NEW GENERATING FUNCTIONS FOR MULTIVARIATE BIORTHOGONAL POLYNOMIALS ON THE N-SPHERE

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ABSTRACT

Certain multivariate biorthogonal polynomials on the N-sphere arise in connection with quantum chromodynamics. These functions can be expressed in terms of Lauricella functions of the first kind, and multi-dimensional generating functions for them are deduced by means of an extension of Bailey's theorem.

Key words: Generating Functions, Biorthogonal Polynomials.

AMS subject classification: 33C30, 33C45, 81V05.

1. Introduction and Notation

The purpose of this study is to give a number of new generating functions for the biorthogonal polynomials on the N-sphere first presented by Lam and Tratnik [8], and discussed at considerable length by Kalnins, Miller and Tratnik [6]. These polynomials, which arose in certain calculations relating to quantum chromodynamics, may be represented as Lauricella functions $F_A^{(n)}$ in the form

$$F_A^{(n)}(M + G - 1, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n) \quad (1.1)$$

and

$$(1 - X)^M F_A^{(n)}(1 - M - g_{n+1}, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X - 1), \dots, x_n/(X - 1)), \quad (1.2)$$

where $G = g_1 + \dots + g_{n+1}$.

The basis of the results given is a multi-dimensional generalization of Bailey's theorem as given by Exton [4], p. 139 which extends the discussion of Bailey [2] and Slater [9], p. 58. This very general formulation may be stated as follows:

$$\begin{aligned} \text{If } B_{m_1, \dots, m_n} &= \sum_{p_1, \dots, p_n=0}^{m_1, \dots, m_n} A_{p_1, \dots, p_n} U_{m_1-p_1, \dots, m_n-p_n} V_{m_1+p_1, \dots, m_n+p_n} \text{ and} \\ C_{m_1, \dots, m_n} &= \sum_{p_1=m_1, \dots, p_n=m_n}^{\infty} D_{p_1, \dots, p_n} U_{p_1-m_1, \dots, p_n-m_n} V_{p_1+m_1, \dots, p_n+m_n}, \\ \text{then, formally,} \end{aligned}$$

$$\sum A_{m_1, \dots, m_n} C_{m_1, \dots, m_n} = \sum B_{m_1, \dots, m_n} D_{m_1, \dots, m_n}. \quad (1.3)$$

It is understood that A, U, D and V are functions of $p_1, \dots, p_n$ only, and any questions of convergence must be dealt with in each individual case as appropriate. The symbol $\sum$ without further qualification denotes a summation with the indices of summation running over all nonnegative integer values.

As usual, we employ the notation

$$(a, n) = a(a+1)(a+2)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a); \quad (a, 0) = 1,$$

and following Tratnik [12] for example, we put

$$x_1 + \dots + x_n = X \quad \text{etc.}$$

The generalization hypergeometric function of one variable is given by

$$\begin{aligned} {}_A F_B(a_1, \dots, a_A; b_1, \dots, b_B; x) &= \sum \frac{(a_1, m) \dots (a_A, m) x^m}{(b_1, m) \dots (b_B, m) m!} \\ &= \sum \frac{((a), m) x^m}{((b), m) m!}, \end{aligned}$$

where, for convenience, the sequence of parameters $a_1, \dots, a_A$ is denoted by (a) etc. Certain multiple hypergeometric functions also figure in the analysis and are given as follows:

The Lauricella function of the first kind

$$\begin{aligned} F_A^{(n)}(a, b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n) \\ = \sum \frac{(a, M)(b_1, m_1) \dots (b_n, m_n) x_1^{m_1} \dots x_n^{m_n}}{(c_1, m_1) \dots (c_n, m_n) m_1! \dots m_n!} \end{aligned}$$

the Lauricella function of the third kind

$$\begin{aligned} F_C^{(n)}(a, b; c_1, \dots, c_n; x_1, \dots, x_n) \\ = \sum \frac{(a, M)(b, M) x_1^{m_1} \dots x_n^{m_n}}{(c_1, m_1) \dots (c_n, m_n) m_1! \dots m_n!} \end{aligned}$$

and Karlsson's [7] generalized Kampé de Fériet function

$$\begin{aligned} F_{C:D}^{A:B} \left[\begin{matrix} (a); (b_1); \dots; (b_n); \\ (c); (d_1); \dots; (d_n); \end{matrix} x_1, \dots, x_n \right] \\ = \sum \frac{((a), M)((b_1), m_1) \dots ((b_n), m_n) x_1^{m_1} \dots x_n^{m_n}}{((c), M)((d_1), m_1) \dots ((d_n), m_n) m_1! \dots m_n!}. \end{aligned}$$

