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A NOTE ON A GENERALIZED STANDARD ORIENTATION DISTRIBUTION IN PDF-COMPONENT FIT METHODS

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A generalized model orientation distribution which was recently introduced into texture analysis is identified as von Mises-Fisher matrix distribution on $SO(3)$ or, equivalently, as Bingham distribution on $S^4_+ \subset IR^4$. The one-one correspondence of the distributions is explicitly given.

KEY WORDS: Orthogonal group $SO(3)$, von Mises-Fisher distribution, singular value decomposition, polar decomposition, Rodrigues parameters, hypersphere S^4, Bingham distribution

1 THE VON MISES-FISHER MATRIX DISTRIBUTION ON $SO(3)$

In (Eschner, 1993) the author derived a generalized standard model orientation distribution (odf) which he suggests and elaborates for application in texture component fit methods, where the corresponding model pole distributions (pdf's) are fitted to experimental pdfs. This note is to add some details which seem to have slipped the attention.

The new generalized standard model odf is defined by Eschner (1993, p. 141, eq. 10) as

$$f(g, g_d, g_0, \alpha_1, \alpha_2, \alpha_3) = \mathrm{C}(\alpha_1, \alpha_2, \alpha_3)\exp[\mathrm{tr}(\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)M(g_d^{-1})M(g)M(g_0^{-1})M(g_d))][dM(g)] \quad (1)$$

where $\mathrm{tr}(A)$ denotes the trace of a square matrix A, $\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ denotes the diagonal matrix with entries $\alpha_1, \alpha_2, \alpha_3$, $M(\mathrm{o})$ denotes the matrix $\mathrm{M} \in SO(3)$ representing any orientation in $G \subset SO(3)$, and $[dM(\mathrm{o})]$ denotes the uniform distribution on $SO(3)$. The generalized standard distribution is derived by geometric reasoning, where g_0 denotes the mode ("center of gravity") of the distribution, and g_d and $\alpha_1, \alpha_2, \alpha_3$ provide a measure of variation with respect to the mode g_0 in terms of an (rotated) ellipsoid in $SO(3)$. $\alpha_1, \alpha_2, \alpha_3$ relate to the ellipsoid's main axes when it is canonically orientated (axes parallel to the axes of the sample coordinate system), while g_d denotes its actual orientation (Eschner, 1993, p. 141).

Applying $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, eq. (1) may be rewritten as

$$f(g, g_d, g_0, \alpha_1, \alpha_2, \alpha_3) = C(\alpha_1, \alpha_2, \alpha_3)\exp[\mathrm{tr}(M(g_d)\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)M^t(g_d)M(g_0)M^t(g))][dM(g)] \quad (2)$$

Simplifying the notation by $M(g_d) = \Delta$, $\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) = D_\alpha$, $M(g_0) = M$, and $M(g) = \mathrm{X}$, eq. (2) is rewritten as

$$\begin{aligned} f(X;\, M,\, K) &= C(\alpha_1, \alpha_2, \alpha_3)\exp[\mathrm{tr}(\Delta D_\alpha \Delta^t M X^t)][dX] \\ &= C(K)\exp[\mathrm{tr}(KMX^t)][dX] \\ &= C(F)\exp[\mathrm{tr}(FX^t)][dX] = f_{\mathrm{vMF}}(X;\, F) \end{aligned} \tag{3}$$

which is the von Mises-Fisher matrix distribution $M_3(F)$ on $SO(3)$ (Downs, 1972; Khatri and Mardia, 1977; Prentice, 1986) with (positive or negative) definite parameter matrix F and $X \in SO(3)$. Let F have singular value decomposition $F = \Delta D_\alpha \Gamma$, with Δ, $\Gamma \in SO(3)$, and $|\alpha_1| < |\alpha_2| < |\alpha_3|$, then $F = \Delta D_\alpha \Delta^t \Delta\Gamma$. The decomposition $F = KM$ with $K = \Delta D_\alpha \Delta^t$, $M = \Delta\Gamma \in SO(3)$, is referred to as polar decomposition into the elliptical component K and the polar component $M \in SO(3)$, cf. (Halmos, 1958; Gantmacher, 1960). If D_α is positive definite, then the distribution has a maximum in M; hence M may be interpreted as the mode of the distribution, and K as a measure of variation with respect to M. If D_α is negative definite, then the distribution has a minimum in M; hence M may be interpreted as the "pole", or rather "center", of an equatorial distribution, and K as a measure of variation with respect to the "girdle" or "fiber" with center M, cf. Schaeben, 1990; 1992. The rank deficient case $\mathrm{rank}(D_\alpha) = 2$ is not particularly interesting and will not be considered in greater detail.

The normalizing constant $C(\alpha_1, \alpha_2, \alpha_3)$ as provided by Eschner (1993, p. 143, eq. 20) can be simplified: it is given by a hypergeometric function (cf. Erdélyi *et al.*,1953) of matrix argument as defined by Herz (1955), $C(\alpha_1, \alpha_2, \alpha_3) = \{{}_0F_1(\frac{3}{2}\,;\, \frac{1}{4}D_\alpha^2)\}^{-1}$, cf. (Downs, 1972; Khatri and Mardia, 1977).

