A STOCHASTIC MODEL FOR THE FINANCIAL MARKET WITH DISCONTINUOUS PRICES¹

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(Received May, 1994; Revised January, 1996)

ABSTRACT

This paper models some situations occurring in the financial market. The asset prices evolve according to a stochastic integral equation driven by a Gaussian martingale. A portfolio process is constrained in such a way that the wealth process covers some obligation. A solution to a linear stochastic integral equation is obtained in a class of cadlag stochastic processes.

Key words: Contingent Claim Valuation, Representation of Martingales, Stochastic Integral Equation, Option Pricing, Portfolio Processes.

AMS (MOS) subject classifications: 60H20, 60H30.

1. Introduction

In the present paper we model investments of an economic agent whose decisions cannot affect market prices (a “small investor”).

Karatzas and Shreve in [7] considered a market model in which prices evolve according to a stochastic differential equation, driven by Brownian motion. Aase [1] and M. Picqué and M. Pontier [9] studied a more general model in which the evolution of asset prices is a combination of a continuous process based on Brownian motion (a semimartingale) and a Poisson point process.

The security price model that we use is a linear stochastic equation driven by a Gaussian martingale. This is a natural generalization, because the market is not continuous and the Brownian motion cannot model jump processes. Moreover, the instants of jumps of a Gaussian martingale are nonrandom.

The techniques we use include the martingale representation theorem and the Girsanov's type theorem. We also find a solution to a linear stochastic integral equation.

2. The Model

We consider a model of a security market where an economic agent is allowed to trade continuously up to some fixed planning horizon $0 \leq T < \infty$. We shall denote by X_t the wealth of this

¹This work was supported by the National Science Foundation of Bulgaria, Grant No. MM-440/94.

agent at time t . Let the process $\mathbf{M} = (M_t, F_t, 0 \leq t \leq T)$ be a Gaussian martingale on a fixed probability space $(\Omega, F, \mathbf{P})$ and the filtration $\mathbf{F} = \{F_t, 0 \leq t \leq T\}$ be the augmentation under $\mathbf{P}$ of a natural filtration $F_t^M = \sigma(M_s, 0 \leq s \leq t)$, $0 \leq t < \infty$. F_0 contains the null sets of $\mathbf{P}$ and $\mathbf{F}$ is right continuous. $\langle M \rangle_t = EM_t^2$, $t \in \mathbb{R}_+ = [0, \infty)$ is the square characteristic of $\mathbf{M}$.

Let us suppose that the agent invests in two assets (or “securities”). One of the assets, called *bond*, has a finite variation on $[0, T]$, and its price model is

$$P_0(t) = \int_{(0, t]} P_0(s-) r(s-) d\langle M \rangle_s, \quad P_0(0) = p_0, \quad 0 \leq t \leq T.$$

The other one, called *stock*, is “risky”. Its price is modeled by the linear stochastic equation

$$P(t) = \int_{(0, t]} P(s-) A(s-) d\langle M \rangle_s + \int_{(0, t]} P(s-) \sigma(s-) dM_s, \quad P(0) = p.$$

Here the interest rate process $r(t) > 0$, $0 \leq t < \infty$ of the bond, the appreciation rate process $A(t)$ of the stock, and volatility process $\sigma(t) > 0$, $0 \leq t < \infty$ will all be nonrandom, F -predictable processes such that

$$\begin{aligned} \int_{(0, \infty)} r^2(s-) d\langle M \rangle_s < \infty, \quad \int_{(0, \infty)} A^2(s-) d\langle M \rangle_s < \infty, \\ \int_{(0, \infty)} \sigma^2(s-) d\langle M \rangle_s < \infty, \quad \mathbf{P}\text{-a.s.} \end{aligned} \tag{1}$$

In addition, $A(t-) \Delta \langle M \rangle_t + \sigma(t-) \Delta M_t > -1$, $t \in (0, T]$, to ensure a limited liability of the stock.

