A SYSTEM OF IMPULSIVE DEGENERATE NONLINEAR PARABOLIC FUNCTIONAL-DIFFERENTIAL INEQUALITIES

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ABSTRACT

A theorem about a system of strong impulsive degenerate nonlinear parabolic functional-differential inequalities in an arbitrary parabolic set is proved. As a consequence of the theorem, some theorems about impulsive degenerate nonlinear parabolic differential inequalities and the uniqueness of a classical solution of an impulsive degenerate nonlinear parabolic differential problem are established.

Key words: Impulsive Parabolic Problems, Diagonal Systems, Functional-Differential Inequalities, Impulsive Conditions, Uniqueness Criterion, Arbitrary Parabolic Sets.

AMS (MOS) subject classifications: 35K65, 35R10, 35K85, 35K60, 35K99.

1. Introduction

In this paper we prove a theorem about strong inequalities for the following diagonal system of degenerate nonlinear parabolic functional-differential inequalities

$$\begin{aligned} F_i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), u_{xx}^i(t, x), u) \\ > F_i(t, x, v(t, x), v_t^i(t, x), v_x^i(t, x), v_{xx}^i(t, x), v) \quad (i = 1, \dots, m), \end{aligned} \quad (1.1)$$

where $(t, x) \in D \setminus \bigcup_{j=1}^s (\{t_j\} \times \mathbb{R}^n)$, $t_0 < t_1 < \dots < t_s < t_0 + T$ and D is a relatively arbitrary set more general than the cylindrical domain $(t_0, t_0 + T) \times \Omega_0 \subset \mathbb{R}^{n+1}$. In the expressions

$$F_i(t, x, w(t, x), w_t^i(t, x), w_x^i(t, x), w_{xx}^i(t, x), w) \quad (i = 1, \dots, m)$$

the symbol w denotes a function

$$w: \tilde{D} \ni (t, x) \rightarrow w(t, x) = (w^1(t, x), \dots, w^m(t, x)) \in \mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set such that $\bar{D} \cap [[t_0, t_0 + T) \times \mathbb{R}^n] \subset \tilde{D} \subset (-\infty, t_0 + T) \times \mathbb{R}^n$,

$$w_x^i(t, x) = \text{grad}_x w^i(t, x) \quad (i = 1, \dots, m) \text{ and } w_{xx}^i(t, x) = \left[\frac{\partial^2 w^i(t, x)}{\partial x_j \partial x_k} \right]_{n \times n} \quad (i = 1, \dots, m). \text{ We}$$

assume that the limits $w(t_j^-, x)$, $w(t_j^+, x)$ ($j = 1, \dots, s$) exist for all admissible $x \in \mathbb{R}^n$, they are finite, all different and $w(t_j, x) = w(t_j^+, x)$ ($j = 1, \dots, s$) for all admissible $x \in \mathbb{R}^n$.

System (1.1) is studied together with impulsive and boundary inequalities. The impulsive inequalities are of the form

$$u(t_j, x) - u(t_j^-, x) \leq v(t_j, x) - v(t_j^-, x) \quad (j = 1, \dots, s). \quad (1.2)$$

As a consequence of the theorem about the strong inequalities for system (1.1), we establish theorems about impulsive degenerate nonlinear parabolic differential inequalities and the uniqueness of a classical solution of an impulsive degenerate nonlinear parabolic differential problem.

The results obtained in the paper are direct generalizations of those given by the author in [2]. To prove the results of this paper, theorems of [2] are used. The paper is a continuation of author's publication [3] about impulsive parabolic problems. The impulsive conditions in the present paper are quite different from those considered in [3]. They are similar to the impulsive conditions used by Bainov, Kamont and Minchev in [1].

2. Preliminaries

We use the notation: $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = [0, \infty)$. For any vectors $z = (z_1, \dots, z_m) \in \mathbb{R}^m$, $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_m) \in \mathbb{R}^m$ we write $z \leq \tilde{z}$ if $z_i \leq \tilde{z}_i$ ($i = 1, \dots, m$).

