BALLS LEFT EMPTY BY A CRITICAL BRANCHING WIENER PROCESS

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ABSTRACT

At time $t = 0$ we have a Poisson random field on $\mathbb{R}^d$. Each particle executes a critical branching Wiener process starting from its position at time $t = 0$. Let R_T be the radius of the largest ball around the origin of $\mathbb{R}^d$ which does not contain any particle at time T . Our goal is to characterize the properties of the stochastic process $\{R_T, T \geq 0\}$.

This article is dedicated to the memory of Professor Roland L. Dobrushin.

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1. Introduction

Consider the following

Model 1:

- (i) a particle starts from the position $0 \in \mathbb{R}^d$ and executes a Wiener process $W(t) \in \mathbb{R}^d$;
- (ii) arriving at time $t = 1$ to the new location $W(1)$, it dies;
- (iii) at death, it is replaced by Y offspring, where

$$\mathbf{P}\{Y = 0\} = \mathbf{P}\{Y = 2\} = 1/2;$$

- (iv) each offspring, starting from where its ancestor dies, executes a Wiener process (from its starting point) and repeats the above given steps and so on. All Wiener processes and offspring numbers are assumed independent of each other.

A more formal definition is given in Chapter 6 of [1], p. 91.

Let $A \subset \mathbb{R}^d$ be a Borel set and let $\lambda(A, t)$ ($t = 0, 1, 2, \dots$) be the number of particles located in A at time t . Then

$$B(t) = \lambda(\mathbb{R}^d, t)$$

is the number of particles living at t and $\{B(t), t = 0, 1, 2, \dots\}$ is a branching process.

We also consider the following

Model 2: At time $t = 0$ we have a Poisson random field of parameter μ , i.e., in a Borel set $A \subset \mathbb{R}^d$, we have k particles with probability

$$\pi(A, k) = \frac{(\mu |A|)^k}{k!} e^{-\mu |A|},$$

where $|A|$ is the Lebesgue measure of A . It is also assumed that the numbers of particles in disjoint Borel sets are independent r.v.'s. Each particle executes a critical branching Wiener process (starting from its position at time $t = 0$) according to Model 1.

A more formal definition is given in Chapter 8 of [1], p. 129. Let $\Lambda(A, t)$ be the number of particles located in A at time t . Then clearly

$$\mathbf{P}\{\Lambda(A, 0) = k\} = \pi(A, k).$$

Let

$$\mathcal{C}(x, r) = \{y: \|y - x\| \leq r\} \subset \mathbb{R}^d$$

and

$$R_T = \sup\{R: \Lambda(\mathcal{C}(0, R), T) = 0\} \quad (T = 0, 1, 2, \dots),$$

i.e., R_T is the radius of the largest ball around the origin of $\mathbb{R}^d$ which does not contain any particle at time T .

We are interested in the limit behavior of R_T as $T \rightarrow \infty$.

In the case $d = 1$, this problem is very simple. In fact we have,

Theorem A: (Theorem 8.2 p. 129 in [1]). *Let $d = 1$. Then for any $\epsilon > 0$ we have*

$$\Lambda(\mathcal{C}(0, T(\log T)^{-1-\epsilon}), T) = 0 \text{ a.s.}$$

for all but finitely many T ,

$$\Lambda(\mathcal{C}(0, \epsilon T), T) \geq 1 \text{ i.o. a.s.,}$$

$$\Lambda(\mathcal{C}(0, \epsilon^{-1}T), T) = 0 \text{ i.o. a.s.,}$$

and

$$\Lambda(\mathcal{C}(0, T(\log T)^{1+\epsilon}), T) \geq 1 \text{ a.s.}$$

for all but finitely many T .

We note that Theorem 8.2 of [1] is formulated in a slightly different way, but the above Theorem A can be obtained directly by the method presented there.

Now we formulate our main result.

Theorem 1: *We have*

$$\Lambda(\mathcal{C}(0, R_1(T, d)), T) \geq 1 \text{ a.s.}$$

for all but finitely many T ,

$$\Lambda(\mathcal{C}(0, R_2(T, d)), T) = 0 \text{ i.o. a.s.,}$$

$$\Lambda(\mathcal{C}(0, R_3(T, d)), T) \geq 1 \text{ i.o. a.s.}$$

and

$$\Lambda(\mathcal{C}(0, R_4(T, d)), T) = 0 \text{ a.s.}$$

for all but finitely many T , where

$$R_1(T, d) = \begin{cases} T(\log T)^{1+\epsilon} & \text{if } d = 1, \\ K(T \log T)^{1/2} & \text{if } d = 2, \\ K(\log T)^{1/(d-2)} & \text{if } d \geq 3, \end{cases}$$

$$R_2(T, d) = \begin{cases} \epsilon^{-1}T & \text{if } d = 1, \\ T^{1/2}(g(T))^{-1} & \text{if } d = 2, \\ K^{-1}(\log \log \log T)^{1/(d-2)} & \text{if } d \geq 3, \end{cases}$$

$$R_3(T, d) = \begin{cases} \epsilon T & \text{if } d = 1, \\ (\log T)^{-1/2+\epsilon} & \text{if } d = 2, \\ K^{-1}(\log T)^{-1/d} & \text{if } d \geq 3, \end{cases}$$

$$R_4(T, d) = \begin{cases} T(\log T)^{-1-\epsilon} & \text{if } d = 1, \\ T^{-1/2}(\log T)^{-1/2-\epsilon} & \text{if } d = 2, \\ T^{-1/d}(\log T)^{-1/d-\epsilon} & \text{if } d \geq 3, \end{cases}$$

K is large enough, $g(T)$ is an arbitrary function with $g(T) \uparrow \infty$ and ϵ is an arbitrary positive number.