Let $m(t)$ denote the number of stocks held at time t . Then the amount invested in the stocks is

$$\Pi(t) = m(t)P(t).$$

The process $(\Pi(t), F_t)$, $0 \leq t \leq T$ describes the investment policy and will be called a *portfolio process*. It is assumed to be measurable, F_t -predictable and

$$\int_{(0, T]} \Pi^2(s-) d\langle M \rangle_s < \infty, \quad \mathbf{P}\text{-a.s.} \tag{2}$$

for every finite number $T > 0$. Note that $\Pi(t)$ can be negative, which amounts to selling the stock short.

On the other hand, $C(t)$, $0 \leq t \leq T$ is a non-negative consumption process, assumed to be nondecreasing and F_t -predictable, such that

$$\int_{(0, T]} C(s-) d\langle M \rangle_s < \infty, \quad \mathbf{P}\text{-a.s.} \tag{3}$$

for every finite number $T > 0$.

The quantity

$$\Pi_0(t) = X_t - \Pi(t),$$

is invested in the bond at any particular time and may also become negative. This is to be interpreted as borrowing at the interest rate $r(t)$.

We assume now that the investor starts with some initial wealth $x \geq 0$, and the wealth at time t satisfies the linear stochastic equation

$$\begin{aligned} X_t = & \int_{(0,t]} \Pi(s-) \sigma(s-) dM_s + \int_{(0,t]} \Pi(s-) [A(s-) - r(s-)] d\langle M \rangle_s \\ & + \int_{(0,t]} [X_s - r(s-) - C(s-)] d\langle M \rangle_s, \quad 0 < t \leq T; \\ & X(0) = x. \end{aligned} \quad (4)$$

Conditions (1), (2), and (3) ensure that the stochastic equation (4) has a unique solution in the class of cadlag adapted processes (see Section 5 and Theorem 3).

3. Characterization of the Portfolio Process

If $A(t) = r(t)$ for every $t \in [0, \infty)$, the drift

$$\int_{(0,t]} \Pi(s-) [A(s-) - r(s-)] d\langle M \rangle_s$$

vanishes from the right-hand side of (4). When $A(t) \neq r(t)$ we introduce a new probability measure $\bar{\mathbf{P}}$ which removes this drift.

Let us denote by ϕ_t the solution of the equation

$$\phi_t = 1 - \int_{(0,t]} \phi_s - \theta(s-) dM_s, \quad 0 \leq t \leq T,$$

where

$$\theta(t) = \frac{A(t) - r(t)}{\sigma(t)}.$$

From our assumptions on A , r , and σ , it follows that $\theta(t)$ is bounded, measurable and adapted to $\{F_t -\}$. Then the exponential supermartingale

$$\phi_t = \exp \left[- \int_{(0,t]} \theta(s-) dM_s^c - \frac{1}{2} \int_{(0,t]} \theta^2(s-) d\langle M^c \rangle_s \right] \cdot \prod_{s \leq t} [1 - \theta(s-) \Delta M_s],$$

is actually a martingale, where

$$\Delta M_t = M_t - M_{t-} \neq \frac{\sigma(t-)}{A(t-) - r(t-)} \quad \text{for } 0 < t \leq T.$$

Here M_t^c and $\langle M^c \rangle_t$ are the continuous parts of the processes M_t and $\langle M \rangle_t$, respectively, for $t \in \mathbb{R}_+$.

We define the new probability measure $\bar{\mathbf{P}}$:

$$\bar{\mathbf{P}}(A) = \mathbf{E}(\phi_T \mathbf{I}_A), \quad A \in F_T \text{ on } (\Omega, F).$$

The probability measures $\mathbf{P}$ and $\bar{\mathbf{P}}$ are mutually absolutely continuous on F_T .

The process

$$\bar{M}_t = M_t + \int_{(0,t]} \theta(s-) d\langle M \rangle_s, \quad 0 \leq t \leq T, \quad (6)$$

is a $\bar{\mathbf{P}}$ -Gaussian martingale [8], and

$$(\langle \bar{M} \rangle_t, \bar{\mathbf{P}}) \equiv (\langle M \rangle_t, \mathbf{P}), \quad 0 \leq t \leq T.$$