By Ω we denote an arbitrary open subset of $(t_0, t_0 + T) \times \mathbb{R}^m$, where $t_0 \in \mathbb{R}$ and $T \in \mathbb{R}_+ \setminus \{0\}$, such that the projection of Ω on the t -axis is the interval $(t_0, t_0 + T)$.

Next, by D we denote the subset of the set $\bar{\Omega} \cap [(t_0, t_0 + T) \times \mathbb{R}^n]$ satisfying the condition that for any $(\tilde{t}, \tilde{x}) \in D$ there exists a number $\rho > 0$ such that

$$\{(t, x) : (t - \tilde{t})^2 + \sum_{i=1}^n (x_i - \tilde{x}_i)^2 < \rho, \quad t < \tilde{t}\} \subset \Omega.$$

It is clear that $\Omega \subset D$.

We define the sets

$$S_t := \{x \in \mathbb{R}^n : (t, x) \in \bar{D}\} \quad \text{for } t \in [t_0, t_0 + T]$$

and

$$\sigma_t := \bar{D} \cap (\{t\} \times \mathbb{R}^n) \quad \text{for } t \in [t_0, t_0 + T].$$

By s we denote a fixed number belonging to $\mathbb{N}$.

Let $t_1, t_2, \dots, t_s$ be arbitrary fixed real numbers such that

$$t_0 < t_1 < \dots < t_s < t_0 + T.$$

We introduce the following sets:

$$D_j := D \cap [(t_j, t_{j+1}) \times \mathbb{R}^n] \quad (j = 0, 1, \dots, s-1),$$

$$D_s := D \cap [(t_s, t_0 + T) \times \mathbb{R}^n],$$

$$D_* := \bigcup_{j=0}^s D_j \quad \text{and} \quad \sigma_* := \bigcup_{j=1}^s \sigma_{t_j}.$$

Let $\tilde{D}$ be an arbitrary set such that

$$\bar{D} \cap [(t_0, t_0 + T) \times \mathbb{R}^n] \subset \tilde{D} \subset (-\infty, t_0 + T) \times \mathbb{R}^n, \quad t_0 + T \leq \infty.$$

By Σ we denote the part of $\partial D \setminus (\sigma_{t_0} \cup \sigma_* \cup \sigma_{t_0+T})$ disjoint with D .

Assumption (A): For each $i \in \{1, \dots, m\}$ let Σ_i be a subset (possibly empty) of Σ and let, for each $(t, x) \in \Sigma_i$, $\ell_i(t, x)$ be a direction. We assume that ℓ_i is orthogonal to the t -axis and some open segment, with one extremity at (t, x) , of the ray with origin at (t, x) in the direction of ℓ_i is contained in D .

Given a subset E of $\sigma_{t_0} \cup \sigma_* \cup \Sigma$ [E of σ_*] and a function $\omega: D_* \rightarrow \mathbb{R}$, we say that ω has *finite t -right-hand [t -left-hand] sided limits* in $E \cup \{\infty\}$ if for every $(\tilde{t}, \tilde{x}) \in E$ and every $\tilde{t} \in P(E)$, and for each sequence $(t^\nu, x^\nu) \in D_*$ such that $t^\nu > \tilde{t}$ [$t^\nu < \tilde{t}$], $t^\nu \rightarrow \tilde{t}$ and $(t^\nu, x^\nu) \rightarrow (\tilde{t}, \tilde{x})$ or $|x^\nu| \rightarrow \infty$, the limit $\lim_{\nu \rightarrow \infty} \omega(t^\nu, x^\nu)$ is finite; here $P(E)$ is the projection of E on the t -axis. Obviously, this limit does not depend on the choice of the sequence (t^ν, x^ν) and it will be denoted by $\omega(\tilde{t}^+, \tilde{x})$ and $\omega(\tilde{t}^-, \infty)$ [$\omega(\tilde{t}^-, \tilde{x})$ and $\omega(\tilde{t}^-, \infty)$], respectively.