Remark: Intuitively it is clear that if $R_1(T)$ ($T = 1, 2, \dots$) is a function going to infinity fast enough, then the ball around the origin, of radius $R_1(T)$ will contain at least one living particle at time T for any T large enough. Theorem 1 claims that in $\mathbb{R}^2$ we might choose $R_1(T) = K(T \log T)^{1/2}$, while in $\mathbb{R}^3$ it is enough to choose $R_1(T) = K \log T$. We are also interested to characterize those functions $R_3(T)$ for which it is still true that the ball, around the origin, of radius $R_3(T)$ contains particles at time T for infinitely many T . Theorem 1 claims that in $\mathbb{R}^2$ we might choose $R_3(T) = (\log T)^{-1/2+\epsilon}$ while in $\mathbb{R}^3$ we might have $R_3(T) = K^{-1}(\log T)^{-1/3}$. The results on R_2 and R_4 tell us how exact are the results on R_1 and R_3 . Unfortunately, it turns out that we have a very big gap.

We also prove two theorems describing some properties of $\lambda(\cdot, \cdot)$ of Model 1, which will be used in the proof of Theorem 1 and which seem to be interesting in themselves.

Let $f(t)$ ($t = 1, 2, \dots$) be a positive, real valued function with $f(t) \rightarrow \infty$ (as $t \rightarrow \infty$), let $\alpha \in \mathbb{R}^d$ and let

$$\mathbb{C} = \mathbb{C}(\alpha T^{1/2}, T^{1/2}(f(T))^{-1}).$$

Then we have,

Theorem 2: In case $d = 1$ we have

$$1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right)^{\frac{c(\alpha)}{2}} \frac{T}{f(T)} \leq \mathbf{E}(\lambda(\mathbb{C}, T) \mid \lambda(\mathbb{C}, T) > 0) \\ \leq 1 + (1 + \epsilon)c(\alpha) \frac{T}{f(T)} + K \frac{T}{f^2(T)}$$

for any $K > 0$, $\epsilon > 0$ if T is large enough, where

$$c(\alpha) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\pi/2} \exp\left(-\frac{\alpha^2}{2} \frac{1 - \sin x}{1 + \sin x}\right) dx.$$

If we also assume that $f(T) \leq T^{1/2}$ and $K \geq 2$ then

$$1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right)^{\frac{c(\alpha)}{2}} \frac{T}{f(T)} \\ \leq \mathbf{E}(\lambda(\mathbb{C}, T) \mid \lambda(\mathbb{C}, T) > 0) \leq 1 + (1 + \epsilon) \frac{c(\alpha)}{2} \left(1 + \frac{2}{K}\right) \frac{T}{f(T)} + K \frac{T}{f^2(T)}.$$

In case $d = 2$ we have

$$1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{4} \frac{T}{f^2(T)} \log f(T)$$

$$\leq \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \leq 1 + \frac{1}{2}(1 + \epsilon) \frac{T}{f^2(T)} \log f(T) + K \frac{T}{f^2(T)}$$

for any $K > 0$ if T is large enough.

If we also assume that $f(T) \leq T^{1/2}$ and $K \geq 2$, then

$$1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{4} \frac{T}{f^2(T)} \log f(T)$$

$$\leq \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0)$$

$$\leq 1 + \frac{1 + \epsilon}{4} \left(1 + \frac{2}{K}\right) \frac{T}{f^2(T)} \log f(T) + K \frac{T}{f^2(T)}.$$

In case $d \geq 3$ for any $K > 0$ if T is large enough, we have

$$1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{d-2} \frac{\omega_d}{(4\pi)^{d/2}} \frac{T}{f^2(T)} K^{-(d-2)/2}$$

$$\leq \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0)$$

$$\leq 1 + (1 + \epsilon) \frac{2}{d-2} \frac{\omega_d}{(4\pi)^{d/2}} \frac{T}{f^2(T)} K^{-(d-2)/2} + K \frac{T}{f^2(T)},$$

where

$$\omega_d = \begin{cases} 2 & \text{if } d = 1, \\ \pi & \text{if } d = 2, \\ \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} & \text{if } d \geq 3 \end{cases}$$

is the volume of a ball in $\mathbb{R}^d$ of radius 1.

Consequences: In case $d = 1$,

$$\frac{c(\alpha)}{2} \leq \liminf_{T \rightarrow \infty} \frac{f(T)}{T} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0)$$

$$\leq \limsup_{T \rightarrow \infty} \frac{f(T)}{T} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) < c(\alpha),$$

provided that

$$\lim_{T \rightarrow \infty} \frac{T}{f(T)} = \infty.$$

If $d = 1$ and

$$\lim_{T \rightarrow \infty} \frac{T}{f(T)} = \beta, \quad 0 \leq \beta < \infty,$$

then

$$1 + \frac{c(\alpha)}{2} \beta \leq \liminf_{T \rightarrow \infty} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0)$$

$$\leq \limsup_{T \rightarrow \infty} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \leq 1 + c(\alpha) \beta.$$

If $d = 1$ and $f(T) \leq T^{1/2}$, then

$$\lim_{T \rightarrow \infty} \frac{f(T)}{T} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) = \frac{c(\alpha)}{2}.$$