With respect to a new probability measure, equation (4) can be rewritten as

$$X_t = \int_{(0,t]} \Pi(s-) \sigma(s-) d\bar{M}_s + \int_{(0,t]} [X_{s-} r(s-) - C(s-)] d\langle \bar{M} \rangle_s, \quad 0 < t \leq T, \quad (7)$$

$$X(0) = x$$

and the solution (see Section 5) for $0 \leq t \leq T$, leads to

$$\frac{X_t}{\Phi(t)} + \int_{(0,t]} \frac{C(s-)}{\Phi(s-)[1+r(s-)\Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s = x + \int_{(0,t]} \frac{\Pi(s-)\sigma(s-)}{\Phi(s-)[1+r(s-)\Delta\langle M \rangle_s]} d\bar{M}_s, \quad (8)$$

where

$$\Phi(t) = \exp \left[\int_{(0,t]} r(s-) d\langle M^c \rangle_s \right] \cdot \prod_{s \leq t} [1 + r(s-) \Delta\langle M \rangle_s]$$

is a unique strong solution of the homogeneous equation corresponding to (7):

$$\Phi(t) = 1 + \int_{(0,t]} \Phi(s-) r(s-) d\langle M \rangle_s.$$

If we suppose that $1 + r(s-) \Delta\langle M \rangle_s \leq 0$ for some $s \in \mathbb{S}$, then $\Delta\langle M \rangle_s \leq -\frac{1}{r(s-)}$. But this is impossible if $r(s)$ is nonnegative. Consequently, $1 + r(s-) \Delta\langle M \rangle_s > 0$ for every $s \in \mathbb{R}_+$.

Let us notice also that

$$\inf_{t \in \mathbb{R}_+} |\Phi(t)| > 0.$$

The right-hand side of (8) is a $\bar{\mathbf{P}}$ -local martingale. If (Π, C) is an admissible pair (i.e., $X_t \geq 0$, $0 \leq t \leq T$ a.s.), the left-hand side is nonnegative, consequently it is a nonnegative supermartingale under $\bar{\mathbf{P}}$. From the supermartingale property we obtain that

$$\bar{\mathbf{E}} \left[\frac{X_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1+r(s-)\Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s \right] \leq x, \quad (10)$$

where $\bar{\mathbf{E}}$ denotes the expectation operator under measure $\bar{\mathbf{P}}$.

This condition is also sufficient for the admissibility in the sense of the following theorem.

Theorem 1: Suppose that $x \geq 0$ and B_T is a nonnegative F_T -measurable random variable, such that

$$\bar{\mathbf{E}} \left[\frac{B_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1+r(s-)\Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s \right] \leq x. \quad (11)$$

Then there exists a portfolio process Π such that the pair (Π, C) is admissible for the initial endowment x and the terminal wealth X_T is at least B_T .

Proof: It is obvious that we can assume equality to hold in (11).

Let us define the nonnegative process

$$\bar{\mu}_t = \bar{\mathbf{E}} \left[\frac{B_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-)[1+r(s-)\Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s \middle| F_t \right], \quad (12)$$

$$\bar{\mu}_0 = x,$$

which is a $\bar{\mathbf{P}}$ -martingale and has “cadlag” paths.

Define the process μ_t , $0 \leq t \leq T$ by

$$\mu_t = \bar{\mu}_t + \int_{(0,t]} \phi_s^{-1} d\langle \phi, \bar{\mu} \rangle_s, \quad (13)$$

where ϕ_t is the density (5). It is well known that the process μ_t is a $\mathbf{P}$ -martingale [5], $\mu_0 = \bar{\mu}_0$, and $\langle \mu \rangle = \langle \bar{\mu} \rangle$.

Now by the martingale representation theorem [8], if $(\mathbf{M}, \mu)$ is a Gaussian process, there exists an F_t -predictable measurable process $h(s)$, such that

$$\int_{(0,T]} h^2(s-) d\langle M \rangle_s < \infty, \quad \mathbf{P}\text{-a.s.}$$

for every finite $T > 0$ and

$$\mu_t = \mu_0 + \int_{(0,t]} h(s-) dM_s, \quad 0 \leq t \leq T. \quad (14)$$

The process (13) can be represented as

$$\mu_t = \bar{\mu}_t - \int_{(0,t]} \theta(s-) h(s-) d\langle M \rangle_s, \quad 0 \leq t \leq T. \quad (15)$$

