$\begin{aligned} \text{Graph}\Psi_i &= C^{-1}([\Omega \times X \setminus \text{Dom}(A_i \cap P_i)] \times X) \\ &\cap C^{-1}(\text{Proj}_{\Omega \times X}[\text{Graph}\overline{B_i} \cap (\Omega \times \Delta_i)] \times X) \\ &\in \Sigma \otimes \mathfrak{B}(X \times X), \end{aligned}$$

and I is countable, we have $\text{Graph}\Psi = \bigcap_{i \in I} \text{Graph}\Psi_i \in \Sigma \otimes \mathfrak{B}(X \times X)$. By Theorem A, there exists a measurable mapping $\psi: \Omega \rightarrow X$ such that $\psi(\omega) \in \Psi(\omega, \psi(\omega))$ for all $\omega \in \Omega$; i.e., $A_i(\omega, \psi(\omega)) \cap P_i(\omega, \psi(\omega)) = \emptyset$ and $\pi_i(\psi_i(\omega)) \in \overline{B_i}(\omega, \psi(\omega))$ for all $\omega \in \Omega$ and for all $i \in I$. \square

As a consequence of Theorem 3, we have the following existence theorem of random qualitative games:

Theorem 4. Let (Ω, Σ) be a measurable space with Σ a Suslin family and $\Gamma = (\Omega; X_i; P_i)_{i \in I}$ a random qualitative game such that I is countable and $X = \prod_{i \in I} X_i$ is paracompact. For each $i \in I$, suppose that the following conditions are satisfied:

- (i) X_i is a non-empty convex and Suslin subset of a locally convex Hausdorff topological vector space;
- (ii) $\text{Dom}P_i \in \Sigma \otimes \mathfrak{B}(X)$;
- (iii) for each $\omega \in \Omega$, $\text{Dom}P_i(\omega, \cdot)$ is open in X ;
- (iv) for each fixed $\omega \in \Omega$, $P_i(\omega, \cdot)$ is L -majorized;
- (v) for each fixed $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there is an $x \in \text{co}(X_0(\omega) \cup \{y\})$ with $x_i \in \text{co}(P_i(\omega, y))$ for all $i \in I$.

Then Γ has a random equilibrium.

Proof. For each $i \in I$, define $A_i, B_i: \Omega \times X \rightarrow 2^{X_i}$ by $A_i(\omega, x) = B_i(\omega, x) = X$ for each

$(\omega, x) \in \Omega \times X$. Then it is easily seen that all hypotheses of Theorem 3 are satisfied. By Theorem 3, the conclusion follows. \square

4. Random Quasi-Variational Inequalities

In this section, by our existence theorems of random equilibria for random generalized games, namely, Theorem 3, some existence theorems of random quasi-variational inequalities and generalized random quasi-variational inequalities are given. Our results not only generalize the results of Tan [36] and Zhang [47], but also they are the stochastic versions of corresponding results in the literatures, e.g., see Aubin [3], Aubin and Ekeland [4], Hildenbrand and Sonnenschein [15], Shih and Tan [33], Tan [35, 36], Zhang [47], Zhou and Chen [48] and the references therein.

Here we emphasize that our arguments for the existence of solutions for non-compact random quasi-variational inequalities are different from the approaches used in the literatures by Tan [36] and Zhang [47].

Theorem 5. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and I be countable. For each $i \in I$, suppose that the following conditions are satisfied:*

- (a) X_i is a non-empty convex and closed Suslin subset of a locally convex Hausdorff topological vector space such that X_i has the property (K) and $X = \Pi_{i \in I} X_i$ is paracompact;
- (b) for each fixed $\omega \in \Omega$, $A_i(\omega, \cdot): X = \Pi_{i \in I} X_i \rightarrow 2^{X_i}$ is upper semicontinuous with non-empty compact and convex values;
- (c) $\psi_i: \Omega \times X \times X_i \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is such that:
 - (c)₁: $x \mapsto \psi_i(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X_i$;
 - (c)₂: $x_i \notin \text{co}\{y \in X_i: \psi_i(\omega, x, y) > 0\}$ for each fixed $(\omega, x) \in \Omega \times X$;
 - (c)₃: for each fixed $\omega \in \Omega$, the set $\{x \in X: \alpha_i(\omega, x) > 0\}$ is open in X , where $\alpha_i: \Omega \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $\alpha_i(\omega, x) = \sup_{y_i \in A_i(\omega, x)} \psi_i(\omega, x, y_i)$ for each $(\omega, x) \in \Omega \times X$;
- (d) $\{(\omega, x) \in \Omega \times X: \alpha_i(\omega, x) > 0\}$, and $\{(\omega, x) \in \Omega \times X: \pi_i(x) \in A_i(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (e) for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there exists $x \in \text{co}(X_0(\omega) \cup \{y\})$ with $x_i \in \text{co}(A_i(\omega, y) \cap \{z \in X_i: \psi_i(\omega, y, z) > 0\})$.

Then there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that for $i \in I$, $\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega))$ and

$$\sup_{y \in A_i(\omega, \phi(\omega))} \psi_i(\omega, \phi(\omega), y) \leq 0$$

for all $\omega \in \Omega$.

Proof. For each $i \in I$, define $P_i: \Omega \times X \rightarrow 2^{X_i}$ by $P_i(\omega, x) = \{y \in X_i: \psi_i(\omega, x, y) > 0\}$ for each $(\omega, x) \in \Omega \times X$. We shall show that $G = (\Omega; X_i; A_i; P_i)_{i \in I}$ satisfies all hypotheses of Theorem 3 with $A_i = B_i$ for all $i \in I$.

