REMARKS ON THE CONTROLLABILITY OF NONLINEAR PERTURBATIONS OF VOLterra INTEGRODIFFERENTIAL SYSTEMS

K. BALACHANDRAN and P. BALASUBRAMANIAM
Bharathiar University
Department of Mathematics
Coimbatore- 641 046, Tamil Nadu, India

(Received August, 1994; Revised March, 1995)

ABSTRACT

Sufficient conditions for the complete controllability of nonlinear perturbations of Volterra integrodifferential systems with implicit derivative are established. The results generalize the results of Dauer and Balachandran [9] and are obtained through the notions of condensing map and measure of noncompactness of a set.

Key words: Controllability, Integrodifferential Systems, Perturbations, Fixed Point Technique.

AMS (MOS) subject classifications: 93B05.

1. Introduction

The controllability of perturbed nonlinear systems has been studied by several authors [2-4, 7-9] with the help of fixed point theorems. Dacka [6] introduced a new method of analysis to study the controllability of nonlinear systems with implicit derivative based on the measure of noncompactness of a set and Darbo's fixed point theorem. This method has been extended to a larger class of perturbed systems by Balachandran [2, 3]. Anichini et al. [1] studied the problem through the notions of condensing map and measure of noncompactness of a set. They used the fixed point theorem due to Sadovskii [11]. In this note, we shall study the controllability of nonlinear perturbations of Volterra integrodifferential systems with implicit derivative by suitably adopting the technique of Anichini et al. [1]. The results generalize the results of Dauer and Balachandran [9].

2. Preliminaries

We first summarize some facts concerning condensing maps; for definitions and results about the measure of noncompactness and related topics, the reader can refer to the paper of Dacka [6]. Let $X$ be a subset of a Banach space. An operator $T:X \to X$ is called condensing if, for any bounded subset $E$ in $X$ with $\mu(E) \neq 0$, we have $\mu(T(E)) < \mu(E)$, where $\mu(E)$ denotes the measure of noncompactness of the set $E$ as defined in [11].

We observe that, as a consequence of the properties of $\mu$, if an operator $T$ is the sum of a compact and a condensing operator, then $T$ itself is a condensing operator. Further, if the operator
$P: X \rightarrow X$ satisfies the condition $|Px - Py| \leq k|x - y|$ for $x, y \in X$, with $0 \leq k < 1$, then the operator $P$ has a fixed point property. However, the condition $|Px - Py| < |x - y|$ for $x, y \in X$ is insufficient to ensure that $P$ is a condensing map or that $P$ will admit a fixed point (Browder [5]). The fixed point property holds in the condensing case (Sadovskii [11]).

Let $C_n(J)$ denote the space of continuous $R^n$ valued functions on the interval $J$. For $x \in C_n(J)$ and $h > 0$, let

$$
\theta(x, h) = \sup \{ |x(t) - x(s)| : t, s \in J \text{ with } |t - s| \leq h \},
$$

and write $\theta(E, h) = \sup \{ \theta(x, h) : x \in E \}$, so that $\theta(E, \cdot)$ is the modulus of continuity of a bounded set $E$. Set $\theta_0(E) = \lim_{h \to 0} \theta(E, h)$. Assume that $\Omega$ is the set of functions $\omega: R^+ \rightarrow R^+$ that are right continuous and nondecreasing such that $\omega(r) < r$ for $r > 0$. Let $J = [t_0, t_1]$.

**Lemma 1:** [11] Let $X \subset C_n(J)$ and let $\beta$ and $\gamma$ be functions defined on $[0, t_1 - t_0]$ such that $\lim_{s \to 0} \beta(s) = \lim_{s \to 0} \gamma(s) = 0$. If a transformation $T: X \rightarrow C_n(J)$ maps bounded sets into bounded sets such that

$$
\theta(T(x), h) < \omega(\theta(x, \beta(h)) + \gamma(h)) \text{ for all } h \in [0, t_1 - t_0]
$$

and $x \in X$ with $\omega \in \Omega$, then $T$ is a condensing mapping.

