AN APPROACH TO THE STOCHASTIC CALCULUS IN THE NON-GAUSSIAN CASE

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ABSTRACT

We introduce and study a class of operators of stochastic differentiation and integration for non-Gaussian processes. As an application, we establish an analog of the Itô formula.

Key words: Non-Gaussian Stochastic Process, Stochastic Integral, Stochastic Derivative, Itô's Formula.

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1. Introduction

Operators of stochastic differentiation $D$ and an extended integration $I = D^*$ play an important role in stochastic calculus. In the Gaussian case and for certain special martingales, $D$ and $I$ can be defined with the aid of an orthogonal expansion (cf., T. Sekiguchi, Y. Shiota [3]). Also, $D$ and $I$ can be defined by means of the usual differentiation with respect to the admissible translation of the probability measure (A.A. Dorogovtsev [2]). In all these situations there are some common features. In this article we consider a general scheme in which the operators $D$ and $I$ are constructed for a non-Gaussian case. Since $I$ plays the role of stochastic integration, an analog of the Itô formula is also established.

2. Stochastic Derivative and the Logarithmic Process

Let $\{\xi(t); t \in [0,1]\}$ be a random process defined on a probability space $(\Omega, \mathcal{F}, P)$. A subset $K$ of $\mathbb{R}^n$ is said to have the conic property if for every $x \in K$, there exists a cone, $C_x$, with the non-empty interior and a neighborhood, $U_x$ of $x$ such that $x \in U_x \cap C_x \subset K$.

Suppose that the support of any finite-dimensional distribution of $\xi$ has the conic property.

Let $\lambda$ be the Lebesgue measure on the Borel $\sigma$-algebra $\mathcal{B}([0,1])$.

Definition 1: A family of the random elements $\{\xi(t); t \in [0,1]\}$ from $L_2(\Omega \times [0,1], P \times \lambda)$ is called a differentiation rule if

1) $\forall t \in [0,1]; \zeta(t) \cdot \chi_{(t,1)} = 0 \ (\text{mod} \ P)$,

2) for every tuple $t_1, \ldots, t_n \in [0,1], a_1, \ldots, a_n \in \mathbb{R}, n \geq 1, G \in \mathcal{F}$, such that

$$ (a_1 \xi(t_1) + \ldots + a_n \xi(t_n)) \chi_G = 0 \ (\text{mod} \ P), $$

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the following equality holds

\[(a_1\zeta(t_1) + \ldots + a_n\zeta(t_n))x \equiv 0 \pmod{p \times \lambda}.

**Definition 2:** Let \(\varphi: \mathbb{R}^n \to \mathbb{R}\) be bounded, continuously differentiable and have a bounded derivative. For a random variable

\[\alpha = \varphi(\xi(t_1), \ldots, \xi(t_n)), \quad t_1, \ldots, t_n \in [0, 1],\]

the sum

\[\varphi'_1(\xi(t_1), \ldots, \xi(t_n))\zeta'(t_1) + \ldots + \varphi'_n(\xi(t_1), \ldots, \xi(t_n))\zeta'(t_n)\]

is called a **stochastic derivative** of \(\alpha\) and denoted by \(D\alpha\) (so \(D\xi(t) = \zeta(t)\)).

In the sequel, denote the set of all random variables from Definition 2 by \(\mathcal{M}\). \(\mathcal{M}\) is a linear subset of \(L_2(\Omega, \mathcal{F}, P)\). Also for \(t \in [0, 1]\), denote by \(\mathcal{M}_t\) the subset of \(\mathcal{M}\) which is only from \(\{\xi(s), 0 \leq s \leq t\}\). Obviously, \(\mathcal{M}_1 = \mathcal{M}_0\).

**Lemma 1:** \(D\) is well-defined on \(\mathcal{M}\).

**Proof:** Consider \(\varphi, \psi: \mathbb{R}^n \to \mathbb{R}\) which satisfy the conditions in Definition 2, and let \(t_1, \ldots, t_n\) be such that

\[\varphi(\xi(t_1), \ldots, \xi(t_n)) = \psi(\xi(t_1), \ldots, \xi(t_n)) \pmod{p}.