It will be seen that

$$\begin{aligned} F_A^{(n)}(a, b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n) \\ = F_{0:1}^{1:1} \left[\begin{matrix} a; b_1; \dots; b_n; \\ -; c_1; \dots; c_n; \end{matrix} x_1, \dots, x_n \right] \end{aligned}$$

and

$$\begin{aligned} F_C^{(n)}(a, b; c_1, \dots, c_n; x_1, \dots, x_n) \\ = F_{0:1}^{2:0} \left[\begin{matrix} a, b; -; \dots; -; \\ -; c_1; \dots; c_n; \end{matrix} x_1, \dots, x_n \right]. \end{aligned}$$

Any values of the parameters for which any of the expressions given in this paper do not make sense are tacitly excluded. For any relevant general information, the reader is referred to Appell et Kampé de Fériet [1], Erdélyi [3], Exton [4], Slater [9], Srivastava and Karlsson [10], and Srivastava and Manocha [11], for example.

2. Main Results

A generating relation of quite general character for the polynomial (1.1) may be deduced from (1.3). Put

$$U = 1/(m_1! \dots m_n!), \quad V = (G-1, M)$$

$$A = \frac{(-x_1)^{m_1} \dots (-x_n)^{m_n}}{(g_1, m_1) \dots (g_n, m_n) m_1! \dots m_n!}$$

and

$$D = \frac{((p), M)((h_1), m_1) \dots ((h_n), m_n) t_1^{m_1} \dots t_n^{m_n}}{((q), M)((k_1), m_1) \dots ((k_n), m_n)}.$$

After some rather tedious manipulation, we obtain the expression

$$\begin{aligned} & \sum \frac{((p), M)(G-1, M)((h_1), m_1) \dots ((h_n), m_n) t_1^{m_1} \dots t_n^{m_n}}{((q), M)((k_1), m_1) \dots ((k_n), m_n) m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(M+G-1, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n) \\ & = \sum \frac{((p), M)(G-1, 2M)((h_1), m_1) \dots ((h_n), m_n) (-x_1 t_1)^{m_1} \dots (-x_n t_n)^{m_n}}{((q), M)(g_1, m_1) \dots (g_n, m_n)((k_1), m_1) \dots ((k_n), m_n) m_1! \dots m_n!} \\ & \cdot F_{Q:K}^{P+1:H} \left[\begin{matrix} (p) - M, G-1+2M; (h_1) + m_1; \dots; (h_n) + m_n; \\ (q) + M; (k_1) + m_1; \dots; (k_n) + m_n; \end{matrix} \middle| t_1, \dots, t_n \right]. \end{aligned} \quad (2.1)$$

This is the required generating function for the polynomial (1.1).

Similarly, on putting

$$U = \frac{1}{(g_{n+1}, M) m_1! \dots m_n!}, \quad V = 1,$$

$$A = \frac{x_1^{m_1} \dots x_n^{m_n} (X-1)^{-M}}{(g_1, m_1) \dots (g_n, m_n) m_1! \dots m_n!}$$

and

$$D = \frac{((p), M)((h_1), m_1) \dots ((h_n), m_n) (X-1)^M t_1^{m_1} \dots t_n^{m_n}}{((q), M)((k_1), m_1) \dots ((k_n), m_n)}$$

we obtain a general generating function for the polynomial (1.2), namely

$$\begin{aligned} & \sum \frac{((p), M)((h_1), m_1) \dots ((h_n), m_n) t_1^{m_1} \dots t_n^{m_n} (X-1)^M}{((q), M)((k_1), m_1) \dots ((k_n), m_n) m_1! \dots m_n! (g_{n+1}, M)} \\ & \cdot F_A^{(n)}(1 - g_{n+1} - M, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X-1), \dots, x_n/(X-1)) \end{aligned}$$

$$\begin{aligned}
&= \sum \frac{((p), M)((h_1), m_1) \dots ((h_n), m_n)(x_1 t_1)^{m_1} \dots (x_n t_n)^{m_n}}{((q), M)(g_1, m_1) \dots (g_n, m_n)((k_1), m_1) \dots ((k_n), m_n) m_1! \dots m_n!} \\
&\cdot F_{Q+1:K}^{P:H} \left[\begin{matrix} (p) + M: (h_1) + m_1; \dots; (h_n) + m_n; \\ (q) + M, g_{n+1}: (k_1) + m_1; \dots; (k_n) + m_n; \end{matrix} t_1(X-1), \dots, t_n(X-1) \right]. \quad (2.1)
\end{aligned}$$

3. Compact Generating Relations

A number of more compact expressions can be obtained as special cases of (2.1) and (2.2). In the first of these, suppose that $P = Q = H = K = 0$, when the inner generalized Kampé de Fériet function on the right becomes

$${}_1F_0(G-1+2M; -; T) = (1-T)^{1-G-2M}$$

as a simple consequence of the binomial theorem.