Suppose $i \in I$ and $\omega \in \Omega$. By (c)₁, for each fixed $y \in X_i$, $(P_i(\omega, \cdot))^{-1}(y) = \{x \in X: \psi_i(\omega, x, y) > 0\}$ is open in X and by (c)₂, $x_i \notin \text{co}P_i(\omega, x)$ for each $x \in X$. This shows that $P_i(\omega, \cdot)$ is lower semicontinuous and is of class L and hence is L -majorized. By the definition of α_i , we note that $\{x \in X: A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\} = \{x \in X: \alpha_i(\omega, x) > 0\}$, so that $\{x \in X: A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X by (c)₃. By (d), we know that $\text{Dom}(A_i \cap P_i) \in \Sigma \otimes \mathfrak{B}(X)$ and

$$\text{Proj}_{\Omega \times X}[\text{Graph}A_i \cap (\Omega \times \Delta_i)] \in \Sigma \otimes \mathfrak{B}(X).$$

Thus $G = (\Omega, X_i, A_i, P_i)_{i \in I}$ satisfies all hypothesis of Theorem 3 with $A_i = B_i$ for each $i \in I$. By

Theorem 3, there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that for each $i \in I$,

$$\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega)) \text{ and } A_i(\omega, \phi(\omega)) \cap P_i(\omega, \phi(\omega)) = \emptyset$$

for all $\omega \in \Omega$, i.e.,

$$\pi_i(\phi(\omega)) \in A_i(\omega, \phi(\omega)) \text{ and } \sup_{y \in A_i(\omega, \phi(\omega))} \phi_i(\omega, \phi(\omega), y) \leq 0$$

for all $\omega \in \Omega$. □

Letting $I = \{1\}$ in Theorem 5, we have the following existence results on random quasi-variational inequalities:

Theorem 6. *Let (Ω, Σ) be a measurable space with Σ a Suslin family. Suppose that the following conditions are satisfied:*

- (a) X is a non-empty closed paracompact convex and Suslin subset of a locally convex Hausdorff topological vector space, and X has the property (K);
- (b) for each fixed $\omega \in \Omega$, $A(\omega, \cdot): X \rightarrow 2^X$ is upper semicontinuous with non-empty compact and convex values;
- (c) $\psi: \Omega \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is such that:
 - (c)₁ $x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X$;
 - (c)₂ $x \notin \text{co}\{y \in X: \psi(\omega, x, y) > 0\}$ for each fixed $(\omega, x) \in \Omega \times X$;
 - (c)₃ for each fixed $\omega \in \Omega$, the set $\{x \in X: \alpha(\omega, x) > 0\}$ is open in X , where $\alpha: \Omega \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $\alpha(\omega, x) = \sup_{y \in A(\omega, x)} \psi(\omega, x, y)$ for each $(\omega, x) \in \Omega \times X$;
- (d) $\{(\omega, x) \in \Omega \times X: \alpha(\omega, x) > 0\}$, and $\{(\omega, x) \in \Omega \times X: x \in A(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (e) for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $y \in X \setminus K(\omega)$ there exist $x \in \text{co}(X_0(\omega) \cup \{y\})$ with $x \in \text{co}(A(\omega, y) \cap \{z \in X: \psi(\omega, y, z) > 0\})$.

Then there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that $\phi(\omega) \in A(\omega, \phi(\omega))$ and

$$\sup_{y \in A(\omega, \phi(\omega))} \psi(\omega, \phi(\omega), y) \leq 0$$

for all $\omega \in \Omega$.

4. Generalized Random Quasi-Variational Inequalities

Let (Ω, Σ) be a measurable space, X a non-empty compact convex subset of a locally convex Hausdorff topological vector E and E^* the dual space of E . Suppose the correspondences $F: \Omega \times X \rightarrow 2^X$, $T: \Omega \times X \rightarrow 2^{E^*}$ and the function $f: \Omega \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ are given. We want to find a measurable mapping $\psi: \Omega \rightarrow X$ which satisfies the following generalized random quasi-variational inequalities:

$$\begin{cases} \psi(\omega) \in F(\omega, \psi(\omega)) \\ \sup_{y \in F(\omega, \psi(\omega))} [\sup_{u \in T(\omega, \psi(\omega))} \text{Re}\langle u, \psi(\omega) - y \rangle + f(\omega, \psi(\omega), y)] \leq 0 \end{cases} \quad (*)$$

for each $\omega \in \Omega$. We also want to find two measurable maps $\psi: \Omega \rightarrow X$ and $\phi: \Omega \rightarrow E^*$ such that

$$\begin{cases} \psi(\omega) \in F(\omega, \psi(\omega)) \text{ and } \phi(\omega) \in T(\omega, \psi(\omega)) \\ \text{Re}\langle \phi(\omega), \psi(\omega) - y \rangle + f(\omega, \psi(\omega), y) \leq 0 \end{cases} \quad (**)$$

for all $y \in F(\omega, \psi(\omega))$ and for all $\omega \in \Omega$.

In this section, by applying results in Section 3, we shall consider the generalized random variational inequality problems (*) and (**) above.