Then, it follows from the assumption about \(\xi\) that for all \(i = 1, \ldots, n\),

\[\varphi'_i(\xi(t_1), \ldots, \xi(t_n)) = \psi'_i(\xi(t_1), \ldots, \xi(t_n)) \pmod{p}.

Thus, the corresponding sums in Definition 2 are equal. The lemma is proved.

**Definition 3:** A random process \(\xi\) is said to have a **logarithmic derivative** with respect to a differentiation rule \(\zeta\) if there exist a random process \(\{p_\Delta, \Delta \in \mathcal{B}\}\) indexed by the Borel subsets of \([0, 1]\) such that

1) \(\forall \Delta \in \mathcal{B}, M p_\Delta^2 < +\infty\);
2) \(\forall \alpha \in \mathcal{M}\) and \(\forall \Delta \in \mathcal{B}\);

\[M \int_{\Delta} D\alpha(\tau)d\tau = M\alpha \cdot p_\Delta.

In the sequel, suppose that the process \(\xi\) satisfies the conditions in Definition 3.

**Definition 4:** Denote for \(t \in [0, 1]\),

\[m(t) = p_{[0, t]}\]

The process \(\{m(t); t \in [0, 1]\}\) is called the **logarithmic process**.

Let for \(t \in [0, 1], \mathcal{F}_t = \sigma(\{\xi(s); s \leq t\})\). Note, that analogous processes were considered in different situation in A. Benassi [1].

**Lemma 2:** For \(0 \leq s \leq t \leq 1\),

\[M(m(t) - m(s))/\mathcal{F}_s = 0 \pmod{p}.

**Proof:** For \(\alpha \in \mathcal{M}_s\) consider

\[M(m(t) - m(s)) \cdot \alpha = M_{[0, t]} \cdot \alpha - M_{[0, s]} \cdot \alpha\]
\[ M \int_0^t D\alpha(\tau) d\tau - M \int_0^s D\alpha(\tau) d\tau = M \int_{(s,t]} D\alpha(\tau) d\tau = \sum_{i=1}^n M \int_{(s,t]} \varphi_i'((\xi(\tau_1), \ldots, \xi(\tau_n))) \cdot \xi(\tau_i) d\tau = 0 \pmod{P}. \]

Since the set \( \mathcal{A}_s \) is dense in \( L_2(\Omega, \mathcal{F}_s, P) \) then the statement of the lemma follows.

For further considerations the following result will be useful.

**Lemma 3:** The operator \( D \) can be closed as a linear operator from \( \mathcal{A}_s \subset L_2(\Omega, \mathcal{F}, P) \) to \( L_2(\Omega \times [0,1], P \times \lambda) \).

**Proof:** Consider a sequence \( \{\alpha_n; n \geq 1\} \subset \mathcal{A}_s \), such that there exists \( \nu \in L_2(\Omega \times [0,1], P \times \lambda) \) for

\[ M \alpha_n^2 \to 0, \quad n \to \infty, \]
\[ M \int_0^1 (D\alpha_n(\tau) - \nu(\tau))^2 \lambda(d\tau) \to 0, \quad n \to \infty. \]

Then, for every \( \Delta \in \mathcal{B} \) and \( \beta \in \mathcal{A}_s \),

\[ M \beta \cdot \int_\Delta \nu(\tau) d\tau = \lim_{n \to \infty} M \beta \cdot \int_\Delta D\alpha_n(\tau) d\tau = \lim_{n \to \infty} (M \int_\Delta (D\alpha_n(\beta)(\tau) d\tau - M \alpha_n \int_\Delta D\beta(\tau) d\tau) = \lim_{n \to \infty} (M \alpha_n \beta \cdot \rho_\Delta - M \alpha_n \int_\Delta D\beta(\tau) d\tau) = \lim_{n \to \infty} M \alpha_n (\beta \cdot \rho_\Delta - \int_\Delta D\beta(\tau) d\tau) = 0 \pmod{P}. \]

So,

\[ \int_\Delta \nu(\tau) d\tau = 0 \pmod{P}. \]

Since \( \Delta \) was arbitrary,

\[ \nu = 0 \pmod{P \times \lambda}. \]

The lemma is proved.