Hence, we see that

$$\begin{aligned}
&\sum \frac{(G-1, M)t_1^{m_1} \dots t_n^{m_n}}{m_1! \dots m_n!} F_A^{(n)}(M+G-1, m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n) \\
&= (1-T)^{1-G} F_C^{(n)}\left(\frac{1}{2}G - \frac{1}{2}, \frac{1}{2}G; g_1, \dots, g_n; -4t_1x_1(1-T)^{-2}, \dots, t_nx_n(1-T)^{-2}\right).
\end{aligned}$$

Next, put $t_1 = \dots = t_n = t$ and take $P = K = 0$ and $H = Q = 1$. After letting $q = H$, we then have

$$\begin{aligned}
&\sum \frac{(G-1, M)(h_1, m_1) \dots (h_n, m_n)t^M}{(H, M)m_1! \dots m_n!} F_A^{(n)}(M+G-1, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n) \\
&= (1-t)^{1-G} F_{1:1}^{2:1}\left[\begin{matrix} \frac{1}{2}G - \frac{1}{2}, \frac{1}{2}G; h_1; \dots; h_n; \\ H; g_1; \dots; g_n; \end{matrix} -4x_1t(1-t)^{-2}, \dots, -4x_nt(1-t)^{-2}\right].
\end{aligned}$$

Further, put $g_1 = h_1, \dots, g_n = h_n$ and obtain

$$\begin{aligned}
&\sum \frac{(G-1, M)t^M}{(H, M)m_1! \dots m_n!} F_A^{(n)}(M+G-1, -m_1, \dots, -m_n; h_1, \dots, h_n; x_1, \dots, x_n) \\
&= (1-t)^{1-G} {}_2F_1\left(\frac{1}{2}G - \frac{1}{2}, \frac{1}{2}G; H; -4tX(1-t)^{-2}\right).
\end{aligned}$$

A general result of quite a different character may be deduced by putting

$$t_n = -t_1 - \dots - t_{n-1},$$

so that, if we let $H = K = 0$, the inner generalized Kampé de Fériet on the right of (2.1) reduces to unity by means of the binomial theorem. Compare Exton [5]. Hence,

$$\begin{aligned}
&\sum \frac{((p), M)(G-1, M)t_1^{m_1} \dots t_{n-1}^{m_{n-1}} (-t_1 - \dots - t_{n-1})^{m_n}}{((q), M)m_1! \dots m_n!} \\
&\cdot F_A^{(n)}(M+G-1, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n) \\
&= F_{Q:1}^{P+2:0}\left[\begin{matrix} (p), \frac{1}{2}G - \frac{1}{2}, \frac{1}{2}G; -; \dots; -; \\ (q); g_1; \dots; g_n; \end{matrix} -4x_1t_1, \dots, -rx_{n-1}t_{n-1}, 4x_n(t_1 + \dots + t_{n-1})\right]. \quad (3.1)
\end{aligned}$$

The previous expansion embodies some degree of flexibility, since the p and q parameters can be chosen arbitrarily in order to simplify the result. For example, if $P = 0$ and $Q = 2$ such that $q_1 = \frac{1}{2}G - \frac{1}{2}$ and $q_2 = \frac{1}{2}G$, the right-hand member of (3.1) reduces to the product of several ${}_0F_1$ series, which are, effectively, Bessel functions. We then see that

$$\begin{aligned} & \sum \frac{(G-1, M) t_1^{m_1} \dots t_{n-1}^{m_{n-1}} (-t_1 - \dots - t_{n-1})^{m_n}}{(\frac{1}{2}G - \frac{1}{2}, M) (\frac{1}{2}G, M) m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(M + G - 1, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n) \\ & = {}_0F_1(-; g_1; -4x_1 t_1) \dots {}_0F_1(-; g_{n-1}; -4x_{n-1} t_{n-1}) {}_0F_1(-; g_n; 4x_n(t_1 + \dots + t_{n-1})). \end{aligned}$$

The second general result, (2.2), may also be used to obtain a number of expansion. Suppose that $P = H = Q = K = 0$, when the inner generalized Kampé de Fériet function on the right takes the form

$${}_0F_1(-; g_{n+1}; T(X-1))$$

and we have the generating function

$$\begin{aligned} & \sum \frac{t_1^{m_1} \dots t_n^{m_n} (X-1)^M}{(g_{n+1}, M) m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(1 - M - g_{n+1}, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X-1), \dots, x_n/(X-1)) \end{aligned}$$