Now we recall some definitions (e.g., see [48]). Let X be a non-empty convex subset of topological vector space E . A function $\psi(\omega, y): X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be

- (1) γ -diagonally quasi-convex (respectively, γ -diagonally quasi-concave) in y , in short γ -DQCX (respectively, γ -DQCV) in y , if for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\gamma \leq \max_{x \in A} \psi(y, x)$ (respectively, $\gamma \geq \inf_{x \in A} \psi(y, x)$);
- (2) γ -diagonally convex (respectively, γ -diagonally concave) in y , in short γ -DCX (respectively, γ -DCV) in y , if for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ with $y = \sum_{i=1}^m \lambda_i y_i$ ($\lambda_i \geq 0$, and $\sum_{i=1}^m \lambda_i = 1$), we have $\gamma \leq \sum_{i=1}^m \lambda_i \psi(y, y_i)$ (respectively, $\gamma \geq \sum_{i=1}^m \lambda_i \psi(y, y_i)$).

Let X and Y be two non-empty convex subsets of E , we also recall that a function $\psi: X \times Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is quasi-convex (respectively, quasi-concave) in y , if for each fixed $x \in X$, for each $A \in \mathcal{F}(Y)$ and each $y \in co(A)$, $\psi(x, y) \leq \max_{z \in A} \psi(x, z)$ (respectively, $\psi(x, y) \geq \min_{z \in A} \psi(x, z)$). Moreover, it is easy to verify that

- (i) if $\psi(x, y)$ is γ -DCX (respectively, γ -DCV) in y , then $\psi(x, y)$ is γ -DQCX (respectively, γ -DQCV) in y ,
- (ii) if $\psi_i: X \times Y \rightarrow \mathbb{R}$ is γ -DCX (respectively, γ -DCV) in y for each $i = 1, 2, \dots, m$, then $\psi(x, y) = \sum_{i=1}^m a_i(x) \psi_i(x, y)$ is also γ -DCX (respectively, γ -DCV) in y , where $a_i: X \rightarrow \mathbb{R}$ with $a_i(x) \geq 0$ and $\sum_{i=1}^m a_i(x) = 1$ for each $x \in X$, and
- (iii) the function $\psi(x, y): X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is 0-DQCV in y if and only if $x \notin co(\{y \in X: \psi(x, y) > 0\})$ for each $x \in X$.

In what follows, we first consider the existence of solutions of problem (*) for which monotonicity is needed.

Theorem 7. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty closed paracompact convex and Suslin subset of a locally convex Hausdorff topological vector space E such that X has the property (K). Suppose that the following conditions are satisfied:*

- (i) $F: \Omega \times X \rightarrow 2^X$ is such that for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$ is upper semicontinuous with non-empty compact and convex values;
- (ii) $T: \Omega \times X \rightarrow 2^{E^*}$ is such that for each fixed $\omega \in \Omega$, $T(\omega, \cdot)$ is monotone (i.e., $Re(u - v, y - x) \geq 0$ for all $u \in T(\omega, y)$ and $v \in T(\omega, x)$ for all $x, y \in X$) with non-empty values and for each one-dimensional flat $L \subset E$, $T(\omega, \cdot)|_{L \cap X}$ is lower semicontinuous from the relative topology of X into the weak*-topology $\sigma(E^*, E)$ of E^* ;
- (iii) $f: \Omega \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is such that $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X$ and for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$ for each $(\omega, x) \in \Omega \times X$;
- (iv) for each fixed $\omega \in \Omega$, the set $\{x \in X: \sup_{y \in F(\omega, x)} [\sup_{u \in T(\omega, y)} Re(u, x - y) + f(\omega, x, y)] > 0\}$ is open in X ;
- (v) $\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} [\sup_{u \in T(\omega, y)} Re(u, x - y) + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathcal{B}(X)$;
- (vi) $\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathcal{B}(X)$;
- (vii) for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in co(X_0(\omega) \cup \{x\})$ with $y \in co(F(\omega, x) \cap \{z \in X: \sup_{u \in T(\omega, z)} Re(u, x - z) + f(\omega, x, z) > 0\})$.

Then there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $y \in F(\omega, \phi(\omega))$ and $\omega \in \Omega$.

Proof. Define a function $\psi: \Omega \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(\omega, x, y) = \sup_{u \in T(\omega, y)} Re\langle u, x - y \rangle + f(\omega, x, y)$$

for each $(\omega, x, y) \in \Omega \times X \times X$. By (iii), $x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X for each $(\omega, y) \in \Omega \times X$. For each $\omega \in \Omega$, since $T(\omega, \cdot)$ is monotone, by (iii), it is easy to verify that $\psi(\omega, x, y)$ is 0-DCV in y by Proposition 3.2 of Zhou and Chen [48]. The conditions (i)-(vi) imply that all hypotheses of Theorem 6 are satisfied. By Theorem 6, there exists a measurable mapping $\Phi: \Omega \rightarrow X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \sup_{u \in T(\omega, y)} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0 \tag{1}$$

for all $\omega \in \Omega$. We shall now prove that

$$\sup_{y \in F(\omega, \phi(\omega))} \sup_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for each $\omega \in \Omega$.

Fix an $\omega \in \Omega$. Let $x \in F(\omega, \phi(\omega))$ be arbitrarily given and let $z_t(\omega) = tx + (1-t)\phi(\omega) = \phi(\omega) - t(\phi(\omega) - x)$ for $t \in [0, 1]$. As $F(\omega, \phi(\omega))$ is convex, we have $z_t(\omega) \in F(\omega, \phi(\omega))$ for $t \in [0, 1]$. Therefore, by (1) we have

$$\sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - z_t(\omega) \rangle + f(\omega, \phi(\omega), z_t(\omega))] \leq 0$$

for all $t \in [0, 1]$.