From equalities (6), (14), and (15) it follows that

$$\begin{aligned} \bar{\mu}_t &= \mu_t + \int_{(0,t]} \theta(s-) h(s-) d\langle M \rangle_s \\ &= \mu_0 + \int_{(0,t]} h(s-) [d\bar{M}_s - \theta(s-) d\langle M \rangle_s] + \int_{(0,t]} \theta(s-) h(s-) d\langle M \rangle_s \\ &= \mu_0 + \int_{(0,t]} h(s-) d\bar{M}_s. \end{aligned} \quad (16)$$

Now,

$$\Pi(t-) = \frac{h(t-) \Phi(t-) [1 + r(t-) \Delta\langle M \rangle_t]}{\sigma(t-)}, \quad 0 < t \leq T \quad (17)$$

is a well-defined portfolio process.

From (12), (16), and (17), we get

$$\begin{aligned} \bar{\mu}_t &= \bar{\mathbf{E}} \left[\frac{B_T}{\Phi(T)} + \int_{(0,T]} \frac{C(s-)}{\Phi(s-) [1 + r(s-) \Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s \middle| F_t \right] \\ &= x + \int_{(0,t]} h(s-) d\bar{M}_s. \end{aligned} \quad (18)$$

By using (18) and (8), we obtain

$$\bar{\mu}_t = \frac{X_t^{\Pi, C, x}}{\Phi(t)} + \int_{(0,t]} \frac{C(s-)}{\Phi(s-) [1 + r(s-) \Delta\langle M \rangle_s]} d\langle \bar{M} \rangle_s, \quad (19)$$

where $X_t^{\Pi, C, x}$ is a solution of equation (7) for the pair (Π, C) and the initial capital $x \geq 0$.

Now, from (18) and (19), it follows that

$$\frac{X_t^{\Pi, C, x}}{\Phi(t)} = \bar{\mathbf{E}} \left[\frac{B_T}{\Phi(T)} + \int_{(t, T]} \frac{C(s-)}{\Phi(s-)[1+r(s-)\Delta\langle M \rangle_s} d\langle \bar{M} \rangle_s \middle| F_t \right]. \quad (20)$$

Consequently, $X_t^{\Pi, C, x}$ is nonnegative and (Π, C) is an admissible strategy. $\square$

4. Valuation of Contingent Claim

Definition: A contingent claim is a nonnegative F_T -measurable random variable B that satisfies

$$0 < \bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \right] \leq x.$$

The hedging price of this contingent claim is defined by

$$U \stackrel{\text{def}}{=} \inf \left\{ x > 0, \exists (\Pi, C) - \text{admissible, such that } X_T^{\Pi, C, x} \geq B \text{ P-a.s.} \right\}.$$

Theorem 2: The value of the contingent claim is attained and

$$U = \bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \right].$$

Proof: Let us suppose that $X_T^{\Pi, C, x} \geq B$ a.s. for some value of $x > 0$ and a suitable pair (Π, C) . Then from (10) it follows that

$$\bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \right] \leq \bar{\mathbf{E}} \left[\frac{X_T^{\Pi, C, x}}{\Phi(T)} \right] \leq x.$$

Consequently, $z = \bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \right] \leq U$.

Let us define the nonnegative random process

$$X_0(t) = \Phi(t) \cdot \bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \middle| F_t \right], \quad 0 \leq t \leq T,$$

where $\bar{m}_t = \bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \middle| F_t \right]$ is a $\bar{\mathbf{P}}$ -Gaussian martingale, such that $\bar{m}_0 = \bar{\mathbf{E}} \left[\frac{B}{\Phi(T)} \right]$.

Analogously to the proof of the Theorem 1, we can apply the generalized Girsanov's theorem and the martingale representation theorem.