Let E be a subset of $\sigma_{t_0} \cup \sigma_* \cup \Sigma$. If for $\tilde{t} \in P(E)$ there is a sequence $(t^\nu, x^\nu) \in D_*$ such that $t^\nu > \tilde{t}$, $t^\nu \rightarrow \tilde{t}$ and $|x^\nu| \rightarrow \infty$ then we denote by $(\tilde{t}, \infty)$ the class of all such sequences. By a function $\varphi: E \cup \{\infty\} \rightarrow \mathbb{R}$ we mean a function defined for $(t, x) \in E$ and (t, ∞) with $t \in P(E)$.

By $PC_m(\tilde{D})$ we denote the space of mappings

$$w: \tilde{D} \ni (t, x) \rightarrow w(t, x) = (w^1(t, x), \dots, w^m(t, x)) \in \mathbb{R}^m,$$

such that, for every $i \in \{1, \dots, m\}$, w^i is continuous in $(D \cup \Sigma_i) \setminus \sigma_*$, has finite t -right-hand sided limits $w^i(t^+, x)$, $w^i(t^+, \infty)$ in $\sigma_{t_0} \cup \sigma_* \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}$, has finite t -left-hand sided limits $w^i(t^-, x)$, $w^i(t^-, \infty)$ in $\sigma_* \cup \{\infty\}$, and $w^i(t, x) := w^i(t^+, x)$ for $(t, x) \in \sigma_{t_0} \cup \sigma_* \cup (\Sigma \setminus \Sigma_i)$ and $w^i(t, \infty) := w^i(t^+, \infty)$ for $t \in P[\sigma_{t_0} \cup \sigma_* \cup (\Sigma \setminus \Sigma_i)]$.

For $w, \tilde{w} \in PC_m(\tilde{D})$ and for every fixed $t < t_0 + T$, we write $w \stackrel{t}{\leq} \tilde{w}$ if $w^i(\tau, x) \leq \tilde{w}^i(\tau, x)$ for $(\tau, x) \in \tilde{D}$, $\tau \leq t$ ($i = 1, \dots, m$). Given the sets Σ_i ($i = 1, \dots, m$) and the directions ℓ_i ($i = 1, \dots, m$) satisfying Assumption (A), a function $w \in PC_m(\tilde{D})$ is said to *belong to* $PC_{m, \Sigma}^{1,2}(\tilde{D})$ if w_t^i, w_x^i, w_{xx}^i ($i = 1, \dots, m$) are continuous in D_* and the derivatives $\frac{dw^i}{d\ell_i}$ ($i = 1, \dots, m$) are finite on Σ_i ($i = 1, \dots, m$), respectively.

By $M_{n \times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$. For each $i \in \{1, \dots, m\}$ by F_i we denote the mapping

$$F_i: D_* \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \times PC_{m, \Sigma}^{1,2}(\tilde{D})$$

$$\ni (t, x, z, p, q, r, w) \rightarrow F_i(t, x, z, p, q, r, w) \in \mathbb{R},$$

where $q = (q_1, \dots, q_n)$ and $r = [r_{jk}]_{n \times n}$.

We use the notation

$$F_i[t, x, w] := F_i(t, x, w(t, x), w_t^i(t, x), w_x^i(t, x), w_{xx}^i(t, x), w) \quad (i = 1, \dots, m)$$

for all $(t, x) \in D_*$ and $w \in PC_{m, \Sigma}^{1,2}(\tilde{D})$.

By Z we denote a fixed subset of $PC_{m, \Sigma}^{1,2}(\tilde{D})$. Functions u and v belonging to Z are called *solutions* of the system

$$F_i[t, x, u] > F_i[t, x, v] \quad (i = 1, \dots, m) \quad (2.1)$$

in D_* , if they satisfy (2.1) for all $(t, x) \in D_*$.