Since for each $x \in X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$, it follows that for $t \in (0, 1]$,

$$\begin{aligned} & t \cdot \left\{ \sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - x \rangle] + f(\omega, \phi(\omega), x) \right\} \\ & \leq \sup_{u \in T(\omega, z_t(\omega))} t \cdot [Re\langle u, \phi(\omega) - x \rangle] + f(\omega, \phi(\omega), tx + (1-t)\phi(\omega)) \\ & = \sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - z_t(\omega) \rangle] + f(\omega, \phi(\omega), z_t(\omega)) \leq 0 \end{aligned}$$

which implies that for $t \in (0, 1]$,

$$\sup_{u \in T(\omega, z_t(\omega))} [Re\langle u, \phi(\omega) - x \rangle] + f(\omega, \phi(\omega), x) \leq 0. \tag{2}$$

Let $z_0 \in T(\omega, \phi(\omega))$ be arbitrarily fixed. For each $\epsilon > 0$, let

$$U_{z_0} = \{z \in E^*: |Re\langle z_0 - z, \phi(\omega) - x \rangle| < \epsilon\}.$$

Then U_{z_0} is a $\sigma(E^*, E)$ -neighborhood of z_0 . Since $T(\omega, \cdot) |_{L \cap X}$ is lower semicontinuous where $L := \{z_t^0(\omega): t \in [0, 1]\}$, and $U_{z_0} \cap T(\omega, \phi(\omega)) \neq \emptyset$, there exists a neighborhood $N(\phi(\omega))$ of $\phi(\omega)$ in L such that if $z \in N(\phi(\omega))$, then $T(\omega, \phi(\omega)) \cap U_{z_0} \neq \emptyset$. But then there exists $\delta \in (0, 1]$ such that $z_t(\omega) \in N(\phi(\omega))$ for all $t \in (0, \delta)$. Fixing any $t \in (0, \delta)$ and $u \in T(\omega, z_t(\omega)) \cap U_{z_0}$, we have

$| \operatorname{Re}\langle z_0 - u, \phi(\omega) - z \rangle | < \epsilon$. This implies that

$$\operatorname{Re}\langle z_0, \phi(\omega) - x \rangle < \operatorname{Re}\langle u, \phi(\omega) - z \rangle + \epsilon.$$

Thus

$$\operatorname{Re}\langle z_0, \phi(\omega) - x \rangle + f(\omega, \phi(\omega), x) < \operatorname{Re}\langle u, \phi(\omega) - z \rangle + f(\omega, \phi(\omega), x) + \epsilon < \epsilon$$

by (2). Since $\epsilon > 0$ is arbitrary, $\operatorname{Re}\langle z_0, \phi(\omega) - x \rangle + f(\omega, \phi(\omega), x) \leq 0$. As $z_0 \in T(\omega, \phi(\omega))$, is arbitrary, we have the following

$$\sup_{z \in T(\omega, \phi(\omega))} [\operatorname{Re}\langle z, \phi(\omega) - x \rangle + f(\omega, \phi(\omega), x)] \leq 0$$

for all $x \in F(\omega, \phi(\omega))$. □

Corollary 8. *Let (Ω, Σ) be a measurable space with Σ a Suslin family, X a non-empty compact convex Suslin subset of a locally convex Hausdorff topological vector space E and $F: \Omega \times X \rightarrow 2^X$ be such that $\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$. If for each fixed $\omega \in \Omega$, $F(\omega, \cdot)$ is upper semicontinuous with non-empty compact convex values, then F has a random fixed point.*

We shall now observe that in Theorem 7, the interaction between the correspondences T and F (namely, the condition (iv)) can be achieved by imposing additional continuity conditions on T and F .

Theorem 9. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty closed paracompact convex and Suslin bounded subset of a locally convex Hausdorff topological vector space E such that X has the property (K). If $F: \Omega \times X \rightarrow 2^X$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values, and $T: \Omega \times X \rightarrow 2^{E^*}$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values, and $T: \Omega \times X \rightarrow 2^{E^*}$ is such that for each given $\omega \in \Omega$, $T(\omega, \cdot)$ is monotone with non-empty values and is lower semicontinuous from the relative topology of X to the strong topology of E^* . Suppose that*

- (i) $f: \Omega \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is such that for each given $\omega \in \Omega$, $(x, y) \mapsto f(\omega, x, y)$ is lower semicontinuous and for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is concave and $f(\omega, x, x) = 0$ for each $(\omega, x) \in \Omega \times X$;
- (ii) the set $\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \sup_{u \in T(\omega, y)} [\operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (iii) $\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (iv) for each given $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in \operatorname{co}(X_0(\omega) \cup \{x\})$ with $y \in \operatorname{co}(F(\omega, x) \cap \{z \in X: \sup_{u \in T(\omega, z)} \operatorname{Re}\langle u, x - z \rangle + f(\omega, x, z) > 0\})$.

Then there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \left[\sup_{u \in T(\omega, \phi(\omega))} \operatorname{Re}\langle u\phi(\omega) - y \rangle + f(\omega, \phi(\omega), y) \right] \leq 0$$

for all $\omega \in \Omega$.

Proof. By Theorem 7, we need only show that for each given $\omega \in \Omega$, the set $\Sigma(\omega): = \{x \in X: \sup_{y \in F(\omega, x)} \left[\sup_{u \in T(\omega, y)} \operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y) \right] > 0\}$

is open in X .