By comparing the processes

$$\begin{aligned} \frac{X_0(t)}{\Phi(t)} &= z + \int_{(0, t]} h(s-) d\bar{M}_s \\ \text{and } \frac{X_t^{\Pi, 0, x}}{\Phi(t)} &\text{ we obtain that} \\ X_0(t) &\equiv X_t^{\Pi, 0, z}, \quad 0 \leq t \leq T. \end{aligned} \quad (21)$$

Consequently, $z \geq U$. $\square$

Remark 1: Let us note that (21) yields

$$X_0(T) \equiv X_T^{\Pi, 0, z} = B, \text{ a.s.,}$$

i.e., the contingent claim is attained with the initial capital U , portfolio Π , and zero consumption. This fact could be used as a starting point for solving appropriate optimal problems.

Remark 2: If $\langle M \rangle_t \equiv t$, we have $(\mathbf{M}, \mathbf{P})$ and $(\bar{\mathbf{M}}, \bar{\mathbf{P}})$ (standard) Wiener processes, and $\mathbf{S}$ empty. Then Theorem 1 and Theorem 2 reduce to the results of Karatzas and Shreve [7] and Cvitanic and Karatzas [2].

Corollary: Let $C(t) \equiv 0$ and let the agent invest in one stock asset. Then, the following representations hold:

$$X_t = \Phi(t) \cdot \bar{\mathbf{E}} \left[\frac{P(T)}{\Phi(T)} \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T; \quad (i)$$

$$X_t = p \cdot \exp \left[\int_{(0,t]} \sigma(s-) d\bar{M}_s - \frac{1}{2} \int_{(0,t]} \sigma^2(s-) d\langle \bar{M}^c \rangle_s + \int_{(0,t]} r(s-) d\langle \bar{M}^c \rangle_s \right] \cdot \prod_{s \leq t} [1 + r(s-) \Delta \langle \bar{M} \rangle_s + \sigma(s-) \Delta \bar{M}_s], \quad 0 \leq t \leq T. \quad (ii)$$

Proof: Representation (i) follows from (20) when $C(t) \equiv 0$.

By Ito's rule it can be proved that $\frac{P(T)}{\Phi(T)}$ is a $\bar{\mathbf{P}}$ -Gaussian martingale and is a unique solution of the following stochastic equation:

$$\frac{P(T)}{\Phi(T)} = p + \int_{(0,T]} \frac{P(s-)}{\Phi(s-)} \cdot \frac{\sigma(s-)}{[1 + r(s-) \Delta \langle M \rangle_s]} d\bar{M}_s, \quad T > 0.$$

Consequently, from representation (i) it follows that $\frac{X_t}{\Phi(t)}$, $0 \leq t \leq T$ is a $\bar{\mathbf{P}}$ -Gaussian martingale and it yields representation (ii). $\square$

5. A Linear Stochastic Integral Equation

In this section we will obtain a solution of the stochastic equation

$$X_t = X_0 + \int_{(0,t]} [S(s)X_{s-} + \sigma(s)] dM_s + \int_{(0,t]} [A(s)X_{s-} + a(s)] db(s), \quad (22)$$

$0 \leq t < \infty$, which is a more general than we anticipate.

Let $\mathbf{M} = (M_t, F_t)$, $M_0 = 0$, $F_t = \sigma(M_s, s \leq t)$, $t \in \mathbb{R}_+ = [0, \infty)$, be a cadlag Gaussian martingale, $b = b(t)$, $t \in \mathbb{R}_+$ -nonrandom, right-continuous function with finite variation on each finite interval. Suppose the function $b(t)$ be a real-valued deterministic function, absolutely continuous with respect to $\langle M \rangle_t = \mathbf{E}M_t^2$ and

$$b(t) = \int_{(0,t]} \gamma_s d\langle M \rangle_s, \quad t \in \mathbb{R}_+,$$

where $\gamma = (\gamma_t, F_{t-})$, $F_{t-} = \sigma(M_s, 0 < s < t)$ is a F -predictable function, $X_0 = (X_0, F_0)$ be a Gaussian random variable, independent of $\mathbf{M}$. $A(t)$, $\sigma(t)$, $a(t)$, and $S(t)$ are nonrandom F -predictable functions, such that

$$\int_{(0,\infty)} \sigma^2(s) d\langle M \rangle_s < \infty, \quad \int_{(0,\infty)} A^2(s) db(s) < \infty,$$

$$\int_{(0,\infty)} a^2(s) db(s) < \infty \quad \text{and} \quad \int_{(0,\infty)} S^2(s) db(s) < \infty, \quad \mathbf{P}\text{-a.s.}$$

We will find a solution of equation (22) in the class of cadlag adapted processes (i.e., pro-

cesses with right-continuous paths and finite left limits) and it provides a fairly explicit representation. According to [4], such a solution exists and it is unique in the sense of **P**-indistinguishability.

