ON WEAK SOLUTIONS OF RANDOM DIFFERENTIAL INCLUSIONS

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ABSTRACT

In the paper we study the existence of solutions of the random differential inclusion

\[
\dot{x}_t \in G(t, x_t) \quad P\mathcal{L}_t, t \in [0, T]\text{-a.e.}
\]

\[x_0 \overset{d}{=} \mu,\]

where \(G\) is a given set-valued mapping value in the space \(K^n\) of all nonempty, compact and convex subsets of the space \(\mathbb{R}^n\), and \(\mu\) is some probability measure on the Borel \(\sigma\)-algebra in \(\mathbb{R}^n\). Under certain restrictions imposed on \(F\) and \(\mu\), we obtain weak solutions of problem (I), where the initial condition requires that the solution of (I) has a given distribution at time \(t = 0\).

Key words: Set-Valued Mappings, Hukuchara's Derivative, Aumann's Integral, Tightness and Weak Convergence of Probability Measures.

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1. Preliminaries

Problems of existence of solutions of differential inclusions were studied by many. In particular, random cases were considered in [3], [5], [7]. This work deals with the inclusion with a purely stochastic initial condition. First, we recall several notions and results needed in the sequel. Let \(K_c(S)\) be the space of all nonempty compact and convex subsets of a metric space \(S\) equipped with the Hausdorff metric \(H\) (see e.g., [1], [4]):

\[H(A, B) = \max(\overline{H}(A, B), \overline{H}(B, A)); A, B \in K_c(S),\]

where \(\overline{H}(A, B) = \sup_{a \in A} \inf_{b \in B} \rho(a, b).\) By \(\| A \|\) we denote the distance \(H(A, 0)\). For \(S\) being a separable Banach space, \((K_c(S), H)\) is a polish metric space.

Let \(I = [0, T], \ T > 0.\) For a given multifunction \(G: I \to K_c(S)\) by \(D_H G(t_0),\) we denote its Hukuchara derivative at the point \(t_0 \in I\) (see e.g., [2], [9]) by the limits (if they exist in \(K_c(S)\))

\[
\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t - h)}{h},
\]

both equal to the same set \(D_H F(t_0) \in K_c(S).\)

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For $S = \mathbb{R}^n$ and $K^n = K_e(\mathbb{R}^n)$, we denote by $C_I = C(I, K^n)$ the space of all $H$-continuous set-valued mappings. In $C_I$ we consider a metric $\rho$ of uniform convergence

$$\rho(F, G) = \sup_{0 \leq t \leq T} H(X(t), Y(t)), \text{ for } X, Y \in C_T.$$ 

Then $C_I$ is a polish metric space.

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space. We recall now the notion of a multivalued stochastic process. The family of set-valued mappings $X = (X_t)_{t \geq 0}$ is said to be a multivalued stochastic process if for every $t \geq 0$, the mapping $X_t: \Omega \rightarrow K^n$ is measurable, i.e., $X_t(U) = \{\omega: X_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$, for every open set $U \subseteq \mathbb{R}^n$ (see e.g., [1, 4]). It can be noted that $U$ can be also chosen as closed or Borel subset. We restrict our interest to the case when $0 \leq t \leq T, T > 0$. If the mapping $t \rightarrow X_t(\omega)$ is continuous ($H$-continuous) with probability on $(P, 1)$, then we say that the process $X$ has continuous "paths."

Let us notice that the set-valued stochastic process $X$ can be though as a random element $X: \Omega \rightarrow C_I$. Indeed, it follows immediately from [3] and from the fact that the topology of the uniform convergence and the compact-open topology in $C_I$ are the same.

Definition 1: A probability measure $\mu$ (on $C_I$) is a distribution of the set-valued process $X = (X_t)_{0 \leq t \leq T}$ if one has $\mu(A) = P(X_t(\omega) \in A)$ for every Borel subset $A$ from $C_I$.

A distribution of $X$ will be denoted by $P^X$.

Definition 2: A set-valued mapping $F: I \times K^n \rightarrow K^n$ is said to be an integrably bounded of the Carathéodory type if:

1) there exists a measurable function $m: I \rightarrow \mathbb{R}^+$ such that $\int_0^T m(t) dt < \infty$ and $\|F(t, \cdot)\| \leq m(t)$ $t$-a.e., $A \in K^n$.