Since X is bounded and $f(\omega, \cdot, \cdot)$ is lower semicontinuous, the function $(u, x, y) \mapsto \text{Re}\langle u, x - y \rangle + f(\omega, x, y)$ is lower semicontinuous from $E^* \times X \times X$ to \mathbb{R} for each fixed $\omega \in \Omega$. Therefore $(x, y) \mapsto \sup_{u \in T(\omega, y)} [\text{Re}\langle u, x - y \rangle + f(\omega, x, y)]$ is also lower semicontinuous by lower semicontinuity of $T(\omega, \cdot)$ and Proposition III-19 of Aubin and Ekeland [4, p.118]. Since $F(\omega, \cdot)$ is lower semicontinuous, $x \mapsto \sup_{y \in F(\omega, x)} \sup_{u \in T(\omega, y)} [\text{Re}\langle u, x - y \rangle + f(\omega, x, y)]$ is lower semicontinuous by Proposition III-19 of [4, p. 118] again for each fixed $\omega \in \Omega$. Thus the set $\Sigma(\omega) := \{x \in X : \sup_{y \in F(\omega, x)} \sup_{u \in T(\omega, x)} [\text{Re}\langle u, x - y \rangle + f(\omega, x, y)] > 0\}$ is open in X . \square

Now we will consider the existence of solutions for the problems (*) and (**) without assuming the monotonicity as in Theorem 9.

Theorem 10. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex and Polish subset of a locally convex Hausdorff topological vector space E . Suppose that:*

- (i) $F: \Omega \times X \rightarrow 2^X$ is such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is upper semicontinuous with non-empty compact and convex values;
- (ii) $T: \omega \times X \rightarrow 2^{E^*}$ is such that $x \mapsto \inf_{u \in T(\omega, x)} \text{Re}\langle u, x - y \rangle$ is lower semicontinuous for each $(\omega, y) \in \Omega \times X$;
- (iii) $f: \Omega \times X \times X \rightarrow \mathbb{R}$ is such that $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X for each fixed $(\omega, y) \in \Omega \times X$; and for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is 0-diagonal concave;
- (iv) for each given $\omega \in \Omega$, the set

$$\{x \in X : \sup_{y \in F(\omega, x)} [\inf_{u \in T(\omega, x)} \text{Re}\langle u, x - y \rangle + f(\omega, x, y)] > 0\}$$

is open in X ;

- (v) $\{(\omega, x) \in \Omega \times X : \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\text{Re}\langle u, x - y \rangle + f(\omega, x, y)] > 0\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (vi) $\{(\omega, x) \in \Omega \times X : x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (vii) for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in \text{co}(X_0(\omega) \cap \{x\})$ with $y \in \text{co}(F(x) \cap \{z \in X : \sup_{u \in T(\omega, z)} \text{Re}\langle u, x - z \rangle + f(\omega, x, z) > 0\})$.

Then there exists a measurable mapping $\phi: \Omega \times X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\inf_{u \in T(\omega, \phi(\omega))} [\text{Re}\langle u, \phi(\omega) - y \rangle] + f(\omega, \phi(\omega), y) \leq 0$$

for all $y \in F(\omega, \phi(\omega))$ and $\omega \in \Omega$.

Suppose that in addition,

- (1) for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is lower semicontinuous and concave and f is measurable;
- (2) there exists a non-empty Polish subset E_0^* of E^* such that $T(\Omega, X) \subset E_0^*$, T is measurable with non-empty strongly compact convex values; and
- (3) F is measurable.

Then there exist a measurable function $\rho: \Omega \rightarrow E^*$ such that $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [\text{Re}\langle \rho(\omega), \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

Proof. Define $\psi: \Omega \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi(\omega, x, y) = \inf_{u \in T(\omega, x)} [\text{Re}\langle u, x - y \rangle + f(\omega, x, y)],$$

for each $(\omega, x, y) \in \Omega \times X \times X$. Then by (ii), (iii) and (iv) we have:

- (a) for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto \psi(\omega, x, y)$ is lower semicontinuous on X and $x \notin \text{co}(\{y \in X: \psi(\omega, x, y) > 0\})$ for each $(\omega, x) \in \Omega \times X$;
- (b) for each $\omega \in \Omega$, the set $\{x \in X: \sup_{y \in F(\omega, x)} \psi(\omega, x, y) > 0\}$ is open in X .

Therefore, F and ψ satisfy all conditions of Theorem 6. By Theorem 6 there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \inf_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

If, in addition, the conditions (1), (2) and (3) hold, we shall find another measurable (single-valued) mapping $\rho: \Omega \rightarrow E^*$ such that $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [Re\langle \rho(\omega), \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for each $\omega \in \Omega$.

Fix any $\omega \in \Omega$. Define $f_1: F(\omega, \phi(\omega)) \times T(\omega, \phi(\omega)) \rightarrow \mathbb{R}$ by

$$f_1(y, u) = Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)$$

for each $(y, u) \in F(\omega, \phi(\omega)) \times T(\omega, \phi(\omega))$. Then for each $y \in F(\omega, \phi(\omega))$, $u \mapsto f_1(y, u)$ is lower semicontinuous and convex and for each fixed $u \in T(\omega, \phi(\omega))$, $y \mapsto f_1(y, u)$ is concave. By Kneser's Minimax Theorem [20],

$$\inf_{u \in T(\omega, \phi(\omega))} \sup_{y \in F(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] =$$

$$\sup_{y \in F(\omega, \phi(\omega))} \inf_{u \in T(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0.$$