Note that

$$c(0) = \left(\frac{\pi}{2}\right)^{1/2}.$$

In case $d = 2$ we have

$$\begin{aligned} \frac{1}{4} &\leq \liminf_{T \rightarrow \infty} \frac{f^2(T)}{T \log f(T)} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ &\leq \limsup_{T \rightarrow \infty} \frac{f^2(T)}{T \log f(T)} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \leq \frac{1}{2}, \end{aligned}$$

provided that

$$\lim_{T \rightarrow \infty} \frac{T}{f^2(T)} \log f(T) = \infty.$$

If $d = 2$ and

$$\lim_{T \rightarrow \infty} \frac{T}{f^2(T)} \log f(T) = \beta,$$

then

$$\begin{aligned} 1 + \frac{\beta}{4} &\leq \liminf_{T \rightarrow \infty} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ &\leq \limsup_{T \rightarrow \infty} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \leq 1 + \frac{\beta}{2}. \end{aligned}$$

If $d = 2$ and $f(T) \leq T^{1/2}$, then

$$\lim_{T \rightarrow \infty} \frac{f^2(T)}{T \log f(T)} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) = \frac{1}{4}.$$

In case $d \geq 3$,

$$\begin{aligned} \frac{2}{d-2} \frac{\omega_d}{(8\pi)^{d/2}} &\leq \liminf_{T \rightarrow \infty} \frac{f^2(T)}{T} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ &\leq \limsup_{T \rightarrow \infty} \frac{f^2(T)}{T} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \leq \frac{4}{d-2} \frac{\omega_d}{(8\pi)^{d/2}} + 2, \end{aligned}$$

provided that

$$\lim_{T \rightarrow \infty} \frac{T}{f^2(T)} = \infty.$$

If $d \geq 3$ and

$$\lim_{T \rightarrow \infty} \frac{T}{f^2(T)} = \beta,$$

then

$$\begin{aligned} 1 + \frac{2}{d-2} \frac{\omega_d}{(8\pi)^{d/2}} \beta &\leq \liminf_{T \rightarrow \infty} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ &\leq \limsup_{T \rightarrow \infty} \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ &\leq 1 + 2\beta + \frac{4}{d-2} \frac{\omega_d}{(8\pi)^{d/2}} \beta. \end{aligned}$$

Theorem 3: Consider the case where

$$d = 1, \quad \lim_{T \rightarrow \infty} \frac{T}{f(T)} = \infty.$$

Then,

$$\frac{1}{(2\pi)^{1/2}} \frac{\exp(-\alpha^2/2)}{c(\alpha)} \leq \liminf_{T \rightarrow \infty} \mathbf{P}\{\lambda(\mathcal{C}, T) > 0 \mid B(T) > 0\}$$

$$\begin{aligned} &\leq \limsup_{T \rightarrow \infty} \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq \left(\frac{2}{\pi}\right)^{1/2} \frac{\exp(-\alpha^2/2)}{c(\alpha)}. \end{aligned} \quad (1.1)$$

If $d = 1$ and

$$\lim_{T \rightarrow \infty} \frac{T}{f(T)} = \beta, \quad 0 \leq \beta < \infty,$$

then

$$\begin{aligned} \frac{\beta}{(2\pi)^{1/2}} \frac{\exp(-\alpha^2/2)}{1 \leq c(\alpha)\beta} &\leq \liminf_{T \rightarrow \infty} \frac{f(T)}{T} \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq \limsup_{T \rightarrow \infty} \frac{f(T)}{T} \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq \frac{\beta}{(2\pi)^{1/2}} \frac{2 \exp(-\alpha^2/2)}{2 + c(\alpha)\beta}. \end{aligned} \quad (1.2)$$

If

$$d = 1 \text{ and } f(T) \leq T^{1/2},$$

then

$$\lim_{T \rightarrow \infty} \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} = \left(\frac{2}{\pi}\right)^{1/2} \frac{e^{-\alpha^2/2}}{c(\alpha)}.$$

Now consider the case

$$d = 2, \lim_{T \rightarrow \infty} \frac{T}{f^2(T)} \log f(T) = \infty.$$

Then,

$$\begin{aligned} \frac{1}{2}e^{-\alpha^2/2} &\leq \liminf_{T \rightarrow \infty} (\log f(T)) \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq \limsup_{T \rightarrow \infty} (\log f(T)) \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq e^{-\alpha^2/2}. \end{aligned} \quad (1.3)$$

If

$$d = 2 \text{ and } \lim_{T \rightarrow \infty} \frac{T}{f^2(T)} \log f(T) = \beta,$$

then

$$\begin{aligned} \frac{\exp(-\alpha^2/2)}{4 + 2\beta} &\leq \liminf_{T \rightarrow \infty} \frac{f^2(T)}{T} \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq \limsup_{T \rightarrow \infty} \frac{f^2(T)}{T} \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} \\ &\leq \frac{\exp(-\alpha^2/2)}{4 + \beta}. \end{aligned} \quad (1.4)$$

If

$$d = 2, \lim_{T \rightarrow \infty} \frac{T}{f^2(T)} \log f(T) = \infty \text{ and } f(T) \leq T^{1/2},$$

then

$$\lim_{T \rightarrow \infty} (\log f(T)) \mathbf{P}\{\lambda(\mathbb{C}, T) > 0 \mid B(T) > 0\} = \exp\left(-\frac{\alpha^2}{2}\right). \quad (1.5)$$