Denote the closure of \( D \) by the same symbol. The domain of \( D \) is denoted by \( W^1 \).

3. **Integral with Respect to the Logarithmic Process and the Procedure of Approximation**

**Definition 5:** The adjoint operator

\[ I = D^*: L_2(\Omega \times [0,1]; P \times \lambda) \to L_2(\Omega, \mathcal{F}, P) \]
is called a stochastic integration with respect to the process \( m \). The domain of \( I \) is denoted by \( \mathcal{D} \).

In the following, suppose that

\[
\forall \Delta \in \mathcal{D}; \rho_\Delta \in W^1,
\]

and, that the correspondence \( \Delta \mapsto \rho_\Delta \) can be extended by the bounded linear operator \( A: L_2([0,1], \lambda) \to W^1 \) (the inner product in \( W^1 \) is defined in the usual way, as a sum of \( L_2 \)-products of random variables and their stochastic derivatives). Note that under this assumption, each \( \varphi \in L_2([0,1]) \) also belongs to \( \mathcal{D} \) and

\[
I(\varphi) = A(\varphi).
\]

To have \( I \) act on random elements of \( L_2([0,1]) \), i.e., to define an extended stochastic integral with respect to the process \( m \), we need the following.

Let \( \{K_n; n \geq 1\} \) be a sequence of symmetric kernels defined on \( [0,1]^2 \) such that

1) \( K_n \in L_2([0,1]^2, \lambda \times \lambda) \),
2) \( A \in L_2([0,1], \lambda) \),

where \( K_n \) is an integral operator in \( L_2([0,1], \lambda) \) with the kernel \( K_n \). Denote for \( n \geq 1 \),

\[
h_n(s,r) = D\left( \int_0^1 K_n(s,\tau)dm(\tau) \right)(r).
\]

It follows from the existence of the operator \( A \) that

\[
\forall n \geq 1; h_n \in L_2([0,1]^2, \lambda \times \lambda) \quad (\text{mod } P).
\]

Consider the following sequences of integral operators with random kernels:

\[
\forall \varphi \in L_2([0,1], \lambda) \text{ and } \forall n \geq 1;
\]

\[
B_n(\varphi)(t) = \int_0^1 \varphi(s) \int_0^1 h_n(s,\tau)K_n(t,\tau)d\tau ds,
\]

\[
C_n(\varphi)(t) = \int_0^t \varphi(s) \int_0^1 h_n(s,\tau)K_n(t,\tau)d\tau ds.
\]

Suppose that for every \( \varphi \) there exist

\[
L_2 - \lim_{n \to \infty} B_n(\varphi) = B(\varphi) \quad \text{and} \quad L_2 - \lim_{n \to \infty} C_n(\varphi) = C(\varphi).
\]

Then the operators \( B \) and \( C \) are strong random linear operators (A.V. Skorokhod [4]) which are continuous in \( L_2 \)-sense.

**Definition 6:** A random element \( x \) from \( L_2([0,1], \lambda) \) is said to belong to the domain of \( B \) (or \( C \)) if the sequence \( \{B_n(x); n \geq 1\} \) converges in \( L_2 \)-sense (\( \{C_n(x); n \geq 1\} \) respectively).

The following statement can be verified.

**Lemma 4:** Let \( H \) be a separable real Hilbert space embedded into \( L_2([0,1], \lambda) \) by the Hilbert-Schmidt operator, and let \( x \) be an essentially bounded random element of \( H \). Then, \( x \in \mathcal{D}(B) \) and
Now, consider the stochastic integration. Suppose that the differentiation rule is such that the highest derivatives are symmetric, i.e.,

\[ D^2 \alpha(\tau_1, \tau_2) = D^2 \alpha(\tau_2, \tau_1) \mod P \times \lambda \times \lambda. \]

The space of random variables which have \( k \)th stochastic derivative will be denoted by \( W^k \).