The family of von Mises-Fisher matrix distributions includes the uniform distribution for $\alpha_1 = \alpha_2 = \alpha_3 = 0$. In general, the distribution is not symmetrical. If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 0$ it reduces to the rotationally invariant unimodal von Mises-Fisher matrix distribution on $SO(3)$

$$f_{\mathrm{vMF}}(\omega(g, g_0);\alpha) = C(\alpha)\exp(\alpha\cos\omega)\ \sin^2\frac{\omega}{2}\,\mathrm{d}\omega \tag{4}$$

with $\omega(g, g_0) = \omega(gg_0^{-1})$ denoting the angle ω of the rotation gg_0^{-1}, which was labeled Gaussian standard odf in texture analysis by Matthies *et al.*, 1987, without reference to the von Mises-Fisher matrix distribution.

If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha < 0$ it reduces to the rotationally invariant equatorial von Mises-Fisher matrix distribution on $SO(3)$ with a maximum in the fiber with center g_0. It should be noted that the set $\{g \in SO(3) \mid g\mathbf{x} = \mathbf{y}\}$ for any given $\mathbf{x}$, $\mathbf{y} \in S^3$ forms a circle on S^4. Let g_0 be the center of the circle $\{g \in SO(3) \mid g\mathbf{x} = \mathbf{y}\}$. Then with $\alpha < 0$

$$\begin{aligned} f_{\mathrm{vMF}}(\omega(g, g_0);\alpha) &= C(\alpha)\exp[\alpha\cos\omega(gg_0^{-1})]\ [\mathrm{d}\omega] \\ &= C(\alpha)\exp[-\alpha\cos(\pi - \omega)]\ [\mathrm{d}\omega] \\ &= C(\alpha)\exp(-\alpha\cos\eta)\ [\mathrm{d}\omega] \\ &= C(\alpha)\exp(-\alpha\mathbf{y}g\mathbf{x})\ [\mathrm{d}\omega] \\ &= C(\kappa)\exp(\kappa\mathbf{y}g\mathbf{x})\ [\mathrm{d}\omega] \\ &= f_{\mathrm{vMF}}(\eta(\mathbf{y}, g\mathbf{x});\, \kappa) \end{aligned} \tag{5}$$

with $\kappa = -\alpha > 0$ and $\eta(\mathbf{y}, g\mathbf{x}) = \arccos(\mathbf{y}g\mathbf{x})$, which was labeled Gaussian standard odf in case of a fiber texture by Matthies *et al.*, 1987, without reference to the von Mises-Fisher matrix distribution, and without discussion of the rôle of the sign of α, i.e. whether the parameter matrix F is positive or negative definite.

2 THE BINGHAM DISTRIBUTION ON S_+^4

Let $\mathbf{x} \in S_+^4$ be the Rodrigues unit vector (Becker and Panchanadeeswaran, 1989; Morawiec and Pospiech, 1989) representing the rotation $X = \mu(\mathbf{x}) \in SO(3)$. $\mathbf{x} \in S_+^4$ is said to be distributed according to the (even) Bingham distribution $B_4(L, A)$ (Bingham, 1964; 1974) if $\mathbf{x}$ has the density

$$f(\mathbf{x};\ L,\ A) = (c_B(L))^{-1}\exp[\mathrm{tr}(LA\mathbf{x}\mathbf{x}^tA^t)][ds] \tag{6}$$

where $[ds]$ represents the Lebesgue invariant area element on S_+^4, $A \in SO(4)$ is a (4 × 4) orthogonal matrix, L is a (4 × 4) diagonal matrix with entries $l_1,....,l_4$, and $c_B(L)$ is a normalizing constant depending on L only. It should be noted that $B_4(L + \lambda I_4, A)$ and $B_4(L, A)$ are indistinguishable for any $\lambda \in IR^1$, i. e. the shape parameters l_i, $i = 1,...,4$, are determined only up to an additive constant. Uniqueness can be imposed by some convention. Different sets of entries of L in eq. (6) give the uniform distribution ($l_1 = l_2 = l_3 = l_4$), a bipolar distribution ($l_2 + l_3 < l_1 + l_4$) with respect to the mode $\mathbf{e}_4 \in IR^4$, or an equatorial distribution ($l_1 + l_4 < l_2 + l_3$) with respect to the center $\mathbf{e}_4 \in IR^4$. The intermediate case $l_1 + l_4 = l_2 + l_3$ corresponds to the rank deficient case $\mathrm{rank}(D_\alpha) = 2$ of the von Mises-Fisher matrix distribution. If $l_1 = l_2 = l_3 < l_4$, then the Bingham distribution reduces to the rotationally invariant (rotationally symmetric) Watson distribution of polar type; if $l_4 < l_1 = l_2 = l_3$, then $\mathbf{x} \in S_+^4$ has a rotationally invariant equatorial Watson distribution. In general the l's are labeled such that $l_1 < l_2 < l_3 < l_4$ in the bipolar case, and $l_4 < l_3 < l_2 < l_1$ in the equatorial case, cf. (Prentice, 1986).