Now consider the case

$$d \geq 3, \lim_{T \rightarrow \infty} \frac{T}{f^2(T)} = \infty.$$

Then,

$$\frac{2^d \omega_d (d-2) e^{-\alpha^2/2}}{8\omega_d + 4(8\pi)^{d/2} (d-2)}$$

$$\begin{aligned}
 &\leq \liminf_{T \rightarrow \infty} (f(T))^{d-2} \mathbf{P}\{\lambda(\mathcal{C}, T) > 0 \mid B(T) > 0\} \\
 &\leq \limsup_{T \rightarrow \infty} (f(T))^{d-2} \mathbf{P}\{\lambda(\mathcal{C}, T) > 0 \mid B(T) > 0\} \\
 &\leq (d-2)2^{d-2} \exp\left(-\frac{\alpha^2}{2}\right).
 \end{aligned} \tag{1.6}$$

If
then

$$\begin{aligned}
 &d \geq 3 \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{T}{f^2(T)} = \beta, \\
 &\frac{2^d \omega_d (d-2) e^{-\alpha^2/2}}{8\beta \omega_d + 2(d-2)(8\pi)^{d/2}(1+\beta)} \\
 &\leq \liminf_{T \rightarrow \infty} \frac{(f(T))^d}{T} \mathbf{P}\{\lambda(\mathcal{C}, T) > 0 \mid B(T) > 0\} \\
 &\leq \limsup_{T \rightarrow \infty} \frac{(f(T))^d}{T} \mathbf{P}\{\lambda(\mathcal{C}, T) > 0 \mid B(T) > 0\} \\
 &\leq \frac{(d-2)2^d \omega_d e^{-\alpha^2/2}}{2(d-2)(8\pi)^{d/2} + 4\omega_d \beta}.
 \end{aligned} \tag{1.7}$$

2. Lemmas

Let

$$\begin{aligned}
 W_1(t) &= \{W_{11}(t), W_{12}(t), \dots, W_{1d}(t)\}, \\
 W_2(t) &= \{W_{21}(t), W_{22}(t), \dots, W_{2d}(t)\}, \\
 W_3(t) &= \{W_{31}(t), W_{32}(t), \dots, W_{3d}(t)\}
 \end{aligned}$$

be independent Wiener processes and let

$$\begin{aligned}
 \Gamma_1(t, s, T) &= \begin{cases} W_1(t) & \text{if } 0 \leq t \leq s, \\ W_1(s) + W_2(t-s) & \text{if } s \leq t \leq T, \end{cases} \\
 \Gamma_2(t, s, T) &= \begin{cases} W_1(t) & \text{if } 0 \leq t \leq s, \\ W_1(s) + W_3(t-s) & \text{if } s \leq t \leq T. \end{cases}
 \end{aligned}$$

Let

$$\gamma(x) = \gamma_{T,s}(x) = \mathbf{P}\{\Gamma_2(T, s, T) = x \mid \Gamma_1(T, s, T) = z\}$$

be the conditional density function of Γ_2 given $\Gamma_1 = z$.

Lemma 1:

$$\gamma(x) = (2\pi\sigma^2)^{-d/2} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right),$$

where

$$\nu = \mathbf{E}(\Gamma_2(T, s, T) \mid \Gamma_1(T, s, T) = z) = \frac{sz}{T}$$

and

$$\sigma^2 = \mathbf{E}((\Gamma_2(T, s, T) - \nu)^2 \mid \Gamma_1(T, s, T) = z) = T \left(1 - \frac{s^2}{T^2}\right).$$

Proof is trivial.

Lemma 2: Let $A \subseteq \mathbb{R}^d$ be a Borel set. Then

$$\mathbf{P}\{\Gamma_2(T, s, T) \in A \mid \Gamma_1(T, s, T) \in A\} = \frac{\int_A \left(\int_A \gamma(x) dx \right) \psi(z) dz}{\int_A \psi(z) dz},$$

where

$$\psi(z) = \psi_T(z) = (2\pi T)^{-d/2} \exp \left(-\frac{z^2}{2T} \right)$$

is the density function of $\Gamma_1(T, s, T)$.

Proof: Since

$$\mathbf{P}\{\Gamma_2 \in A \mid \Gamma_1 \in A\} = \frac{\int_A \mathbf{P}\{\Gamma_2 \in A \mid \Gamma_1 = z\} \psi(z) dz}{\int_A \psi(z) dz},$$

Lemma 2 follows.

Lemma 3: *Let*

$$\kappa_T = T - K \frac{T}{f^2(T)} \quad (K > 0)$$

and

$$P(T) = \sum_{s=1}^{\kappa_T} \mathbf{P}\{\Gamma_2 \in \mathbb{C} \mid \Gamma_1 \in \mathbb{C}\}.$$

Then in the case $d = 1$, for any $\epsilon > 0$, there exists a $T_0 = T_0(\epsilon) > 0$ such that

$$(1 - \epsilon) \left(1 - \frac{1}{K} \right) c(\alpha) \leq \frac{f(T)}{T} P(T) \leq (1 + \epsilon) c(\alpha)$$

if $T \geq T_0$, where

$$c(\alpha) = \left(\frac{2}{\pi} \right)^{1/2} \int_0^{\pi/2} \exp \left(-\frac{\alpha^2}{2} \frac{1 - \sin x}{1 + \sin x} \right) dx.$$

In the case $d = 2$, for any $\epsilon > 0$, there exists a $T_0 = T_0(\epsilon) > 0$ such that

$$(1 - \epsilon) \left(1 - \frac{1}{K} \right) \frac{1}{2} \frac{T}{f^2(T)} \log f(T) \leq P(T) \leq (1 + \epsilon) \frac{1}{2} \frac{T}{f^2(T)} \log f(T)$$

if $T \geq T_0$.