The functions F_i ($i = 1, \dots, m$) are said to be *parabolic* with respect to $w \in PC_{m, \Sigma}^{1,2}(\tilde{D})$ in D_* if for every $r = [r_{jk}]$, $\tilde{r} = [\tilde{r}_{jk}] \in M_{n \times n}(\mathbb{R})$ and $(t, x) \in D_*$ the following implications hold:

$$\begin{aligned} r \leq \tilde{r} &\Rightarrow F_i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), r, u) \\ &\leq F_i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), \tilde{r}, u) \quad (i = 1, \dots, m), \end{aligned} \quad (2.2)$$

where $r \leq \tilde{r}$ means that the inequality $\sum_{j,k=1}^n (r_{jk} - \tilde{r}_{jk}) \lambda_j \lambda_k \leq 0$ is satisfied for each $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

3. Theorem about Impulsive Functional-Differential Inequalities

Theorem 3.1. *Assume that:*

1. *The functions F_i ($i = 1, \dots, m$) are weakly increasing with respect to $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m$ ($i = 1, \dots, m$), respectively. Moreover, F_i ($i = 1, \dots, m$) are weakly increasing with respect to w in the sense of the relation $\stackrel{t}{\leq}$ for all $t \in (t_0, t_0 + T)$ and*

$$F_i(t, x, z, p, q, r, w) \geq F_i(t, x, z, \tilde{p}, q, r, w) \quad (i = 1, \dots, m)$$

for all $(t, x) \in D_$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$, $w \in Z$.*

2. *For the given sets Σ_i ($i = 1, \dots, m$) and the directions ℓ_i ($i = 1, \dots, m$) satisfying Assumption (A), for the given functions $a_i: \Sigma_i \rightarrow \mathbb{R}_+$ ($i = 1, \dots, m$) and for the given functions $\phi_i: \Sigma_i \times \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, \dots, m$) of the variables (t, x, ξ) and weakly increasing with respect to ξ , functions u and v belonging to Z satisfy the inequalities*

$$u(t, x) < v(t, x) \quad \text{for } (t, x) \in \tilde{D} \setminus \bar{D}, \quad (3.1)$$

$$u^i(t, x) < v^i(t, x) \quad \text{for } (t, x) \in \sigma_{t_0} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\} \quad (i = 1, \dots, m), \quad (3.2)$$

$$u(t, x) - u(t^-, x) < v(t, x) - v(t^-, x) \quad \text{for } (t, x) \in \sigma_*, \quad (3.3)$$

$$\phi_i(t, x, u^i(t, x)) - \phi_i(t, x, v^i(t, x)) < a_i(t, x) \frac{d[u^i(t, x) - v^i(t, x)]}{d\ell_i} \quad (3.4)$$

$$\text{for } (t, x) \in \Sigma_i \quad (i = 1, \dots, m)$$

and the condition

$$u^i(t, x) \neq v^i(t, x) \quad \text{for } (t, x) \in \Sigma_i \quad (i = 1, \dots, m). \quad (3.5)$$

3. F_i ($i = 1, \dots, m$) are parabolic with respect to u in D_* and u, v are solutions of system (2.1) in D_* .

Then,

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D}. \quad (3.6)$$

Proof. To prove Theorem 3.1 consider the following problem:

$$\left. \begin{aligned} &F_i[t, x, u] > F_i[t, x, v] \text{ for } (t, x) \in D_0 \quad (i = 1, \dots, m), \\ &u(t, x) < v(t, x) \text{ for } (t, x) \in (\tilde{D} \setminus \bar{D}) \cap ((-\infty, t_1) \times \mathbb{R}^n), \\ &u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in [\sigma_{t_0} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}] \\ &\quad \cap [t_0, t_1] \times \mathbb{R}^n \quad (i = 1, \dots, m), \\ &\phi_i(t, x, u^i(t, x)) - \phi_i(t, x, v^i(t, x)) < a_i(t, x) \frac{d[u^i(t, x) - v^i(t, x)]}{d\ell_i} \\ &\text{for } (t, x) \in \Sigma_i \cap [(t_0, t_1) \times \mathbb{R}^n] \quad (i = 1, \dots, m). \end{aligned} \right\} \quad (3.7)$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.7), by Theorem 2.1 from [2] applied to set D_0 , we obtain the inequality

$$u(t, x) < v(t, x) \text{ for } (t, x) \in D_0. \quad (3.8)$$

By (3.8) and by the fact that $u, v \in PC_m(\tilde{D})$,

$$u(t^-, x) \leq v(t^-, x) \text{ for } (t, x) \in \sigma_{t_1}. \quad (3.9)$$

From (3.3) and (3.9), we have

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \sigma_{t_1}. \quad (3.10)$$