**Lemma 5:** For every bounded \( \alpha_1, \ldots, \alpha_n \in W^2 \) and for every \( \varphi_1, \varphi_2, \ldots, \varphi_n \in L^2([0; 1], \lambda) \), the sum 

\[ x = \sum_{i=1}^{n} \alpha_i \varphi_i \in \mathcal{D} \]

and

\[ I(x) = \sum_{i=1}^{n} \alpha_i I(\varphi_i) - \sum_{i=1}^{n} \int_{0}^{1} D\alpha_i(\tau) \varphi_i(\tau) d\tau, \]

\[ MI(x) = 0, \]

\[ MI(x)^2 = M \left\{ \int_{0}^{1} (Bx)(\tau) x(\tau) d\tau + tr(Dx \cdot Dx) \right\}. \]

**Proof:** First consider \( x = \alpha \cdot \varphi \). For every \( \beta \in \mathcal{M} \),

\[ M \int_{0}^{1} D\beta(\tau) \cdot x(\tau) d\tau = M \alpha \int_{0}^{1} D\beta(\tau) \varphi(\tau) d\tau \]

\[ = M \int_{0}^{1} (D(\alpha \beta)(\tau) - \beta D\alpha(\tau)) \varphi(\tau) d\tau \]

\[ = M \alpha \beta I(\varphi) - M \beta \int_{0}^{1} D\alpha(\tau) \varphi(\tau) d\tau \]

\[ = M \beta [\alpha I(\varphi) - \int_{0}^{1} D\alpha(\tau) \varphi(\tau) d\tau]. \]

So, \( \alpha \cdot \varphi \in \mathcal{D} \) and

\[ I(\alpha \cdot \varphi) = \alpha \cdot I(\varphi) - \int_{0}^{1} D\alpha(\tau) \varphi(\tau) d\tau. \]

Consequently,

\[ I(\sum_{i=1}^{n} \alpha_i \varphi_i) = \sum_{i=1}^{n} \alpha_i I(\varphi_i) - \sum_{i=1}^{n} \int_{0}^{1} D\alpha_i(\tau) \varphi_i(\tau) d\tau \]

\[ = \sum_{i=1}^{n} \alpha_i I(\varphi_i) - tr(D \sum_{i=1}^{n} \alpha_i \varphi_i). \]

To prove that \( MI(x) = 0 \) it is sufficient to see that \( D1 = 0 \) and use the equation \( I = D^* \). Now, consider the following chain of equalities:

\[ MI(x)^2 = M \left[ \sum_{i_1, i_2 = 1}^{n} \alpha_{i_1} \alpha_{i_2} I(\varphi_{i_1}) I(\varphi_{i_2}) - 2 \sum_{i_1, i_2 = 1}^{n} \alpha_{i_1} I(\varphi_{i_1}) \int_{0}^{1} D\alpha_{i_2}(\tau) \varphi_{i_2}(\tau) d\tau \right] \]
\[ M \left[ \sum_{i_1^2 = 1}^{n} \alpha_{i_1} \int_0^1 D(I(\varphi_{i_1}))(\tau) \varphi_{i_2}(\tau) d\tau + \sum_{i_1^2 = 1}^{n} \alpha_{i_2} \int_0^1 D(I(\varphi_{i_2}))(\tau) \varphi_{i_1}(\tau) d\tau \right] \]

\[ = M \left[ \sum_{i_1^2 = 1}^{n} \alpha_{i_1} \int_0^1 D(I(\varphi_{i_1}))(\tau) \varphi_{i_2}(\tau) d\tau + \sum_{i_1^2 = 1}^{n} \alpha_{i_2} \int_0^1 D(I(\varphi_{i_2}))(\tau) \varphi_{i_1}(\tau) d\tau \right] \]

Note that, due to the previous lemma, \( x \in \mathcal{B} \), and

\[ B_n(x) = \sum_{i = 1}^{n} \alpha_i \int_0^1 \varphi_i(s) \int_0^1 D \left( \int_0^1 K_n(s, \tau) dm(\tau) \right) (\tau) K_n(\cdot, \tau) ds d\tau, \quad n \geq 1. \]
\[ B(x) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i \int_{0}^{1} K_n(\tau, \varphi_i(s)K_n(s, \tau)ds \right) dm(\tau) (\tau) d\tau \]
\[ = \sum_{i=1}^{n} \alpha_i \cdot D(I(\varphi_i)). \]

Consequently,
\[ \sum_{i_1, i_2=1}^{n} \alpha_{i_1} \alpha_{i_2} \int_{0}^{1} D(I(\varphi_{i_1}))(\varphi_{i_2})(\tau)d\tau = \int_{0}^{1} B(x)(\tau)x(\tau)d\tau. \]

The lemma is proved.