Recall [6] that the random process **M** and the deterministic nondecreasing function $\langle M \rangle$ have their jumps at the same nonrandom moments of time which form a countable set $\mathbb{S} \subset \mathbb{R}_+ \setminus \{0\}$. Let us notice now that the function $b(t)$ and the process X_t have their jumps at the same moments.

Let us suppose that the function $b = b(t)$, $t \in \mathbb{R}_+$ has no more than a countable subset of jumps $\{0 \leq s_0 < s_1 < \dots < s_k < \dots < \infty\} \subseteq \mathbb{S}$ with

$$\Delta b(s_k) = -\frac{1 + S(s_k)\Delta M_{s_k}}{A(s_k)}, \quad k \geq 1.$$

It is obvious that

$$\begin{aligned} \Delta X_{s_k} &= [A(s_k)X_{s_k^-} + a(s_k)]\Delta b(s_k) + [S(s_k)X_{s_k^-} + \sigma(s_k)]\Delta M_{s_k} \\ &= -X_{s_k^-} - \frac{a(s_k)}{A(s_k)}[1 + S(s_k)\Delta M_{s_k}] + \sigma(s_k)\Delta M_{s_k}. \end{aligned}$$

Consequently,

$$X_{s_k} = \sigma(s_k)\Delta M_{s_k} - \frac{a(s_k)}{A(s_k)}[1 + S(s_k)\Delta M_{s_k}]. \quad (23)$$

We will find a solution of equation (22) on the interval $[s_k, s_{k+1})$, $k \geq 0$, with an initial condition X_{s_k} , independent of increments $M_t - M_{s_k}$, $s_k \leq t < s_{k+1}$, $k \geq 0$, according to (23) and the conditions imposed on X_0 .

The homogeneous equation corresponding to (22) is

$$\begin{aligned} \Phi(t, s_k) &= 1 + \int_{(s_k, t]} \Phi(s-, s_k)A(s)db(s) + \int_{(s_k, t]} \Phi(s-, s_k)S(s)dM_s, \\ \Phi(s_k, s_k) &= 1, \quad k \geq 0 \end{aligned}$$

and has a unique solution [3]:

$$\begin{aligned} \Phi(t, s_k) &= \exp \left\{ \int_{(s_k, t]} A(s)db(s) + \int_{(s_k, t]} S(s)dM_s - \frac{1}{2} \int_{(s_k, t]} S^2(s)d\langle M^c \rangle_s \right\} \\ &\cdot \prod_{s_k < s \leq t} \{1 + A(s)\Delta b(s) + S(s)\Delta M_s\} \cdot \exp\{-A(s)\Delta b(s) - S(s)\Delta M_s\}; \\ \Phi(t, s_k) &= \exp \left\{ \int_{(s_k, t]} A(s)db^c(s) + \int_{(s_k, t]} S(s)dM_s^c - \frac{1}{2} \int_{(s_k, t]} S^2(s)d\langle M^c \rangle_s \right\} \\ &\cdot \prod_{s_k < s \leq t} \{1 + A(s)\Delta b(s) + S(s)\Delta M_s\}, \end{aligned} \quad (24)$$

where $b^c(t)$ is the continuous path of the function $b(t)$, $s_k \leq t < s_{k+1}$, $k \geq 0$.

Let us notice that if

$$A(t)\Delta b(t) + S(t)\Delta M_t \neq -1$$

on (s_k, s_{k+1}) , from the solution of (24) it follows

$$s_k \leq t < s_{k+1} \wedge T \quad |\Phi(t, s_k)| > 0 \quad (25)$$

with some $T \in [s_k, \infty)$, $k \geq 0$.