Put $H = 1$, $K = 0$ and $t_1 = \dots = t_n = t$, and obtain the result

$$\begin{aligned} & \sum \frac{((p), M)(h_1, m_1) \dots (h_n, m_n) t^M (X-1)^M}{((q), M)(g_{n+1}, M) m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(1 - M - g_{n+1}, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X-1), \dots, x_n/(X-1)) \\ & = \sum \frac{((p), M)(h_1, m_1) \dots (h_n, m_n) x_1^{m_1} \dots x_n^{m_n} t^M}{((q), M)(g_1, m_1) \dots (g_n, m_n) m_1! \dots m_n!} \\ & \cdot {}_{P+1}F_{Q+1}((P) + M, H + M; (q) + M, g_{n+1}; t(X-1)). \end{aligned}$$

Further, let $P = 0$, $Q = 1$ and $q = h$, when it is found that

$$\begin{aligned} & \sum \frac{(h_1, m_1) \dots (h_n, m_n) t^M (X-1)^M}{(H, M)(g_{n+1}, M) m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(1 - M - g_{n+1}, m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X-1), \dots, x_n/(X-1)) \\ & = {}_0F_1(-; g_1; x_1 t) \dots {}_0F_1(-; g_n; x_n t) {}_0F_1(-; g_{n+1}; t(X-1)). \end{aligned} \tag{3.2}$$

In (2.2), let $H = K = 0$ and $t_n = -t_1 - \dots - t_{n-1}$, when, as in (3.1), the inner generalized Kampé de Fériet function reduces to unity, so that

$$\begin{aligned} & \sum \frac{((p), M) t_1^{m_1} \dots t_{n-1}^{m_{n-1}} (-t_1 - \dots - t_{n-1})^{m_n} (X-1)^M}{((q), M)(g_{n+1}, M) m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(1 - M - g_{n+1}, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X-1), \dots, x_n/(X-1)) \end{aligned}$$

$$= F_{Q:1}^{P:0} \left[\begin{matrix} (p): -; \dots; -; \\ (q): g_1; \dots; g_n; \end{matrix} x_1 t_1, \dots, x_{n-1} t_{n-1}, -x_n(t_1 + \dots + t_{n-1}) \right].$$

Put $P = Q = 0$, and obtain the result

$$\begin{aligned} & \sum \frac{t_1^{m_1} \dots t_{n-1}^{m_{n-1}} (-t_1 - \dots - t_{n-1})^{m_n} (X-1)^M}{((q), M)(t_{n+1}, M)m_1! \dots m_n!} \\ & \cdot F_A^{(n)}(1 - M - g_{n+1}, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1/(X-1), \dots, x_n/(X-1)) \\ & = {}_0F_1(-; g_1; x_1 t_1) \dots {}_0F_1(-; g_{n-1}; x_{n-1} t_{n-1}) {}_0F_1(-; g_n; -x_n(t_1 + \dots + t_{n-1})). \quad (3.3) \end{aligned}$$

The formulas (3.2) and (3.3) again may be expressed in terms of Bessel functions on account of the occurrence of the ${}_0F_1$ series on the right in each case.

References

- [1] Appell, P. and Kampé de Fériet, J., *Fonctions Hypergéométriques et Hypersphériques*, Gauthier Villars, Paris 1926.
- [2] Bailey, W.N., Identities of Rogers-Ramanujan type, *Proc. London Math. Soc.* **50** (1949), 1-10.
- [3] Erdélyi, A., *Higher Transcendental Functions*, McGraw Hill, New York 1953.
- [4] Exton, H., *Multiple Hypergeometric Functions*, Ellis Horwood, Chicester, UK 1976.
- [5] Exton, H., Generating relations for Tratnik's multivariate biorthogonal continuous Hahn polynomials, *J. Math. Phys.* **33** (1992), 524-527.
- [6] Kalnins, E.G., Miller, W. and Tratnik, M.V., Families of orthogonal and biorthogonal polynomials on the N-sphere, *SIAM J. Math. Anal.* **22** (1991), 272-294.
- [7] Karlsson, P.W., Reduction of certain generalized Kampé de Fériet functions, *Math. Scand.* **32** (1973), 265-268.
- [8] Lam, C.S. and Tratnik, M.V., Conformally invariant operator-product expansions of any number of operators of arbitrary spin, *Canad. J. Phys.* **63** (1985), 1427-1437.
- [9] Slater, L.J., *Generalized Hypergeometric Functions*, Cambridge University Press 1966.
- [10] Srivastava, H.M. and Karlsson, P.W., *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, Chichester, UK 1985.
- [11] Srivastava, H.M. and Manocha, H.L., *A Treatise on Generating Functions*, Ellis Horwood, Chichester, UK 1984.
- [12] Tratnik, M.V., Multivariate biorthogonal Hahn polynomials, *J. Math. Phys.* **30** (1989), 627-634.