CONVERGENCE OF A RANDOM ITERATION SCHEME TO A RANDOM FIXED POINT

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ABSTRACT

This paper discusses the convergence of random Ishikawa iteration scheme to a random fixed point for a certain class of random operators.

Key words: Random fixed point, random Ishikawa iteration, Tricomi’s condition, Hilbert space, random operator.

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1. Introduction

In recent years the study of random fixed points have attracted a great amount of attention. Discussions on random fixed points may be found in works such as [1], [2], [6], and [7]. We review the following concepts which are essentials for the purpose of our discussion. Throughout this paper, \((T, \Sigma)\) denotes a measurable space, and \(H\) denotes a separable Hilbert space.

A function \(f: T \to H\) is said to be measurable if \(f^{-1}(B) \in \Sigma\) for every Borel subset \(B\) of \(H\).

Let \(C\) be any subset of \(H\). A function \(f: T \to C\) is said to be measurable if \(f^{-1}(B \cap C) \in \Sigma\) for every Borel subset \(B\) of \(H\).

A function \(F: T \times H \to H\) is said to be \(H\)-continuous if \(F(t, \cdot): H \to H\) is continuous for every \(t \in T\).

A function \(F: T \times H \to H\) is said to be a random operator if \(F(\cdot, x): T \to H\) is measurable for every \(x \in H\).

A measurable function \(g: T \to H\) is said to be a random fixed point of \(F: T \times H \to H\) if \(F(t, g(t)) = g(t)\) for all \(t \in T\).

The Ishikawa iteration scheme was obtained in [5]. We define the random Ishikawa iteration scheme in an analogous manner as follows:

Let \(g_0: T \to H\) be any measurable function. The functions below are iteratively defined as follows:

\[
g_{n+1}(t) = \alpha_n F(t, h_n(t)) + (1 - \alpha_n) g_n(t), \quad n \geq 0, \quad t \in T. \tag{1}
\]

where

\[
h_n(t) = \beta_n F(t, g_n(t)) + (1 - \beta_n) g_n(t), \quad n \geq 0, \quad t \in T. \tag{2}
\]

and \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences of real numbers such that
\[0 < \alpha_n, \beta_n < 1 \text{ for all } n \geq 0\]  
(3)

\[\lim_{n \to \infty} \sup \beta_n = M < 1\]  
(4)

and

\[\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty.\]  
(5)

Let \( C \subset H, k: C \to C \) is said to satisfy Tricomi’s condition in \( C \) if \( p \in C \) and \( k(p) = p \) imply

\[\| k(x) - p \| \leq \| x - p \| \text{ for all } x \in C.\]  
(6)

The following lemma was proved in [5]:

**Lemma:** \( H \) is a Hilbert space; therefore for any \( x, y, z \in H \) and any real \( \lambda \)

\[\| \lambda x + (1 - \lambda)y - z \| = \| x - z \| + (1 - \lambda) \| y - z \| - \lambda(1 - \lambda) \| x - y \|^2\]  
(7)

2. Main Result

**Theorem 1:** Let \( H \) be a separable Hilbert space. \( F: T \times H \to H \) is an \( H \)-continuous random operator in which case there exists \( f: T \to H \) (not necessarily measurable) such that

\[\| F(t, x) - f(t) \| \leq \| x - f(t) \|\]  
(8)

for all \( t \in T \) and \( x \in H \).

Then, for any measurable function \( g_0: T \to H \), the sequence of functions \( \{g_n\} \) defined by the random Ishikawa iteration scheme, if convergent, converges to a random fixed point of \( F \).

**Proof:** For any \( t \in T \),

\[\| g_{n+1}(t) - f(t) \|^2 \]

\[= \| \alpha_n F(t, h_n(t)) + (1 - \alpha_n) g_n(t) - f(t) \|^2\]

\[= \alpha_n \| F(t, h_n(t)) - f(t) \|^2 + (1 - \alpha_n) \| g_n(t) - f(t) \|^2\]

\[-\alpha_n(1 - \alpha_n) \| F(t, h_n(t)) - g_n(t) \|^2 \]  
(by (7))

\[\leq \alpha_n \| F(t, h_n(t)) - f(t) \|^2 + (1 - \alpha_n) \| g_n(t) - f(t) \|^2 \]  
(by (3))

\[\leq \alpha_n \| h_n(t) - f(t) \|^2 + (1 - \alpha_n) \| g_n(t) - f(t) \|^2 \]  
(by (8))

\[\leq \| g_n(t) - f(t) \|^2 - \alpha_n \beta_n(1 - \beta_n) \| F(t, g_n(t)) - g_n(t) \|^2 \]  
(by using (2), (7) and (8)).

It further implies that

\[\sum_{n=0}^{N} \alpha_n \beta_n(1 - \beta_n) \| F(t, g_n(t)) - g_n(t) \|^2 \]

\[\leq \| g_0(t) - f(t) \|^2 - \| g_{N+1}(t) - f(t) \|^2\]

\[\leq \| g_0(t) - f(t) \|^2 < \infty.\]  
(9)
For \( M' \) satisfying \( M < M' < 1 \), there exists a positive integer \( m_0 \) such that \( \beta_m < M' \) for all \( m \geq m_0 \) (by (4)). Therefore, \( 1 - \beta_m > 1 - M' > 0 \) for all \( m \geq m_0 \), or
\[
\sum_{m = m_0}^{\infty} \alpha_m \beta_m (1 - \beta_m) \geq (1 - M') \sum_{m = m_0}^{\infty} \alpha_m \beta_m = \infty
\] (10)

(9) and (10) imply,
\[
\lim_{n \to \infty} \inf || F(t, g_n(t)) - g_n(t) || = 0 \text{ for all } t \in T.
\] (11)

Hence, if \( \{g_n(t)\} \) converges, for example to \( g(t) \), \( F(t, g_n(t)) \) also converges to \( g(t) \).