From this lemma and from the fact that \( I \) is a closed operator, it follows that every random element \( x \) that satisfies the conditions of Lemma 4 and has a stochastic derivative belongs to \( \mathcal{F} \), and the equalities from Lemma 5 are valid.

The famous particular case of this situation is as follows. Let \( H \) be a Sobolev space of the first order on \([0, 1]\). Then elements of \( H \) have usual derivatives with respect to parameters from \([0, 1]\). Suppose that \( x \) satisfies the conditions of Lemma 4 and that \( Dx \) is a.s. a nuclear operator. Then,
\[ I(x) = x(1)m(1) - \int_{0}^{1} m(t)x'(t)dt - trDx. \]

Note also that in this case,
\[ \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} K(t, \tau) dm(\tau) dm(t) - n \int_{0}^{1} x(t)dm(t) = (1) \]

This expansion enables one to establish the Itô formula.

**Theorem (The Itô formula):** Let a function \( F: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) have a continuous bounded derivative of the first and second order, and let the random process \( x \) satisfy the conditions:
1) \( x \) has the second stochastic derivative;
2) for every \( \tau \in [0, 1] \), \( x \) and \( Dx(\cdot)(\tau) \) satisfy all integrability conditions (considered above);
3) \( Dx(\cdot)(\cdot) \in C([0, 1]^2) \) (mod \( P \));
4) \( x, Dx \) and \( D^2x \) are bounded.
Then, the following random process
\[ z(t) = \int_{0}^{t} x(\tau)dm(\tau), \quad t \in [0, 1] \]
is well-defined and it holds true that
\[ F(t, z(t)) = F(0, 0) + \int_{0}^{t} F'_1(s, z(s))ds \]
\[ + \int_{0}^{t} F'_2(s, z(s))x(s)dm(s) + \int_{0}^{t} x(s)F''(s, z(s))C(x)(s)ds \]
\[ + \int_{0}^{t} x(s)F''(s, z(s)) \cdot \int_{0}^{s} Dx(r)(s)dm(r)ds. \]
The proof follows directly from the expansion (1) and approximation arguments.

4. Examples

**Example 1:** (Wiener case) Let \( \xi(t) = w(t), \quad t \in [0, 1] \) be a Wiener process. Consider the differentiation rule of the form \( \zeta(t) = \chi_{[0, t]}, \quad t \in [0, 1] \). Then the stochastic derivative \( D \) which is obtained from this rule is a well-known stochastic derivative of \( L^2 \)-integrable Wiener functionals (T. Sekiguchi, Y. Shiota [3]) and \( m(t) = w(t), \quad t \in [0, 1] \).

Now the operator \( B \) is the identity operator and \( C = \frac{1}{2}B \). Then, from the previous theorem we can obtain the Itô formula for the extended stochastic integral in the Gaussian case:

\[
F(t, z(t)) = F(0, 0) + \int_0^t F'_1(s, z(s))ds + \int_0^t F'_2(s, z(s))dw(s) \\
+ \frac{1}{2} \int_0^t F''_{22}(s, z(s)) \cdot z(s)^2 ds + \int_0^t x(s)F''_{22}(s, z(s)) \cdot \int_0^s Dz(r)dw(r)ds.
\]

**Example 2:** Let the distribution of the process \( \xi \) in the space \( C([0, 1]) \) be absolutely continuous with respect to the Wiener measure with the density \( p \). Suppose, that

1) \( 0 < \inf p \leq \sup p < \infty \),
2) \( p \) has a bounded continuous derivative on \( C([0, 1]) \).