2) $F(t, \cdot)$ is $H$-continuous $t$-a.e.

3) $F(\cdot, A)$ is a measurable multifunction for every $A \in K^n$.

Let us consider now the multivalued random differential equation:

$$D_H X_t = F(t, X_t) P.1, t \in [0, T] \text{-a.e.}$$

$$X_0 \overset{d}{=} \mu$$

where the initial condition requires that the set-valued solution process $X = (X_t)_{t \in I}$ has a given distribution $\mu$ at the time $t = 0$. By a weak solution of (II) we understand a system $(\Omega, \mathcal{F}, P(X_t)_{t \in I})$ where $(X_t)_{t \in I}$ is a set-valued process on some probability space $(\Omega, \mathcal{F}, P)$ such that (II) is met.

We state the following theorem (see e.g. [6]).

Theorem 1: Let $F: I \times K^n \rightarrow K^n$ be an integrably bounded set-valued function of the Carathéodory type and let $\mu$ be an arbitrary probability measure on the space $K^n$. Then there exists a weak solution of (II).

2. Weak Solutions of Random Differential Inclusions

As an application of Theorem 1, we show the existence of a weak solution of the random differential inclusion

$$\dot{x}_t = G(t, x_t) P.1, t \in [0, T] \text{-a.e.}$$

$$x_0 \overset{d}{=} \mu.$$
The weak solution of (I) is understood similarly as above, where \( \mu \) is now a given probability measure on \( \mathbb{R}^n \).

Let \( \mathcal{T}_0 \) denote the family of nonempty open subsets of \( \mathbb{R}^n \), and let \( C = \{ C_V; V \in \mathcal{T}_0 \} \), where \( C_V = \{ K \in K^n: K \cap V \neq \emptyset \} \). Then we have that \( \mathcal{B}^n = \sigma(C) \) (see e.g. Proposition 3.1 [4]), where \( \mathcal{B}^n \) is a Borel \( \sigma \)-field induced by the metric space \( (K^n, H) \).

**Lemma 1:** The following hold true:

i) \( K^n \subseteq C \),

ii) if \( A_1, A_2, \ldots \subseteq C \) then \( \bigcup_{n=1}^{\infty} A_n \subseteq C \),

iii) if \( C_{V_1} \subseteq C_{V_2} \subseteq \ldots \) then \( V_1 \subseteq V_2 \subseteq \ldots \).

**Proof:** The property i) is obvious. Let \( V_1, V_2, \ldots \subseteq \mathcal{T}_0 \) be such that \( A_n = C_{V_n} \) for \( n = 1, 2, \ldots \). To establish ii), let us observe that \( \bigcup_{n=1}^{\infty} A_n = C_{\bigcup_{n=1}^{\infty} V_n} \).

Let us suppose that iii) does not hold. Then for some \( k \geq 1 \), \( V_k \not\subseteq V_{k+1} \). Hence there exists a point \( x \in V_k \) such that \( x \notin V_{k+1} \). But then \( \{x\} \subseteq C_{V_k} \) and \( \{x\} \notin C_{V_{k+1}} \) contradicts to \( C_{V_k} \subseteq C_{V_{k+1}} \).

To obtain our main result we need the following lemma:

**Lemma 2:** If \( \mu \) is a probability measure on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^n) \), then there exists a probability measure \( \bar{\mu} \) on the space \( K^n \) such that \( \bar{\mu}(C_V) = \mu(V) \), \( V \in \mathcal{T}_0 \).

**Proof:** Let \( C \) be the family generating Borel \( \sigma \)-field \( \mathcal{B} \). We define a set-function \( \nu \) on \( C \) by \( \nu(C_V) = \mu(V) \). Let us observe that \( \nu \) is well-defined. Indeed, if \( C_{V_1} \subseteq C_{V_2} \) and \( \mu(V_1) \neq \mu(V_2) \) then \( V_1 \not\subseteq V_2 \). Hence \( V_1 \setminus V_2 \neq \emptyset \) or \( V_2 \setminus V_1 \neq \emptyset \). Without loss of generality we may assume the first case. Then there exists \( x \in V_2 \) such that \( x \notin V_1 \). But then \( \{x\} \subseteq C_{V_2} \) and \( \{x\} \notin C_{V_1} \) which contradicts with an equality \( C_{V_1} = C_{V_2} \). Similarly, it can be shown that if the sets \( C_{V_1} \) and \( C_{V_2} \) are disjoint, then the sets \( V_1, V_2 \) have the same property too. Hence we get \( \nu(C_{V_1} \cup C_{V_2}) = \nu(C_{V_1}) + \nu(C_{V_2}) \) for disjoint \( C_{V_1} \) and \( C_{V_2} \). From Lemma 1 we conclude that, if \( C_{V_1} \subseteq C_{V_2} \subseteq \ldots \), then \( \bigcup_{n=1}^{\infty} C_{V_n} \subseteq C \) and \( \nu\left(\bigcup_{n=1}^{\infty} C_{V_n}\right) = \lim_{n \to \infty} \nu(C_{V_n}) \).