Since $T(\omega, \phi(\omega))$ is compact, there exists $u_0 \in T(\omega, \phi(\omega))$ such that

$$\sup_{y \in F(\omega, \phi(\omega))} [Re\langle u_0, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0.$$

Now we define $\Phi, T_1: \Omega \rightarrow 2^X$ by

$$\Phi(\omega) = \{u \in T(\omega, \phi(\omega)): \sup_{y \in F(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0\}$$

and

$$T_1(\omega) = T(\omega, \phi(\omega))$$

for each $\omega \in \Omega$. Note that $\Phi(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Since T and ϕ are measurable, T_1 is also measurable by Lemma 3 in [28, p. 55]. Define $g_1: \Omega \times X \times X \times E_0^* \rightarrow \mathbb{R}$ by

$$g_1(\omega, x, y, u) = Re\langle u, x - y \rangle + f(\omega, x, y)$$

for each $(\omega, x, y, u) \in \Omega \times X \times X \times E_0^*$. Then g_1 is measurable. Also we define $g_2: \Omega \times X \times E_0^* \rightarrow \mathbb{R}$ by

$$g_2(\omega, y, u) = Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)$$

for each $(\omega, y, u) \in \Omega \times X \times E_0^*$. Now define $F_1: \Omega \times 2^X$ by

$$F_1(\omega) = F(\omega, \phi(\omega))$$

for each $\omega \in \Omega$. Since ϕ is measurable and F is also measurable, g_2 and F_1 are measurable by Lemma 3 in [28, p. 55] again. Let $g_3: \Omega \times E_0^* \rightarrow \mathbb{R}$ by

$$g_3(\omega, u) = \sup_{y \in F(\omega, \phi(\omega))} g_2(\omega, y, u) = \sup_{y \in F(\omega, \phi(\omega))} [Re\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)]$$

for each $(\omega, u) \in \Omega \times E_0^*$. We shall show that g_3 is measurable. Since F_1 is measurable by Theorem B, there exists a countable family of measurable mappings $p_n: \Omega \rightarrow X$ such that $F_1(\omega) = cl\{p_n(\omega): n = 1, 2, \dots\}$ for each $\omega \in \Omega$. Since ϕ is measurable, for each fixed $(u, y) \in E^* \times X$, the mapping $\omega \mapsto Re\langle u, \phi(\omega) - y \rangle$ is measurable. Note that the mapping $(u, y) \mapsto Re\langle u, \phi(\omega) - y \rangle$ is continuous, so that the mapping $(\omega, u, y) \mapsto Re\langle u, \phi(\omega) - y \rangle$ is measurable by Theorem III.14 of Castaing and Valadier [9, p. 70]. For each $n \in \mathbb{N}$, the function $g'_n: \Omega \times E^* \rightarrow \mathbb{R}$, defined by

$$g'_n(\omega, u) = Re\langle u, \phi(\omega) - p_n(\omega) \rangle + f(\omega, \phi(\omega), p_n(\omega))$$

for each $(\omega, u) \in \Omega \times E^*$, is measurable. Therefore, for each $n \in \mathbb{N}$, the mapping $(\omega, u) \mapsto Re\langle u, \phi(\omega) - p_n(\omega) \rangle + f(\omega, \phi(\omega), p_n(\omega))$ is also measurable. Since for each $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is lower semicontinuous, it follows that for each $r \in \mathbb{R}$,

$$\{(\omega, u) \in \Omega \times E^*: g_3(\omega, u) \leq r\} = \bigcap_{n=1}^{\infty} \{(\omega, u) \in \Omega \times E^*: g'_n(\omega, u) \leq r\} \in \Sigma \otimes \mathfrak{B}(E^*).$$

Therefore the function g_3 is measurable so that the set $M_0 = \{(\omega, u) \in \Omega \times E_0^*: g_3(\omega, u) \leq 0\} \in \Sigma \otimes \mathfrak{B}(E^*)$. Hence $Graph\Phi = (GraphT_1) \cap M_0 \in \Sigma \otimes \mathfrak{B}(E_0^*)$. By Theorem B, there exists a measurable mapping $\rho: \Omega \rightarrow E_0^*$ such that $\rho(\omega) \in \Phi(\omega)$ for each $\omega \in \Omega$. By the definition of Φ , the measurable mapping ρ satisfies the following:

$$\begin{cases} \phi(\omega) \in F(\omega, \phi(\omega)) \text{ and } \rho(\omega) \in T(\omega, \phi(\omega)) \\ \sup_{y \in F(\omega, \phi(\omega))} [Re\langle \rho(\omega), \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0. \end{cases} \quad \square$$

Note that if $T: \Omega \times X \rightarrow 2^{E^*}$ is such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact values, then by Lemma 2 of Kim and Tan in [19, p. 140] or Theorem 1 of Aubin in [3, p. 67], the condition (ii) of Theorem 10 is satisfied. Thus Theorem 10 is a stochastic version of Theorem 3 of Shih and Tan in [33, p. 340]. Recall that for a topological vector space E , the strong topology on its dual space E^* is the topology on E^* generated by the family $\{U(B; \epsilon): B \text{ is a non-empty bounded subset of } E \text{ and } \epsilon > 0\}$ as a base for the neighborhood system at zero, where $U(B; \epsilon) = \{f \in E^*: \sup_{x \in B} |Re\langle f, x \rangle| < \epsilon\}$.