Inequalities (3.1), (3.8), (3.2) and (3.10) imply that

$$u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in (\tilde{D} \setminus \Sigma_i) \cap [(-\infty, t_1] \times \mathbb{R}^n] \quad (i = 1, \dots, m). \quad (3.11)$$

By (3.11), (3.5) and the fact that u^i ($i = 1, \dots, m$) is continuous in Σ_i ($i = 1, \dots, m$), we get

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D} \cap [(-\infty, t_1] \times \mathbb{R}^n]. \quad (3.12)$$

Now, set the following problem:

$$\left. \begin{aligned}
& F_i[t, x, u] > F_i[t, x, v] \text{ for } (t, x) \in D_1 \quad (i = 1, \dots, m), \\
& u(t, x) < v(t, x) \text{ for } (t, x) \in [\tilde{D} \cap ((-\infty, t_2) \times \mathbb{R}^n)] \setminus \bar{D}_1, \\
& u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in [\sigma_{t_1} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}] \\
& \quad \cap [t_1, t_2] \times \mathbb{R}^n \quad (i = 1, \dots, m), \\
& \phi_i(t, x, u^i(t, x)) - \phi_i(t, x, v^i(t, x)) < a_i(t, x) \frac{d[u^i(t, x) - v^i(t, x)]}{d\ell_i} \\
& \text{for } (t, x) \in \Sigma_i \cap [(t_1, t_2) \times \mathbb{R}^n] \quad (i = 1, \dots, m).
\end{aligned} \right\} \quad (3.13)$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.13), by Theorem 2.1 from [2] applied to set D_1 , we arrive at the inequality

$$u(t, x) < v(t, x) \text{ for } (t, x) \in D_1. \quad (3.14)$$

By (3.14) and by the fact that $u, v \in PC_m(\tilde{D})$,

$$u(t^-, x) \leq v(t^-, x) \text{ for } (t, x) \in \sigma_{t_2}. \quad (3.15)$$

From (3.3) and (3.15), we have

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \sigma_{t_2}. \quad (3.16)$$

Inequalities (3.1), (3.12), (3.14), (3.2) and (3.16) imply that

$$u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in [\tilde{D} \cap ((-\infty, t_2] \times \mathbb{R}^n)] \setminus [\Sigma_i \cap [(t_1, t_2) \times \mathbb{R}^n]]. \quad (3.17)$$

By (3.17), (3.5) and the fact that u^i ($i = 1, \dots, m$) is continuous in Σ_i ($i = 1, \dots, m$), we get

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D} \cap ((-\infty, t_2] \times \mathbb{R}^n). \quad (3.18)$$

Repeating the above procedure $s - 2$ times, we obtain

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \sigma_{t_s} \quad (3.19)$$

and

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D} \cap ((-\infty, t_s] \times \mathbb{R}^n). \quad (3.20)$$

Finally, consider the problem

$$\left. \begin{aligned}
 &F_i[t, x, u] > F_i[t, x, v] \text{ for } (t, x) \in D_s \quad (i = 1, \dots, m), \\
 &u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D} \setminus \bar{D}_s, \\
 &u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in [\sigma_{t_s} \cup (\Sigma \setminus \Sigma_i) \cup \{\infty\}] \\
 &\cap [t_s, t_0 + T) \times \mathbb{R}^n \quad (i = 1, \dots, m), \\
 &\phi_i(t, x, u^i(t, x)) - \phi_i(t, x, v^i(t, x)) < a_i(t, x) \frac{d[u^i(t, x) - v^i(t, x)]}{d\ell_i} \\
 &\text{for } (t, x) \in \Sigma_i \cap [(t_s, t_0 + T) \times \mathbb{R}^n] \quad (i = 1, \dots, m).
 \end{aligned} \right\} \quad (3.21)$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.21), by Theorem 2.1 from [2] applied to set D_s , we get the inequality

$$u(t, x) < v(t, x) \text{ for } (t, x) \in D_s. \quad (3.22)$$

Inequalities (3.1), (3.20), (3.22), (3.2) and (3.19) imply that

$$u^i(t, x) < v^i(t, x) \text{ for } (t, x) \in \tilde{D} \setminus [\Sigma_i \cap [(t_s, t_0 + T) \times \mathbb{R}^n]] \quad (i = 1, \dots, m). \quad (3.23)$$