Since \( F: T \times H \to H \) is an \( H \)-continuous random operator and \( H \) is separable, \( \{g_n\} \) is a sequence of measurable functions [4]. Therefore, \( g = \lim_{n \to \infty} g_n \) is measurable. Furthermore, \( F \) is \( H \)-continuous; thus, for all \( t \in T \),
\[
g(t) = \lim_{n \to \infty} g_n(t) = \lim_{n \to \infty} F(t, g_n(t))
\]
\[
= F(t, \lim_{n \to \infty} g_n(t)) = F(t, g(t)).
\]

Hence, \( g \) is a random fixed point of \( F \).

Theorem 2: Let \( C \subset H \) be a convex and compact subset and \( F: T \times C \to C \) satisfies
a) \( F \) is \( H \)-continuous;
b) \( \) there exists \( f: T \to C \) such that \( || F(t, x) - f(t) || \leq || x - f(t) || \) for all \( t \in T \) and \( x \in C \); and
\( c) \ F(t, \cdot): C \to C \) satisfies Tricomi's condition in \( C \) for every \( t \in T \).

Then, for any measurable function \( g_0: T \to C \), the sequence \( \{g_n\} \) of measurable functions constructed by the random Ishikawa iteration scheme converges to a random fixed point of \( F \).

Proof: Since \( C \) is convex and compact, \( H \) is a separable Hilbert space and \( F \) is an \( H \)-continuous random operator and \( \{g_n\} \) is a sequence of measurable functions from \( C \) to \( C \). Proceeding exactly as in Theorem 1, we obtain (as in (11))
\[
\lim_{n \to \infty} \inf || F(t, g_n(t)) - g_n(t) || = 0.
\]
Therefore, for fixed \( t \in T \), there exists \( \{g_{n_i}(t)\} \subset \{g_n(t)\} \) such that
\[
\lim_{i \to \infty} || F(t, g_{n_i}(t)) - g_{n_i}(t) || = 0.
\] (12)
Since \( C \) is compact, there exists \( \{g_{n_{i_k}}(t) \subset \{g_{n_i}(t)\} \), such that \( \{g_{n_{i_k}}(t) \) is convergent.
Let
\[
\lim_{k \to \infty} g_{n_{i_k}}(t) = g(t).
\] (13)
From (12) and (13), and since \( \{g_{n_{i_k}}(t) \subset \{g_{n_i}(t)\} \), we have
\[
\lim_{k \to \infty} F(t, g_{n_{i_k}}(t)) = g(t),
\]
or
\[
F(t, g(t)) = g(t) \text{ (since } F \text{ is } H\text{-continuous}).
\] (14)
Hence, for fixed \( t \in T \), \( g(t) \) is a fixed point of \( F(t, \cdot) \).

For any fixed \( t \in T \),
\[
|| g_{n+1}(t) - g(t) ||^2 = || \alpha_n F(t, h_n(t)) + (1 - \alpha_n) g_n(t) - g(t) ||^2
\]
\[
= \alpha_n || F(t, h_n(t)) - g(t) ||^2 + (1 - \alpha_n) || g_n(t) - g(t) ||^2
\]
\[-\alpha_n(1-\alpha_n) || F(t, h_n(t)) - g_n(t) ||^2 \text{ (by (7))} \]
\[\leq \alpha_n || h_n(t) - g(t) ||^2 + (1-\alpha_n) || g_n(t) - g(t) ||^2 \text{ (by (3) and Tricomi’s condition (6))} \]
\[= \alpha_n || \beta_n F(t, g_n(t)) + (1-\beta_n) g_n(t) - g(t) ||^2 + (1-\alpha_n) || g_n(t) - g(t) ||^2 \]
\[= \alpha_n || g_n(t) - g(t) ||^2 + (1-\beta_n) || g_n(t) - g(t) ||^2 \]
\[= \alpha_n || g_n(t) - g(t) ||^2 + (1-\alpha_n) || g_n(t) - g(t) ||^2 \text{ (by (7))} \]
\[+ (1-\alpha_n) || g_n(t) - g(t) ||^2 \text{ (by (3) and Tricomi’s condition (6))} \]
\[= \alpha_n || g_n(t) - g(t) ||^2 + (1-\alpha_n) || g_n(t) - g(t) ||^2 \]
\[= || g_n(t) - g(t) ||^2. \]

Therefore, for \( t \in T \),

\[ || g_{n+1}(t) - g(t) || \leq || g_n(t) - g(t) ||. \tag{15} \]

Since \( \{g_{n_{i_k}}(t)\} \rightarrow g(t) \), given \( \epsilon > 0 \), there exists \( n_{i_{k_0}} \) such that

\[ || g_{n_{i_{k_0}}}(t) - g(t) || < \epsilon. \]

By virtue of (15),

\[ || g_n(t) - g(t) || < \epsilon \text{ for all } n \geq n_{i_{k_0}}. \]

Therefore, for \( t \in T, \lim_{n \rightarrow \infty} g_n(t) = g(t) \). Since \( \{g_n\} \) is a sequence of measurable functions, \( g \) is also measurable. Thus, \( g: T \rightarrow C \) is a random fixed point of \( F: T \times C \rightarrow C \).

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References


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