In the case $d \geq 3$, for any $\epsilon > 0$, there exists a $T_0 = T_0(\epsilon) > 0$ such that

$$\begin{aligned} (1 - \epsilon) \left(1 - \frac{1}{K} \right) \frac{\omega_d}{(4\pi)^{d/2}} \frac{2}{d-2} \frac{T}{f^2(T)} K^{-(d-2)/2} \\ \leq P(T) \leq (1 + \epsilon) \frac{\omega_d}{(4\pi)^{d/2}} \frac{2}{d-2} \frac{T}{f^2(T)} K^{-(d-2)/2} \end{aligned}$$

if $T \geq T_0$.

Proof: By Lemma 2,

$$P(T) = \frac{\sum_{s=1}^{\kappa_T} \int_{\mathbb{C}} \int_{\mathbb{C}} \gamma(x) dx \psi(z) dz}{\int_{\mathbb{C}} \psi(z) dz}.$$

Let

$$x = \alpha T^{1/2} + u \frac{T^{1/2}}{f(T)},$$

$$z = \alpha T^{1/2} + v \frac{T^{1/2}}{f(T)}.$$

Then,

$$u \in \mathbb{C}(0, 1) \text{ and } v \in \mathbb{C}(0, 1)$$

if

$$x \in \mathbb{C} \text{ and } z \in \mathbb{C}.$$

Hence,

$$\begin{aligned} \exp\left(-\frac{z^2}{2T}\right) &= \exp\left(-\frac{\alpha^2}{2}\right) \exp\left(-\frac{1}{2}\left(\frac{2(\alpha, v)}{f(T)} + \frac{v^2}{f^2(T)}\right)\right) = \\ &= \exp\left(-\frac{\alpha^2}{2}\right) \left(1 + O\left(\frac{1}{f(T)}\right)\right), \end{aligned}$$

$$\int_{\mathbb{C}} \psi(z) dz = (2\pi T)^{-d/2} |\mathbb{C}| \exp\left(-\frac{\alpha^2}{2}\right) \left(1 + O\left(\frac{1}{f(T)}\right)\right)$$

and

$$P(T) = \left(1 + O\left(\frac{1}{f(T)}\right)\right) \frac{1}{|\mathbb{C}|} (2\pi)^{-d/2} \sum_{s=1}^{\kappa_T} \sigma^{-d} \int_{\mathbb{C}} \int_{\mathbb{C}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right) dx dz.$$

Observe that

$$\begin{aligned} \frac{x-\nu}{\sigma} &= \alpha \left(\frac{T-s}{T+s}\right)^{1/2} + (u-v) \frac{T}{f(T)((T-s)(T+s))^{1/2}} + v \left(\frac{T-s}{T+s}\right)^{1/2} \frac{1}{f(T)}, \\ \left(\frac{x-\nu}{\sigma^2}\right)^2 &= \alpha^2 \frac{T-s}{T+s} + (u-v)^2 \frac{T^2}{f^2(T)(T^2-s^2)} + O\left(\frac{1}{f(T)}\right) \end{aligned}$$

and

$$\frac{T}{\sigma^2 f^2(T)} = \frac{T^2}{(T^2-s^2)f^2(T)} \leq \frac{1}{K},$$

provided that

$$1 \leq s \leq \kappa_T.$$

Since

$$dx dz = \frac{T^d}{(f(T))^{2d}} du dv,$$

we have

$$\begin{aligned} &\int_{\mathbb{C}} \int_{\mathbb{C}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right) dx dz \\ &= \left(1 + O\left(\frac{1}{f(T)}\right)\right) \frac{T^d}{(f(T))^{2d}} \exp\left(-\frac{\alpha^2}{2} \frac{T-s}{T+s}\right) \int_{\mathbb{C}(0,1)} \int_{\mathbb{C}(0,1)} \exp\left(-\frac{(u-v)^2}{2} \frac{T}{\sigma^2 f^2(T)}\right) du dv \\ &\leq \left(1 + O\left(\frac{1}{f(T)}\right)\right) \frac{T^d}{(f(T))^{2d}} \omega_d^2 \exp\left(-\frac{\alpha^2}{2} \frac{T-s}{T+s}\right) \end{aligned}$$

and

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right) dx dz > \left(1 + O\left(\frac{1}{f(T)}\right)\right) \left(1 - \frac{1}{K}\right) \frac{T^d}{(f(T))^{2d}} \omega_d^2 \exp\left(-\frac{\alpha^2}{2} \frac{T-s}{T+s}\right).$$

Hence

$$\left(1 + O\left(\frac{1}{f(T)}\right)\right) \left(1 - \frac{1}{K}\right) I \leq P(T) \leq \left(1 + O\left(\frac{1}{f(T)}\right)\right) I,$$

where

$$I = \omega_d (2\pi)^{-d/2} \frac{T^{d/2}}{(f(T))^d} \sum_{s=1}^{\kappa_T} \sigma^{-d} \exp\left(-\frac{\alpha^2}{2} \frac{T-s}{T+s}\right)$$

$$\begin{aligned}
&= \omega_d(2\pi)^{-d/2} \frac{T}{(f(T))^d} \sum_{s=1}^{\kappa_T} \frac{1}{T} \left(1 - \frac{s^2}{T^2}\right)^{-d/2} \exp\left(-\frac{\alpha^2}{2} \frac{1-s/T}{1+s/T}\right) \\
&\sim \omega_d(2\pi)^{-d/2} \frac{T}{(f(T))^d} \int_0^{1-K(f(T))^{-2}} (1-u^2)^{-d/2} \exp\left(-\frac{\alpha^2}{2} \frac{1-u}{1+u}\right) du.
\end{aligned}$$