**Lemma 2:** [1, 11] Let $X \subset C_n([t_0, t_1])$, let $I = [0, 1]$, and let $S \subset X$ be a bounded closed convex set. Let $H: I \times S \rightarrow X$ be a continuous operator such that, for any $\alpha \in I$, the map $H(\alpha, \cdot): S \rightarrow X$ is condensing. If $x \neq H(\alpha, x)$ for any $\alpha \in I$ and any $x \in \partial S$ (the boundary of $S$), then $H(1, \cdot)$ has a fixed point.

Finally, it is possible to show that for any bounded and equicontinuous set $E$ in $C_n^1(J)$, the following relations holds:

$$
\mu_{C_n^1}(E) \equiv \mu_1(E) = \mu(DE) \equiv \mu_{C_n}(DE)
$$

where $DE = \{ \dot{x} : x \in E \}$.

3. Main Results

Consider the nonlinear perturbations of the Volterra integrodifferential system of the form

$$
\dot{x}(t) = g(t, x) + \int_{t_0}^t h(t, s, x(s))ds + B(t, x(t))u(t) + f(t, x(t), \dot{x}(t), (Sx)(t), u(t)), \ldots, t \in J = [t_0, t_1]
$$

$x(t_0) = x_0$, where the operator $S$ is defined by

$$
(Sx)(t) = \int_0^t k(t, s, x(s))ds.
$$

Here, $x(t) \in R^n$, $u(t) \in R^m$ and the functions $g, h, f, B$ and $k$ satisfy the following hypotheses:

i) $g: J \times R^n \rightarrow R^n$ is continuous and continuously differentiable with respect to $x$.

ii) $h: J \times J \times R^n \rightarrow R^n$ is continuous and continuously differentiable with respect to $x$.

iii) $B(t, x(t))$ is a continuous family of matrices on $J \times R^n$. 

iv) \( f: J \times R^n \times R^n \times R^n \times R^n \to R^n \) is continuous.

v) \( k: J \times J \times R^n \to R^n \) is continuous.

Let \( x(t, t_0, x_0) \) be the unique solution of the equation

\[
\dot{x}(t) = g(t, x) + \int_{t_0}^{t} h(t, s, x(s))ds
\]

existing on some interval \( J \).

Define

\[
G(t, t_0, x_0) = g_x(t, x(t, t_0, x_0))
\]

and

\[
H(t, s, t_0, x_0) = h_x(t, s, x(s, t_0, x_0)).
\]

Then \( X(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0) \) exist and is the solution of

\[
\dot{y}(t) = G(t, t_0, x_0)y(t) + \int_{t_0}^{t} H(t, s; t_0, x_0)y(s)ds
\]

such that \( X(t_0, t_0, x_0) = I \).

Then the solution of the equation (1) is given by [10]

\[
x(t) = x(t, t_0, x_0) + \int_{t_0}^{t} X(t, s, x(s))[B(s, x)u(s) + f(s, x(s), \dot{x}(s), (Sx)(s), u(s))]ds
\]

\[
+ \int_{t_0}^{t} \int_{s}^{t} [X(t, \tau, x(\tau)) - R(t, \tau; s, x(s))]h(\tau, s, x(s))d\tau ds
\]

where \( R(t, s; t_0, x_0) \) is the solution of the equation

\[
\frac{\partial R}{\partial s}(t, s; t_0, x_0) + R(t, s; t_0, x_0)G(s, t_0, x_0) + \int_{s}^{t} R(t, \tau; t_0, x_0)H(\tau, s; t_0, x_0)d\tau = 0
\]

such that \( R(t, t; t_0, x_0) = I \) on the interval \( t_0 < s \leq t \) and

\[
R(t, t_0; t_0, x_0) = X(t, t_0, x_0).
\]

We say the system (1) is completely controllable on \( J \) if, for any \( x_0, x_1 \in R^n \), there exists a continuous control function \( u(t) \) defined on \( J \) such that the solution of (1) satisfies \( x(t_1) = x_1 \).