Let now define the function $\Phi(t)$, $t \in \mathbb{R}_+$, where

$$\Phi(t) = \Phi(t, s_k), \quad s_k \leq t < s_{k+1}, \quad k \geq 0.$$

It follows from (25) that the function $\Phi^{-1}(t)$, $t \in \mathbb{R}_+$ is correct defined and bounded on every finite interval $[0, T]$, $T \in \mathbb{R}_+$. Consequently, for every $t \in \mathbb{R}_+$, it holds true that

$$\int_{(0, t]} \Phi^{-2}(s) d\langle M \rangle_s < \infty. \quad (26)$$

Theorem 3: *The unique solution of the equation (22) is given by*

$$\begin{aligned} X_t^k = \Phi(t, s_k) & \left[X_{s_k} + \int_{(s_k, t]} \frac{\sigma(s)}{\Phi(s)} dM_s + \int_{(s_k, t]} \frac{a(s)}{\Phi(s)} db(s) \right. \\ & \left. - \int_{(s_k, t]} \frac{S(s)\sigma(s)}{\Phi(s)} d\langle M^c \rangle_s \right], \quad s_k \leq t < s_{k+1}, \quad k \geq 0. \end{aligned} \quad (27)$$

Proof: Observe that (25) ensures that the process X_t^k is well defined. We will show that the process X_t^k from (27) is a solution of equation (22) over the interval $[s_k, s_{k+1})$, $k \geq 0$.

We apply Ito's rule to (27) on the interval (s_k, s_{k+1}) :

$$\begin{aligned} X_t^k &= X_{s_k} + \int_{(s_k, t]} X_{s-} A(s) db(s) + \int_{(s_k, t]} X_{s-} S(s) dM_s \\ &+ \int_{(s_k, t]} \frac{1}{1 + A(s)\Delta b(s) + S(s)\Delta M_s} [\mathbf{I}_{\{\Delta b(s)=0\}} + \mathbf{I}_{\{\Delta b(s) \neq 0\}}] \\ &\cdot \left[\sigma(s) dM_s + a(s) db(s) - S(s)\sigma(s) \mathbf{I}_{\{\Delta b(s)=0\}} d\langle M^c \rangle_s \right] + \int_{(s_k, t]} S(s)\sigma(s) d\langle M^c \rangle_s \\ &+ \sum_{s_k < s \leq t} \left[1 - \frac{1}{1 + A(s)\Delta b(s) + S(s)\Delta M_s} \right] [\sigma(s)\Delta M_s + a(s)\Delta b(s)]; \\ X_t^k &= X_{s_k} + \int_{(s_k, t]} X_{s-} A(s) db(s) + \int_{(s_k, t]} X_{s-} S(s) dM_s \\ &+ \int_{(s_k, t]} \sigma(s) dM_s + \int_{(s_k, t]} a(s) db(s). \end{aligned}$$

If $s_k < t < s_{k+1}$, then

$$X_t = X_{s_k} + \int_{(s_k, t]} [X_s - A(s) + a(s)] db(s) + \int_{(s_k, t]} [X_s - S(s) + \sigma(s)] dM_s.$$

The last representation and (23) lead to (22). $\square$

Remark 3: The coefficients $A(t)$, $\sigma(t)$, $S(t)$, and $a(t)$ of equation (22) are f -predictable functions. It will be convenient for applications to represent solution (27) in the form

$$\begin{aligned} X_t^k = \Phi(t, s_k) & \left[X_{s_k} + \int_{(s_k, t]} \frac{\sigma(s)}{\Phi(s-)[1 + A(s)\Delta b(s) + S(s)\Delta M_s]} dM_s \right. \\ & + \int_{(s_k, t]} \frac{a(s)}{\Phi(s-)[1 + A(s)\Delta b(s) + S(s)\Delta M_s]} db(s) \\ & \left. - \int_{(s_k, t]} \frac{S(s)\sigma(s)}{\Phi(s-)[1 + A(s)\Delta b(s) + S(s)\Delta M_s]} d\langle M^c \rangle_s \right], \quad t \in [s_k, s_{k+1}), \quad k \geq 0. \end{aligned}$$

Acknowledgement

The author would like to thank Professor Svetlozar Rachev from the UC Santa Barbara for his helpful and stimulating discussions and the referee for constructive comments.

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