In the case $d = 1$,

$$\begin{aligned}
I &\sim \left(\frac{2}{\pi}\right)^{1/2} \frac{T}{f(T)} \int_0^1 (1-u^2)^{-1/2} \exp\left(-\frac{\alpha^2}{2} \frac{1-u}{1+u}\right) du \\
&= \left(\frac{2}{\pi}\right)^{1/2} \frac{T}{f(T)} \int_0^{\pi/2} \exp\left(-\frac{\alpha^2}{2} \frac{1-\sin x}{1+\sin x}\right) dx.
\end{aligned}$$

Hence, for $d = 1$ and for any $K > 0$, we have

$$\left(1 + O\left(\frac{1}{f(T)}\right)\right) \left(1 - \frac{1}{K}\right) c(\alpha) \frac{T}{f(T)} \leq P(T) \leq \left(1 + O\left(\frac{1}{f(T)}\right)\right) c(\alpha) \frac{T}{f(T)}.$$

Hence, Lemma 3 is proved for $d = 1$.

In the case $d \geq 2$,

$$\begin{aligned}
I &\sim \omega_d(2\pi)^{-d/2} \frac{T}{(f(T))^d} \int_{K(f(T))^{-2}}^1 (v(2-v))^{-d/2} \exp\left(-\frac{\alpha^2}{2} \frac{v}{2-v}\right) dv \\
&\sim \omega_d(4\pi)^{-d/2} \frac{T}{(f(T))^d} \int_{K(f(T))^{-2}}^1 v^{-d/2} dv.
\end{aligned}$$

Hence, in the case $d = 2$,

$$I \sim \frac{1}{2} \frac{T}{f^2(T)} \log f(T)$$

and we have Lemma 3 for $d = 2$.

In the case $d \geq 3$,

$$I \sim \frac{2\omega_d}{d-2} (4\pi)^{-d/2} \frac{T}{(f(T))^2} \frac{1}{K^{(d-2)/2}}.$$

Lemma 4: Let X, Y be i.i.d.r.v.'s with

$$\mathbf{P}\{X \geq 0\} = \mathbf{P}\{Y \geq 0\} = 1,$$

$$\mathbf{P}\{X > 0\} = \mathbf{P}\{Y > 0\} = p \quad (0 \leq p \leq 1).$$

Then

$$\mathbf{E}(X + Y \mid X + Y > 0) = \frac{1}{2-p} \mathbf{E}Y + \mathbf{E}(X \mid X > 0).$$

Proof:

$$\begin{aligned}
\mathbf{E}(X + Y \mid X + Y > 0) &= \frac{2}{\mathbf{P}\{X + Y > 0\}} \int_{\{X + Y > 0\}} X d\mathbf{P} = \frac{2}{2p - p^2} \mathbf{E}X = \frac{2p}{2p - p^2} \mathbf{E}(X \mid X > 0) \\
&= \left(1 + \frac{p}{2-p}\right) \mathbf{E}(X \mid X > 0) = \mathbf{E}(X \mid X > 0) + \frac{1}{2-p} \mathbf{E}Y.
\end{aligned}$$

Lemma 4 is proved.

Lemma 5:

$$\mathbf{E}(\lambda(\mathbb{C}, T) \mid B(T) > 0) \sim \frac{T}{2} \mid \mathbb{C} \mid (2\pi T)^{-d/2} \exp\left(-\frac{\alpha^2}{2}\right)$$

$$= \frac{T}{2}(2\pi)^{-d/2}(f(T))^{-d}\omega_d \exp\left(-\frac{\alpha^2}{2}\right).$$

Proof: Clearly,

$$\mathbf{E}(\lambda(\mathcal{C}, T) \mid B(T)) = B(T)(2\pi T)^{-d/2} \int_{\mathcal{C}} \exp\left(-\frac{x^2}{2T}\right) dx \sim B(T) \mid \mathcal{C} \mid (2\pi T)^{-d/2} \exp\left(-\frac{\alpha^2}{2}\right).$$

Since

$$\mathbf{E}B(T) = 1 \text{ and } \mathbf{P}\{B(T) > 0\} \sim \frac{2}{T},$$

we have

$$\mathbf{E}\lambda(\mathcal{C}, T) = \mathbf{E}\mathbf{E}(\lambda(\mathcal{C}, T) \mid B(T)) \sim (2\pi T)^{-d/2} \mid \mathcal{C} \mid \exp\left(-\frac{\alpha^2}{2}\right)$$

and

$$\mathbf{E}(\lambda(\mathcal{C}, T) \mid B(T) > 0) = \frac{1}{\mathbf{P}\{B(T) > 0\}} \int_{\{B(T) > 0\}} \lambda(\mathcal{C}, T) d\mathbf{P} = \frac{1}{\mathbf{P}\{B(T) > 0\}} \mathbf{E}\lambda(\mathcal{C}, T)$$

and consequently we have Lemma 5.

$$\mathbf{Lemma\ 6:} \quad \mathbf{P}\{\lambda(\mathcal{C}, T) > 0\} \leq \mathbf{P}\{B(T) > 0\} \sim \frac{2}{T}.$$

Proof is trivial.