Now if we impose the upper semicontinuity condition to correspondence T , then we have the following:

Theorem 11. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex and Polish bounded subset of a locally convex Hausdorff topological vector space E . Suppose*

- (i) $F: \Omega \times X \rightarrow 2^X$ is random continuous with non-empty compact and convex values;
- (ii) $T: \Omega \times X \rightarrow 2^{E^*}$ is such that for each given $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact and convex values;
- (iii) $f: \Omega \times X \times X \rightarrow \mathbb{R}$ is such that
 - (a) for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X ;

- (b) for each fixed $(\omega, x) \in \Omega \times X$, $y \mapsto f(\omega, x, y)$ is 0-diagonally concave;
- (iv) $\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} \operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y) > 0\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (v) for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in \operatorname{co}(X_0(\omega) \cup \{x\})$ with $y \in \operatorname{co}(F(x) \cap \{z \in X: \sup_{u \in T(\omega, z)} \operatorname{Re}\langle u, x - z \rangle + f(\omega, x, z) > 0\})$.

Then,

- (a) for each fixed $\omega \in \Omega$, the set $\{x \in X: \sup_{y \in F(\omega, x)} [\inf_{u \in T(\omega, x)} \operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y)] > 0\}$ is open in X ;
- (b) $\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$;
- (c) there exists a measurable mapping $\phi: \Omega \rightarrow X$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$ and

$$\inf_{u \in T(\omega, \phi(\omega))} [\operatorname{Re}\langle u, \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $y \in F(\omega, \phi(\omega))$ and $\omega \in \Omega$.

Proof. (a) Fix $\omega \in \Omega$. Since X is a bounded subset of the locally convex Hausdorff topological vector space E , and E^* is equipped with the strong topology, the function $\psi_1: E^* \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, defined by

$$\psi_1(u, x, y) = \operatorname{Re}\langle u, x - y \rangle$$

for each $(u, x, y) \in E^* \times X \times X$, is continuous. Since $T(\omega, \cdot): X \rightarrow 2^{E^*}$ is upper semicontinuous with non-empty strongly compact values, by Theorem 1 of Aubin [3, p. 67], the function $\psi_2: X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$\psi_2(x, y) = \inf_{u \in T(\omega, x)} \operatorname{Re}\langle u, x - y \rangle$$

for each $(x, y) \in X \times X$, is also lower semicontinuous. Thus the mapping $(x, y) \mapsto \inf_{u \in T(\omega, x)} \operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y)$ is lower semicontinuous by (iii). As $F(\omega, \cdot): X \rightarrow 2^X$ is lower semicontinuous with non-empty values, by Proposition III-19 in [4, p. 118], the mapping $x \mapsto \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y)]$ is lower semicontinuous from X to $\mathbb{R} \cup \{-\infty, +\infty\}$ for each fixed $\omega \in \Omega$, so that the set

$$\Sigma(\omega) = \{x \in X: \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y)] > 0\}$$

is open in X .

(b) Since F is random continuous with closed values, by Theorem 3.5 in [17, p. 57] and Lemma 2.5 of Tan and Yuan [37], the set $\{(\omega, x) \in \Omega \times X: x \in F(\omega, x)\} \in \Sigma \otimes \mathfrak{B}(X)$.

Thus all hypotheses of Theorem 10 are satisfied, the conclusion follows. □

If both correspondences T and F are measurable, we have the following:

Theorem 12. Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty convex and Polish bounded subset of a locally convex Hausdorff topological vector space E . Suppose that

- (i) $F: \Omega \times X \rightarrow 2^X$ is measurable such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values;
- (ii) $T: \Omega \times X \rightarrow 2^{E^*}$ is measurable such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact and convex values;

- (iii) $f: \Omega \times X \times X \rightarrow \mathbb{R}$ is measurable such that
 - (a) for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X ;
 - (b) for each fixed $(\omega, x) \in \Omega \times X$, $f(\omega, x, x) = 0$ and $y \mapsto f(\omega, x, y)$ is lower semicontinuous and concave;
- (iv) for each $\omega \in \Omega$, there exist a non-empty compact convex subset $X_0(\omega)$ of X and a non-empty compact subset $K(\omega)$ of X such that for each $x \in X \setminus K(\omega)$ there exists $y \in \text{co}(X_0(\omega) \cup \{x\})$ with $y \in \text{co}(F(\omega) \cap \{z \in X: \sup_{u \in T(\omega, z)} \text{Re}\langle u, x - z \rangle + f(\omega, x, z) > 0\})$.

Then there exist measurable maps $\phi: \Omega \rightarrow X$ and $\rho: \Omega \rightarrow E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [\text{Re}\langle \rho(\omega), \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

Proof. By Theorem 10 and Theorem 11, it remains to prove that $\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} \text{Re}\langle u, x - y \rangle + f(\omega, x, y) > 0\} \in \Sigma \otimes \mathfrak{B}(X)$.

Since T and F are measurable, by Theorem 4.2 (e) of Wagner [44], there exist two countable families of measurable maps $p_n: \Omega \times X \rightarrow X$ and $q_n: \Omega \times X \rightarrow E^*$ such that $F(\omega, x) = \text{cl}\{p_n(\omega, x): n = 1, 2, \dots\}$ and $T(\omega, x) = \text{cl}\{q_n(\omega, x): n = 1, 2, \dots\}$ for each $(\omega, x) \in \Omega \times X$. We define $g_0: E^* \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$g_0(u, x, y) = \text{Re}\langle u, x - y \rangle$$

for each $(u, x, y) \in E^* \times X \times X$. Then g_0 is continuous and is measurable. Therefore the function $g'_0: \Omega \times E^* \times X \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g'_0(\omega, u, x, y) = \text{Re}\langle u, x - y \rangle + f(\omega, x, y)$$