Consider the differentiation rule from Example 1: \( \zeta(t) = \chi_{[0, t]}, \quad t \in [0, 1] \). Then the stochastic derivative of the random variable \( \alpha \) from the family \( \mathcal{M} (M) \) is of type

\[
D\alpha = D\varphi(\xi(t_1), \ldots, \xi(t_n)) = \sum_{i=1}^n \varphi_i \chi_{[0, t_i]}.
\]

Hence, for the Borel subset

\[
M \int \Delta D\alpha(\tau)d\tau = M \sum_{i=1}^n \varphi_i(\delta_{t_i}) \int \chi_\Delta(\tau)d\tau).
\]

Here \( \delta_{t} \) is Dirac \( \delta \)-function with respect to the point \( t \). Denote by \( u_\Delta \) the function

\[
u_\Delta(s) = \int_0^s \chi_\Delta(\tau)d\tau, \quad s \in [0, 1],
\]

by \( \nu \) the distribution of \( \xi \), and by \( \mu \) the Wiener measure. Also, denote by \( \Phi \) the following function on \( C([0, 1]) \):

\[
\forall v \in C([0, 1]), \Phi(v) = \varphi(v(t_1), \ldots, v(t_n)).
\]

Then,

\[
M \int \Delta D\alpha(\tau)d\tau = \int \Phi(\nu ; u_\Delta) \nu(du) = \int \Phi'(v ; u_\Delta)p(v)\mu(du)
\]

\[
= \int \langle (p(v)\Phi(v)') ; u_\Delta \rangle \mu(du) - \int \langle p'(v) ; u_\Delta \cdot \Phi(v) \mu(du) = \int \Phi(v)p(v) \cdot \int d\nu(\tau)\mu(du)
\]
Here the symbol of integration is used for the integration through all \( C([0,1]) \), and the integral

\[
\int \Delta d\tau
\]

is a measurable linear functional on \( C([0,1]) \) with respect to the measure \( \nu \sim \mu \). Note also that the function

\[
\int \Delta d\tau - \langle (ln p(\xi))'; u_\Delta \rangle
\]

is square-integrable with respect to the measure \( \nu \). Consequently, \( \xi \) has a logarithmic derivative, and

\[
\rho_\Delta = \int \Delta d\xi(\tau) - \langle (ln p(\xi))'; u_\Delta \rangle.
\]

Hence, the logarithmic process is of the form

\[
m(t) = \xi(t) - \int_0^t Dln p(\xi) d\tau.
\]

Now the second stochastic derivatives are symmetric. So to estimate the second moment of the extended stochastic integral only the operator \( B \) is essential. To describe the operators \( B \) and \( C \) let us find the stochastic derivative of the integral

\[
\int_0^1 f(\tau) d\tau
\]

Using the approximation by step functions, it can be verified that

\[
D \left( \int_0^1 f(\tau) d\tau \right) (s) = f(s) + \int_0^1 f(\tau) \cdot D^2ln p(\xi)(\tau, s) d\tau, \quad s \in [0,1].
\]

Consequently, for the \( n \geq 1 \),

\[
B_n(\varphi)(t) = \int_0^1 \varphi(s) \int_0^1 K_n(s, \tau) + \int_0^1 K_n(s, r)D^2ln p(\xi)(r, \tau) dr \cdot K_n(t, \tau) d\tau ds.
\]

Hence,

\[
B(\varphi)(t) = \varphi(t) + \int_0^1 D^2ln p(\xi)(t, s) \varphi(s) ds.
\]
In a similar way,

\[ C(\varphi)(t) = \frac{1}{2} \varphi(t) + \int_0^t D^2 \ln p(\xi)(s, t) \varphi(s) ds. \]

Now the second moment of the extended stochastic integral and the Itô formula have the form

\[ M \left( \int_0^1 x(t) dm(t) \right)^2 = M \int_0^1 x^2(t) dt + M \int_0^1 D^2 \ln p(\xi)(t, s) x(t) x(s) dt ds + M (\text{tr}(Dx))^2; \]

\[ F(t, z(t)) = F(0, 0) + \int_0^t F'_1(s, z(s)) ds + \int_0^t F''_1(s, z(s)) x(s) dm(s) \]

\[ + \frac{1}{2} \int_0^t F''_2(s, z(s)) z^2(s) ds + \int_0^t F''_2(s, z(s)) x(s) \int_0^s D^2 \ln p(\xi)(\tau, s) x(\tau) d\tau ds \]

\[ + \int_0^t x(s) F''_2(s, z(s)) \cdot \int_0^s Dx(\tau) dm(\tau) ds. \]

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