Moreover, \( \nu(K^n) = 1 \). Finally let us observe that \( \nu \) is \( \sigma \)-subadditive. Next we define another set function \( \bar{\nu} \) as follows:

\[
\bar{\nu}(A) = \inf\{\nu(D): A \subseteq D, D \in C\}, \quad A \subseteq K^n.
\]

Standard calculations show that \( \bar{\nu} \) is an outer measure on \( K^n \). Thus from the Caratheodory Theorem, \( \bar{\nu} \) is a probability measure on the \( \sigma \)-field of \( \bar{\nu} \)-measurable subsets in \( K^n \). Setting \( \bar{\mu} = \bar{\nu}|_{\mathcal{B}^n} \), we obtain a desired probability measure.

We now present the following existence theorem.

**Theorem 2:** Let us suppose that \( G: I \times \mathbb{R}^n \to K^n \) is an integrably bounded multifunction of the Caratheodory type. Then for any probability measure \( \mu \) on \( \mathbb{R}^n \), there exists a weak solution of problem (I).

**Proof:** Lemma 2 yields the existence of a probability measure \( \bar{\mu} \) on the metric space \( (K^n, H) \) with the property: \( \bar{\mu}(C_V) = \mu(V) \), \( V \in \mathcal{T}_0 \). Let \( F: I \times K^n \to K^n \) be a multifunction defined by \( F(t, A) = \overline{\partial}G(t, A) \), for \( A \subseteq K^n \). Hence from Lemma 1.1 [9], the set-valued mapping \( F \) is integrably bounded of the Caratheodory type too. Consequently, by Theorem 1, there exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and the set-valued stochastic process \( X = (X_t)_{0 \leq t \leq T} \) (on it) with
continuous "paths" and with values in $K^n$ which is a weak solution of the equation

$$D_H X_t = F(t, X_t) \quad P.1, t \in [0, T]$$

$$X_0 \overset{d}{=} \tilde{\mu}.$$  

From Kuratowski and Ryll-Nardzewski Selection Theorem [4] we can choose $\xi: \Omega \rightarrow \mathbb{R}^n$ as a measurable selection of $X_0$. Then by Theorem 4 [5] (see also [3]), there exists a stochastic process $x = (x_t)_{0 \leq t \leq T}$ as a selection of $X$ that is a solution (in strong sense) of the random differential inclusion:

$$\dot{x}_t \in G(t, x_t) \quad P.1, \ t \in [0, T]$$

$$x_0 \in U \quad P.1,$$

where $U(\omega) = \{\xi(\omega)\}$ for $\omega \in \Omega$.

To complete the proof, it is sufficient to show that $x_0 \overset{d}{=} \mu$. Let us notice that $\{\omega: x_0(\omega) \in V\} \subset \{\omega: X_0 \cap V \neq \emptyset\} \subset V$. Because of $X_0 \overset{d}{=} \tilde{\mu}$ and $\tilde{\mu}(C_V) = \mu(V)$ we have

$$P_{x_0}(V) \leq \mu(V).$$  

(*)

Using regularity properties of probability measures (on a separable metric space) (see e.g., Th. 1.2 [8]), we have that

$$P_{x_0}(B) = \inf\{P_{x_0}(V): B \subset V, V \in \mathcal{F}_0\}$$

and $\mu(B) = \inf\{\mu(V): B \subset V, V \in \mathcal{F}_0\}$ for every Borel subset $B$ of $\mathbb{R}^n$. Hence from inequality (*) we get $P_{x_0}(B) \leq \mu(B)$. But $P_{x_0}$ and $\mu$ are probability measures. Therefore they have to be equal.

References