The normalizing constant $c_B(L)$ is given by a confluent hypergeometric function (cf. Erdélyi *et al.*, 1953) of matrix argument, $c_B(L) = {}_1F_1(\frac{1}{2};\ 2;\ L)$, cf. (Bingham, 1964; 1974).

Employing the one-one correspondence between $SO(3)$ and S_+^4 the following theorem has been deduced. $X \in SO(3)$ has a von Mises-Fisher matrix distribution $M_3(F)$ if and only if $\mathbf{x} \in S_+^4$ has a Bingham distribution $B_4(L,A)$ with their parameters related as

$$\mathrm{tr}(FX^t) = \mathbf{x}^tALA^t\mathbf{x} \tag{7}$$

when $\mathbf{x} \in S_+^4$ and $\mu(\mathbf{x}) = X \in SO(3)$ are equivalent representations of the same orientation (Prentice, 1986).

To be more explicit and make eq. (7) more instructive, let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ denote the columns of the unit matrix $I_4 \in IR^4 \times IR^4$, and $\mu(\mathbf{e}_t) = E_t$, $t = 1,..., 4$, then

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{8}$$

$$E_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{9}$$

$$E_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \tag{11}$$

Note that $E_1 + E_2 + E_3 + E_4 = 0$.

Now let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ denote the columns of the orthogonal matrix A. Then

$$\mu(\mathbf{a}_t) = \Delta E_t \Gamma, \qquad t = 1,\ldots,4 \tag{12}$$

Furthermore,

$$\begin{aligned} \alpha_1 &= (l_4 + l_1 - l_2 - l_3)/4 \\ \alpha_2 &= (l_4 + l_2 - l_1 - l_3)/4 \\ \alpha_1 &= (l_4 + l_3 - l_1 - l_2)/4 \end{aligned} \tag{13}$$

or

$$\begin{aligned} l_1 &= \alpha_1 - \alpha_2 - \alpha_3 \\ l_2 &= \alpha_2 - \alpha_1 - \alpha_3 \\ l_3 &= \alpha_3 - \alpha_1 - \alpha_2 \\ l_4 &= \alpha_1 + \alpha_2 + \alpha_3 \end{aligned} \tag{14}$$

with $l_1 + l_2 + l_3 + l_4 = 0$.

Thus

$$4F = \sum_{t=1}^{4} l_t \Delta E_t \Gamma \tag{15}$$

Obviously, $\alpha_1 < \alpha_2 < \alpha_3$ if and only if $l_1 < l_2 < l_3 < l_4$, and $\alpha_1 > \alpha_2 > \alpha_3$ if and only if $l_1 > l_2 > l_3 > l_4$, and D_α is positive (negative) definite if and only if L is bipolar (equatorial), cf. (Prentice 1986).

The one-one correspondence of the von Mises-Fisher matrix and the hyperspherical Bingham distribution follows by a simple trigonometric identity in the special case of rotational invariance. Under these assumptions the Bingham distribution reduces to the hyperspherical Watson distribution which reads

$$W_4(\omega;\lambda) = C_4(\lambda)\exp(\lambda\cos^2\frac{\omega}{2})\ \sin^2\frac{\omega}{2}\,\mathrm{d}\omega \tag{16}$$

and with $2\cos^2\frac{\omega}{2} = 1 + \cos\omega$ it is obviously equivalent to a rotationally invariant von Mises-Fisher matrix distribution by

$$f_{\mathrm{vMF}}(\omega\,;\,\kappa) = C_M(\kappa)\exp(\kappa\cos\omega)\ \sin^2\frac{\omega}{2}\,\mathrm{d}\omega \tag{17}$$

with $\kappa = \lambda/2$, and $C_M(\kappa) = C_4(2\kappa)\exp(\kappa)$.

3 CONCLUSIONS

The von Mises-Fisher matrix distribution was explicitly suggested for applications in texture analysis to generate model odfs which are not central in Schaeben (1990). Well known methods exist to calculate the normalization constants or to estimate the parameters of a set of data sampled from a Bingham or von Mises-Fisher matrix distribution, discussed in the context of texture analysis by Schaeben (1993). Even though the model pdfs corresponding to a von Mises-Fisher matrix distribution are not of the same type (Eschner, 1993), these methods may be helpful to know and apply to the problem of best fit of the model pdfs to experimental pdfs. Moreover, the Bingham distribution or von Mises-Fisher matrix distribution provide prerequisites for statistical tests of uniformity and rotational symmetry.

For texture analysts, the text book (Fisher *et al.*, 1987) may prove itself an invaluable source of some more "new" types of model odfs to be introduced into texture analysis.

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