3. Proofs of Theorems 2 and 3

Having the condition $\{\lambda(\mathcal{C}, T) > 0\}$ we have two particles at time $t = 1$ and we know that at least one of them has at least one living offspring located in $\mathcal{C}$ at time T . Let $\lambda_{11}(\mathcal{C}, T - 1)$, respectively $\lambda_{12}(\mathcal{C}, T - 1)$ be the number of those offspring of the first respectively, second particle which are located in $\mathcal{C}$ at time T . Clearly,

$$\lambda(\mathcal{C}, T) = \lambda_{11}(\mathcal{C}, T - 1) + \lambda_{12}(\mathcal{C}, T - 1).$$

Then by Lemma 4,

$$\begin{aligned} & \mathbf{E}(\lambda(\mathcal{C}, T) \mid \Lambda(\mathcal{C}, T) > 0) \\ &= \mathbf{E}(\lambda_{11}(\mathcal{C}, T - 1) \mid \lambda_{11}(\mathcal{C}, T - 1) > 0) + \frac{1}{2 - p_1} \mathbf{E}\lambda_{12}(\mathcal{C}, T - 1), \end{aligned} \quad (3.1)$$

where

$$p_1 = \mathbf{P}\{\lambda_{11}(\mathcal{C}, T - 1) > 0\} = \mathbf{P}\{\lambda_{12}(\mathcal{C}, T - 1) > 0\}.$$

Consider that particle at time $t = 1$ which has at least one offspring living at time T and located in $\mathcal{C}$. (In the case both particles have such an offspring, consider one of them.) This particle has at time $t = 2$ two offspring and we know that at least one of them has at least one offspring located in $\mathcal{C}$ at time T . Let $\lambda_{21}(\mathcal{C}, T - 2)$ respectively, $\lambda_{22}(\mathcal{C}, T - 2)$ be the number of those offspring of the first respectively, second particle which are located in $\mathcal{C}$ at time T . Clearly,

$$\lambda_{11}(\mathcal{C}, T - 1) = \lambda_{21}(\mathcal{C}, T - 2) + \lambda_{22}(\mathcal{C}, T - 2).$$

Then by Lemma 4,

$$\begin{aligned} & \mathbf{E}(\lambda_{11}(\mathcal{C}, T - 1) \mid \lambda_{11}(\mathcal{C}, T - 1) > 0) \\ &= \mathbf{E}(\lambda_{21}(\mathcal{C}, T - 2) \mid \lambda_{21}(\mathcal{C}, T - 2) > 0) + \frac{1}{2 - p_2} \mathbf{E}\lambda_{22}(\mathcal{C}, T - 2), \end{aligned} \quad (3.2)$$

where

$$p_2 = \mathbf{P}\{\lambda_{21}(\mathcal{C}, T - 2) > 0\} = \mathbf{P}\{\lambda_{22}(\mathcal{C}, T - 2) > 0\}.$$

(3.1) and (3.2), combined, imply

$$\begin{aligned} & \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) = \mathbf{E}(\lambda_{21}(\mathcal{C}, T - 2) \mid \lambda_{21}(\mathcal{C}, T - 2) > 0) \\ &+ \frac{1}{2 - p_1} \mathbf{E}\lambda_{12}(\mathcal{C}, T - 1) + \frac{1}{2 - p_2} \mathbf{E}\lambda_{22}(\mathcal{C}, T - 2). \end{aligned}$$

Continuing this procedure we obtain

$$\begin{aligned} \mathbf{E}(\lambda(\mathbb{C}, T) \mid \lambda(\mathbb{C}, T) > 0) &= \sum_{s=1}^T \frac{1}{2-p_s} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) \\ &= \sum_{s=1}^{\kappa_T} \frac{1}{2-p_s} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) + \sum_{s=\kappa_T+1}^{T-1} \frac{1}{2-p_s} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) + 1 = I + II + 1, \end{aligned} \quad (3.3)$$

where

$$p_s = \mathbf{P}\{\lambda_{s1}(\mathbb{C}, T-s) > 0\} = \mathbf{P}\{\lambda_{s2}(\mathbb{C}, T-s) > 0\}$$

and

$$\kappa_T = T - K \frac{T}{f^2(T)}.$$

Clearly,

$$\mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) = \mathbf{P}\{\Gamma_2(T, s, T) \in \mathbb{C} \mid \Gamma_1(T, s, T) \in \mathbb{C}\}, \quad (3.4)$$

$$\begin{aligned} 0 \leq II &\leq \sum_{s=\kappa_T+1}^{T-1} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) = \sum_{s=\kappa_T+1}^{T-1} \mathbf{P}\{\Gamma_2 \in \mathbb{C} \mid \Gamma_1 \in \mathbb{C}\} \\ &\leq T - \kappa_T = K \frac{T}{f^2(T)} \end{aligned} \quad (3.5)$$

and

$$\frac{1}{2} \sum_{s=1}^{\kappa_T} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) \leq I \leq \sum_{s=1}^{\kappa_T} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s). \quad (3.6)$$

Then by Lemma 3, (3.4) and (3.6) if $d = 1$, we have

$$(1-\epsilon) \left(1 - \frac{1}{K}\right) \frac{c(\alpha)}{2} \frac{T}{f(T)} \leq I \leq (1+\epsilon) c(\alpha) \frac{T}{f(T)}. \quad (3.7)$$

(3.3), (3.5) and (3.7) imply

$$\begin{aligned} 1 + (1-\epsilon) \left(1 - \frac{1}{K}\right) \frac{c(\alpha)}{2} \frac{T}{f(T)} &\leq \mathbf{E}(\lambda(\mathbb{C}, T) \mid \lambda(\mathbb{C}, T) > 0) \\ &\leq 1 + (1+\epsilon) c(\alpha) \frac{T}{f(T)} + K \frac{T}{f^2(T)} \end{aligned} \quad (3.8)$$

for any $K > 0$.