for each $(\omega, u, x, y) \in \Omega \times E^* \times X \times X$, is also measurable since f is measurable. Now fix any $x \in \mathbb{N}$, note that $p_n: \Omega \times X \rightarrow X$ is measurable and f is measurable. For each $j \in \mathbb{N}$, the function $g_j^n: \Omega \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g_j^n(\omega, x) = \text{Re}\langle q_j(\omega, x), x - p_n(\omega, x) \rangle + f(\omega, x, p_n(\omega, x))$$

for each $(\omega, x) \in \Omega \times X$, is measurable by Lemma 3 in [28, p. 55]. Therefore the mapping $g_n: \Omega \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g_n(\omega, x) = \inf_{j \in \mathbb{N}} g_j^n(\omega, x) = \inf_{j \in \mathbb{N}} [\text{Re}\langle q_j(\omega, x), x - p_n(\omega, x) \rangle + f(\omega, x, p_n(\omega, x))]$$

for each $(\omega, x) \in \Omega \times X$, is measurable. Note that $g: \Omega \times X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$g(\omega, x) = \sup_{n \in \mathbb{N}} g_n(\omega, x)$$

for each $(\omega, x) \in \Omega \times X$, is also measurable. Since for each $(\omega, x) \in \Omega \times X$, the mapping $y \mapsto f(\omega, x, y)$ is lower semicontinuous, the set

$$\begin{aligned} & \{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} [\text{Re}\langle u, x - y \rangle + f(\omega, x, u)] > 0\} \\ &= \{(\omega, x) \in \Omega \times X: \sup_{n \in \mathbb{N}} \inf_{j \in \mathbb{N}} [\text{Re}\langle q_j(\omega, x), x - p_n(\omega, x) \rangle + f(\omega, x, p_n(\omega, x))] > 0\} \\ &= \{(\omega, x): g(\omega, x) > 0\} \in \Sigma \otimes \mathfrak{B}(X). \end{aligned}$$

Therefore, we have

$$\{(\omega, x) \in \Omega \times X: \sup_{y \in F(\omega, x)} \inf_{u \in T(\omega, x)} \operatorname{Re}\langle u, x - y \rangle + f(\omega, x, y) > 0\} \in \Sigma \otimes \mathfrak{B}(X). \quad \square$$

Corollary 13. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty compact convex subset of a Banach space E whose dual space E^* is separable. Suppose that*

- (i) $F: \Omega \times X \rightarrow 2^X$ is measurable such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values;
- (ii) $T: \Omega \times X \rightarrow 2^{E^*}$ is measurable such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact and convex values;
- (iii) $f: \Omega \times X \times X \rightarrow \mathbb{R}$ is measurable such that
 - (a) for each fixed $(\omega, y) \in \Omega \times X$, $x \mapsto f(\omega, x, y)$ is lower semicontinuous on X ;
 - (b) for each fixed $(\omega, x) \in \Omega \times X$, $f(\omega, x, x) = 0$ and $y \mapsto f(\omega, x, y)$ is lower semicontinuous and concave.

Then there exist measurable maps $\phi: \Omega \rightarrow X$ and $\rho: \Omega \rightarrow E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} [\operatorname{Re}\langle \rho(\omega), \phi(\omega) - y \rangle + f(\omega, \phi(\omega), y)] \leq 0$$

for all $\omega \in \Omega$.

By allowing f to be zero in Corollary 13, we have the following:

Corollary 14. *Let (Ω, Σ) be a measurable space with Σ a Suslin family and X a non-empty compact convex subset of a Banach space E whose dual space E^* is separable. Suppose that*

- (i) $F: \Omega \times X \rightarrow 2^X$ is measurable such that for each $\omega \in \Omega$, $F(\omega, \cdot)$ is continuous with non-empty compact and convex values;
- (ii) $T: \Omega \times X \rightarrow 2^{E^*}$ is measurable such that for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty strongly compact and convex values.

Then there exist measurable map $\phi: \Omega \rightarrow X$ and $\rho: \Omega \rightarrow E^*$ such that $\phi(\omega) \in F(\omega, \phi(\omega))$, $\rho(\omega) \in T(\omega, \phi(\omega))$ and

$$\sup_{y \in F(\omega, \phi(\omega))} \operatorname{Re}\langle \rho(\omega), \phi(\omega) - y \rangle \leq 0$$

for all $\omega \in \Omega$.

Theorem 11 is also a stochastic version of Theorem 4 of Shih and Tan in [33, p. 341] (and its improvements due to Kim [18] and due to Shih and Tan [33, Theorem 2, p. 69-70] (with $M = 0$)).

Theorem 11 generalizes a theorem of Tan [36, p. 326] in the following ways:

- (1) the correspondence T is upper semicontinuous instead of being continuous, and
- (2) the function f need not be random continuous.

In the case where $F(x) = X$ and $T(x) = 0$ for each $x \in X$, Theorem 11 also improves Theorem 9.2.3 of Zhang [47, p. 304] with weaker continuity and measurability conditions. We also remark that our arguments used in proving the existence of solutions for generalized random quasi-variational inequalities in this section are different from those used by Tan [36] and Zhang [47], etc.

Quasi-variational inequalities and generalized quasi-variational inequalities have many applications in mathematical economics, game theory and optimization and other applied science (e.g., see [3-4], [7], [15] and [25]). Random quasi-variational inequalities and generalized random quasi-variational inequalities will also have many applications in random mathematical economics, random game theory and related fields.

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