Define the matrix \( W \) by

\[
W(t, t_0, x) = \int_{t_0}^{t} X(t, s, x(s))B(s, x(s))B^*(s, x(s))X^*(t, s, x(s))ds,
\]

where the star denotes the matrix transpose. Further define

\[
q(t, t_0, x) = \int_{t_0}^{t} \int_{s}^{t} [X(t, \tau, x(\tau)) - R(t, \tau; s, x(s))]h(\tau, s, x(s))d\tau ds.
\]
The main results concerning the controllability of the system (1) is given in the following theorem.

**Theorem:** Let the system (1) satisfy all the above conditions (i) to (v) and assume the additional conditions

(a) \( \limsup_{|x| \to \infty} \frac{|f(t,x,y, Sx, u)|}{|x|} = 0, \)

(b) there exists a continuous nondecreasing function \( \omega: \mathbb{R}^+ \to \mathbb{R}^+ \), with \( \omega(r) < r \), such that

\[
|f(t,x,y, Sx, u) - f(t,x,z, Sx, u)| < \omega(|y - z|) \text{ for all } (t,x,y, Sx, u) \in J \times \mathbb{R}^{3n} \times \mathbb{R}^n
\]

(c) there exists a positive constant \( \delta \) such that

\[
\det W(t_0, t_1, x) \geq \delta \text{ for all } x.
\]

Then the system (1) is completely controllable on \( J \).

**Proof:** Define the nonlinear transformation

\[
T: C_m(J) \times C_n^1(J) \to C_m(J) \times C_n^1(J)
\]

by

\[
T(u,x)(t) = (T_1(u,x)(t), T_2(u,x)(t))
\]

where the pair of operators \( T_1 \) and \( T_2 \) are defined by

\[
T_1(u,x)(t) = B^*(t,x)X^*(t_1,t,x)W^{-1}(t_1,t_0,x)[x_1 - x(t_1,t_0,x_0) - q(t_1,t_0,x) - \int_{t_0}^{t_1} X(t_1,s,x(s))f(s,x(s),\dot{x}(s),(Sx)(s),u(s))ds]
\]

\[
T_2(u,x)(t) = x(t,t_0,x_0) + q(t,t_0,x) + \int_{t_0}^{t} X(t,s,x(s))B(s,x(s))T_1(u,x)(s)ds + \int_{t_0}^{t} X(t,s,x(s))f(s,x(s),\dot{x}(s),(Sx)(s),T_1(u,x)(s))ds.
\]

Since all the functions involved in the definition of the operator \( T \) are continuous, \( T \) is continuous. Moreover, by direct differentiation with respect to \( t \), a fixed point for the operator \( T \) gives rise to a control \( u \) and a corresponding function \( x = x(t) \), solution of the system (1) satisfying \( x(t_0) = x_0, x(t_1) = x_1 \). Let

\[
\eta^0 = (u^0,x^0) \in C_m(J) \times C_n^1(J)
\]

\[
\eta = (u,x) \neq 0 \in C_m(J) \times C_n^1(J)
\]

and consider the equation

\[
\eta^0 = \eta - \alpha T(\eta),
\]

where \( \alpha \in [0,1] \). This equation can be equivalently written as
\[ \begin{align*}
\dot{u} &= u^0 + \alpha T_1(u, x) \\
\dot{x} &= x^0 + \alpha T_2(u, x).
\end{align*} \tag{2} \tag{3} \]