By (3.23), (3.5) and the fact that $u^i (i = 1, \dots, m)$ is continuous in Σ_i ($i = 1, \dots, m$), we have

$$u(t, x) < v(t, x) \text{ for } (t, x) \in \tilde{D}. \quad (3.24)$$

4. Theorems about Impulsive Differential Inequalities

From the proof of Theorem 3.1 it is easy to see that the following theorem is true:

Theorem 4.1. Assume that:

1. $\tilde{D} = \bar{D}$ and the functions

$$G_i: D_* \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \ni (t, x, z, p, q, r) \rightarrow G_i(t, x, z, p, q, r) \in \mathbb{R}$$

are weakly increasing with respect to $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m$ ($i = 1, \dots, m$), respectively, and

$$G_i(t, x, z, p, q, r) \geq G_i(t, x, z, \tilde{p}, q, r) \quad (i = 1, \dots, m)$$

for all $(t, x) \in D_*$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$.

2. For the given sets $\Sigma_i (i = 1, \dots, m)$ and the directions $\ell_i (i = 1, \dots, m)$ satisfying Assumption (A), for the given functions $a_i: \Sigma_i \rightarrow \mathbb{R}_+$ ($i = 1, \dots, m$) and for the given functions $\phi_i: \Sigma_i \times \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, \dots, m$) of the variables (t, x, ξ) and weakly increasing with respect to ξ , functions u and v belonging to $Z \subset PC_{m, \Sigma}^{1, 2}(\bar{D})$ satisfy inequalities (3.2)-(3.4).

3. $G_i (i = 1, \dots, m)$ are parabolic with respect to u in D_* , and u, v are solutions of the system

$$G_i(t, x, u(t, x), u_t^i(t, x), u_x^i(t, x), u_{xx}^i(t, x))$$

By (4.1), (4.3) and by the fact that G is weakly decreasing with respect to z , we obtain

$$\begin{aligned}
 & G(t, x, u(t, x), u_t(t, x), u_x(t, x), u_{xx}(t, x)) \\
 & - G(t, x, v^\epsilon(t, x), v_t^\epsilon(t, x), v_x^\epsilon(t, x), v_{xx}^\epsilon(t, x)) \\
 & > G(t, x, v(t, x), v_t(t, x), v_x(t, x), v_{xx}(t, x)) \\
 & - G(t, x, v^\epsilon(t, x), v_t^\epsilon(t, x), v_x^\epsilon(t, x), v_{xx}^\epsilon(t, x)) \\
 & \geq 0 \text{ for } (t, x) \in D_*.
 \end{aligned}$$

Moreover, from assumption 2 of Theorem 4.2 and from (4.3) it follows that

$$\begin{aligned}
 u(t, x) & < v^\epsilon(t, x) \text{ for } (t, x) \in \sigma_{t_0} \cup (\Sigma \setminus \widehat{\Sigma}) \cup \{\infty\}, \\
 \phi(t, x, u(t, x)) & - \phi(t, x, v^\epsilon(t, x)) \\
 & < \phi(t, x, u(t, x)) - \phi(t, x, v(t, x)) \\
 & \leq a(t, x) \frac{d[u(t, x) - v(t, x)]}{d\ell} \\
 & = a(t, x) \frac{d[u(t, x) - v^\epsilon(t, x)]}{d\ell} \text{ for } (t, x) \in \widehat{\Sigma}
 \end{aligned}$$

and

$$\begin{aligned}
 u(t_j, x) - u(t_j^-, x) & \leq v(t_j, x) - v(t_j^-, x) \\
 & < [v(t_j, x) + (j+1)\epsilon] - [v(t_j^-, x) + j\epsilon] \\
 & = v^\epsilon(t_j, x) - v^\epsilon(t_j^-, x) \text{ for } x \in S_{t_j} \quad (j = 1, 2, \dots, s).
 \end{aligned}$$

Then we have the inequality

$$u(t, x) < v^\epsilon(t, x) \text{ for } (t, x) \in D_*$$

because functions u and v^ϵ satisfy all the assumptions of Theorem 4.1. Hence (4.2) holds.

Remark 4.1. From the proof of Theorem 4.2 it is easy to see that if function G from Theorem 4.2 is strictly decreasing with respect to z and weakly decreasing with respect to p in D_* then Theorem 4.2 is true if strong inequality (4.1) is replaced by the weak inequality

$$\begin{aligned}
 & G(t, x, u(t, x), u_t(t, x), u_x(t, x), u_{xx}(t, x)) \\
 & \geq G(t, x, v(t, x), v_t(t, x), v_x(t, x), v_{xx}(t, x)), \quad (t, x) \in D_*.
 \end{aligned}$$

Theorem 4.2 and Remark 4.1 imply the following theorem about the uniqueness of a classical solution of a mixed impulsive parabolic differential problem:

Theorem 4.3. *Assume that:*

1. $\tilde{D} = \bar{D}$ and the function G from Theorem 4.2 is strictly decreasing with respect to z and weakly decreasing with respect to p in D_* .
2. The set $\hat{\Sigma} \subset \Sigma$ and the direction ℓ satisfy Assumption (A), $a: \hat{\Sigma} \rightarrow \mathbb{R}_+$ is a given function, the function $\phi: \hat{\Sigma} \times \mathbb{R} \rightarrow \mathbb{R}$ of the variables (t, x, ξ) is strictly increasing with respect to ξ , and $f: \sigma_{t_0} \cup (\Sigma \setminus \hat{\Sigma}) \cup \{\infty\} \rightarrow \mathbb{R}$, $g: \sigma_* \rightarrow \mathbb{R}$, $h: \hat{\Sigma} \rightarrow \mathbb{R}$ are given functions.

Then in the class of all functions w belonging to $PC_{1,\Sigma}^{1,2}(\bar{D})$ and such that function G is parabolic with respect to w in D_* there exists at most one function satisfying the following mixed impulsive parabolic differential problem:

$$G(t, x, w(t, x), w_t(t, x), w_x(t, x), w_{xx}(t, x)) = 0, \quad (t, x) \in D_*,$$

$$w(t, x) = f(t, x), \quad (t, x) \in \sigma_{t_0} \cup (\Sigma \setminus \hat{\Sigma}) \cup \{\infty\},$$

$$w(t, x) - w(t^-, x) = g(t, x), \quad (t, x) \in \sigma_*,$$

$$g(t, x, w(t, x)) - a(t, x) \frac{dw(t, x)}{d\ell} = h(t, x), \quad (t, x) \in \hat{\Sigma}.$$

5. Remarks

Remark 5.1. Since the functions F_i ($i = 1, \dots, m$) from Theorem 3.1 are weakly decreasing with respect to p then these functions may be, particularly, defined by the following formulae:

$$F_i(t, x, z, p, q, r, w) = f_i(t, x, z, q, r, w) - c_i(t, x)p \quad (i = 1, \dots, m),$$

where $(t, x) \in D_*$, $z \in \mathbb{R}^m$, $p \in \mathbb{R}$, $q \in \mathbb{R}^n$, $r \in M_{n \times n}(\mathbb{R})$, $w \in Z$, and $c_i(t, x) \geq 0$ ($i = 1, \dots, m$) for $(t, x) \in D_*$.

The same remarks are true for functions G_i ($i = 1, \dots, m$) and G from Theorems 4.1-4.3.

Therefore, the *degenerate* parabolic problems from this paper are more general than the parabolic problems, in the normal form with respect to p , corresponding to the considered degenerate parabolic problems.

Remark 5.2. Theorems 4.2 and 4.3 are formulated only for the *differential* parabolic problems and for $m = 1$ because assuming, simultaneously, that F_i ($i = 1, \dots, m$) from Theorem 3.1 are weakly increasing with respect to $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n$, w and weakly decreasing with respect to $z_1, \dots, z_n$, w we can consider only the differential problems, where $m = 1$.

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