From condition (i), for any \( \epsilon > 0 \) there exists \( R > 0 \) such that if \( |x| > R \) then \( |f(t, x, y, (S)(x), u)| < \epsilon |x| \). Then from (2) we get

\[ |u| \leq |u^0| + |\alpha| |B| |X| |W^{-1}| x_1 | + |x(t_1, t_0, x_0)| + |q(t_1, t_0, x)| + |X| \epsilon |x| \delta \]

\[ \leq |u^0| + k_1 + |B| |X|^2 |W^{-1}| \epsilon \delta |x| \] \tag{4}

where \( \delta = t_1 - t_0 \) and

\[ k_1 = |B| |X| |W^{-1}| x_1 | + |x(t_1, t_0, x_0)| + |q(t_1, t_0, x)|]. \]

From this inequality and from (3), by applying the Gronwall Lemma, we obtain

\[ |x| \leq |x^0| + |x(t, t_0, x_0)| + |T_1(u, x)| |X| |B| \delta + |q(t, t_0, x)| \exp(|X| \epsilon \delta) \]

\[ \leq |x^0| + |x(t, t_0, x_0)| + (k_1 + |B| |X|^2 |W^{-1}| \epsilon \delta |x|) |X| |B| \delta \]

\[ + |q(t, t_0, x)| \exp(|X| \epsilon \delta). \] \tag{5}

Taking the derivative of (3) with respect to \( t \), we obtain

\[ \dot{x} = \frac{dx^0}{dt} + \alpha \frac{d}{dt} T_2(u, x(t)) \]

and that results in

\[ |\dot{x}| \leq |\dot{x}^0| + |g(t, x)| + \int_{t_0}^{t} |h(t, s, x(s))| \, ds + |B| |x(t)| |T_1(u, x(t))| \]

\[ + |f(t, x(t), \dot{x}(t), (Sx)(t), u(t))| \]

\[ \leq |\dot{x}^0| + |g(t, x)| + \int_{t_0}^{t} |h(t, s, x(s))| \, ds + |B| |k_1 + |B| |X|^2 |W^{-1}| \epsilon \delta |x| + \epsilon |x| \]

\[ = |\dot{x}^0| + k_2 + |x||B|^2 |X|^2 |W^{-1}| \epsilon \delta + \epsilon \] \tag{6}

where \( k_2 = |g(t, x)| + \delta |h(t, s, x(s))| + |B| |k_1| \).

From (4)

\[ |u| - |B| |X|^2 |W^{-1}| \epsilon \delta |x| \leq |u^0| + k_1 \] \tag{7}

and from (5)

\[ |x| \left| \exp(- |X| \epsilon) - |B|^3 |X|^3 |W^{-1}| \epsilon \delta^2 \right| \leq |x^0| + k_3 \] \tag{8}
where \( k_3 = |x(t, t_0, x_0)| + k_1 |X| |B| \delta + |q(t, t_0, x)| \) and from (6)
\[
|\dot{x}| - |x||B|^2|X|^2|W^{-1}|\epsilon \delta + \epsilon| \leq k_2 + |\dot{x}_0|.
\] (9)

Taking the sum of all the inequalities (7), (8) and (9), we obtain
\[
|u| - |x| \{ |B| |X|^2 |W^{-1}| \epsilon \delta - \exp(-|X|\epsilon \delta) + |B|^2 |X|^3 |W^{-1}| \epsilon \delta^2 \\
+ |B|^2 |X|^2 |W^{-1}| \epsilon \delta + |\dot{x}| \leq |u_0| + |z_0| + |\dot{z}_0| + k
\]
where \( k = k_1 + k_2 + k_3 \).

That is,
\[
|u| - \lambda |x| + |\dot{x}| \leq |u_0| + |z_0| + |\dot{z}_0| + k
\]
where \( \lambda = |B| |X|^2 |W^{-1}| \epsilon \delta \{ 1 + |B| |X| \delta + |B| \} + \epsilon - \exp(-|X|\epsilon \delta) \).

Then, for suitable positive constants \( a, b, c \) we can write
\[
|u| - [(\epsilon a - \exp(-\epsilon b)) |x| + |\dot{x}| \leq |u_0| + |z_0| + |\dot{z}_0| + c,
\]
so we divide by \(|u| + |x| + |\dot{x}| \) and from the arbitrariness of \( \epsilon \), we get the existence of a ball \( S \) in \( C_m(J) \times C^1_n(J) \) sufficiently large such that
\[
|\eta - \alpha T(\eta)| > 0 \text{ for } \eta = (u, x) \in \partial S.
\]

We want to show that \( T \) is a condensing map. To this aim, we note that \( T_1:C_n(J) \rightarrow C_m(J) \) is a compact operator and then, if \( E \) is a bounded set, \( \mu(T_1(E)) = 0 \). Then it will be enough to show that \( T_2 \) is a condensing operator. For that, let us consider the modulus of continuity of \( DT_2(u, x)(\cdot) \). Now, for \( t, s \in J \), we have
\[
|DT_2(u, x)(t) - DT_2(u, x)(s)| \leq |g(t, x(t)) - g(s, x(s))| + \int_{t_0}^t h(t, \tau, x(\tau))d\tau
\]
\[
- \int_{t_0}^s h(s, \tau, x(\tau))d\tau + |B(t, x(t)) T_1(u, x)(t) - B(s, x(s)) T_1(u, x)(s)|
\]
\[
+ |f(t, x(t), \dot{x}(t), (Sx)(t), T_1(u, x)(t)) - f(s, x(s), \dot{x}(s), (Sx)(s), T_1(u, x)(s))|.
\]

For the first three terms of the right hand side of the inequality, we may given the upper estimate as \( \beta_0(|t - s|) \) with \( \lim_{h \to 0} \beta_0(h) = 0 \) and it may be chosen independent of the choice of \((u, x)\). For the fourth term, we can given the following estimate:
\[
|f(t, x(t), \dot{x}(t), (Sx)(t), T_1(u, x)(t)) - f(s, x(s), \dot{x}(s), (Sx)(s), T_1(u, x)(s))|
\]
\[
\leq |f(t, x(t), \dot{x}(t), (Sx)(t), T_1(u, x)(t)) - f(t, x(t), \dot{x}(s), (Sx)(t), T_1(u, x)(t))|
\]
\[
+ |f(t, x(t), \dot{x}(s), (Sx)(t), T_1(u, x)(t)) - f(s, x(s), \dot{x}(s), (Sx)(s), T_1(u, x)(s))|.
\]

For the first term we have the upper estimate \( \omega(|\dot{x}(t) - \dot{x}(s)|) \) whereas for the second term,
we may find an estimate
\[ \beta_1(\mid t-s \mid) \text{ with } \lim_{h \to 0} \beta_1(h) = 0. \]

Hence
\[ \theta(DT_2(u,x),h) \leq \omega(\theta(DE,h) + \beta(h)) \]
where \( \beta = \beta_0 + \beta_1. \) Therefore, by Lemma 1, we get
\[ \theta_0(DT_2(E)) < \theta_0(DE). \]

Hence, from
\[ 2\mu_1(T_2(E)) = 2\mu(DT_2(E)) = \theta_0(DT_2(E)) < \theta_0(DE) \]
\[ = 2\mu(DE) = 2\mu_1(E), \]
it follows that \( \mu_1(T_2(E)) < \mu_1(E). \) Then the existence of a fixed point of the operator \( T \) follows from Lemma 2. In other words, there exists functions \( u \in C_m(J) \) and \( x \in C^1_n(J) \) such that
\[ T(u,x) = (u,x) \]
and
\[ u(t) = T_1(u,x)(t), \quad x(t) = T_2(u,x)(t). \]
These functions are the required solutions. Further, it is easy to verify that the function \( x(\cdot) \) given by the systems (1) satisfies the boundary conditions \( x(t_0) = x_0 \) and \( x(t_1) = x_1. \) Hence, the system (1) is completely controllable.

Acknowledgements

The authors are grateful to Professor Jewgeni Dshalalow for his kind help. This work is supported by a grant from CSIR, New Delhi.

References


Submit your manuscripts at http://www.hindawi.com