Note that if

$$f(T) \leq T^{1/2} \text{ and } s \leq \kappa_T,$$

then by Lemma 6,

$$p_s \leq \frac{2}{T-s} \leq \frac{2f^2(T)}{KT} \leq \frac{2}{K}$$

and

$$I \leq \frac{1}{2} \left(1 + \frac{2}{K}\right) \sum_{s=1}^{\kappa_T} \mathbf{E}\lambda_{s2}(\mathbb{C}, T-s) \quad (3.9)$$

if $K \geq 2$.

If we assume that $d = 1$, $f(T) \leq T^{1/2}$ and $K \geq 2$, then by (3.3), (3.4), (3.5), (3.6) and (3.9), we have

$$\begin{aligned} 1 + (1-\epsilon) \left(1 - \frac{1}{K}\right) \frac{c(\alpha)}{2} \frac{T}{f(T)} &\leq \mathbf{E}(\lambda(\mathbb{C}, T) \mid \lambda(\mathbb{C}, T) > 0) \\ &\leq 1 + (1+\epsilon) \frac{1}{2} \left(1 + \frac{2}{K}\right) c(\alpha) \frac{T}{f(T)} + K \frac{T}{f^2(T)}. \end{aligned} \quad (3.10)$$

Hence, we have Theorem 2 in the case $d = 1$.

In the case $d = 2$, Lemma 3, (3.4) and (3.6) imply

$$(1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{4} \frac{T}{f^2(T)} \log f(T) \leq I \leq \frac{1}{2} (1 + \epsilon) \frac{T}{f^2(T)} \log f(T). \quad (3.11)$$

(3.3), (3.5) and (3.11) imply

$$\begin{aligned} & 1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{4} \frac{T}{f^2(T)} \log f(T) \\ & \leq \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \leq 1 + \frac{1}{2} (1 + \epsilon) \frac{T}{f^2(T)} \log f(T) + K \frac{T}{f^2(T)} \end{aligned}$$

for any $K > 0$.

If we assume that $d = 2$, $f(T) \leq T^{1/2}$ and $K \geq 2$, then by (3.3), (3.4), (3.5), (3.6) and (3.9) we have

$$\begin{aligned} & 1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{4} \frac{T}{f^2(T)} \log f(T) \leq \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ & \leq 1 + \frac{1 + \epsilon}{4} \left(1 + \frac{2}{K}\right) \frac{T}{f^2(T)} \log f(T) + K \frac{T}{f^2(T)}. \end{aligned} \quad (3.12)$$

Hence, we have Theorem 2 in the case $d = 2$.

In the case $d \geq 3$, Lemma 3, (3.4) and (3.6) imply

$$\begin{aligned} & (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{2} \frac{\omega_d}{(4\pi)^{d/2}} \frac{2}{d-2} \frac{T}{f^2(T)} K^{-(d-2)/2} \\ & \leq I \leq (1 + \epsilon) \frac{\omega_d}{(4\pi)^{d/2}} \frac{2}{d-2} \frac{T}{f^2(T)} K^{-(d-2)/2}. \end{aligned} \quad (3.13)$$

(3.3), (3.5) and (3.13) imply

$$\begin{aligned} & 1 + (1 - \epsilon) \left(1 - \frac{1}{K}\right) \frac{1}{d-2} \frac{\omega_d}{(4\pi)^{d/2}} \frac{T}{f^2(T)} K^{-(d-2)/2} \leq \mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0) \\ & \leq 1 + (1 + \epsilon) \frac{\omega_d}{(4\pi)^{d/2}} \frac{2}{d-2} \frac{T}{f^2(T)} K^{-(d-2)/2} + K \frac{T}{f^2(T)}. \end{aligned}$$

Hence, we have Theorem 2 in the case $d \geq 3$.

Theorem 3 is a simple outcome of the consequences of Theorem 2, Lemma 5 and the following:

Lemma 7:

$$\mathbf{P}\{\lambda(\mathcal{C}, T) > 0 \mid B(T) > 0\} = \frac{\mathbf{E}(\lambda(\mathcal{C}, T) \mid B(T) > 0)}{\mathbf{E}(\lambda(\mathcal{C}, T) \mid \lambda(\mathcal{C}, T) > 0)},$$

whose proof is trivial.

4. Proof of Theorem 1

Let $B(A, T)$ be the number of those particles which are located in $A \subset \mathbb{R}^d$ at time $t = 0$ and which have at least one offspring living at time T . The following lemma is trivial.

Lemma 8:

$$\mathbf{P}\{B(A, T) = k\} = \frac{\nu^k}{k!} e^{-\nu} \quad (k = 0, 1, 2, \dots)$$

and

$$\mathbf{E} \exp(-zB(A, T)) = \exp(\nu(e^{-z} - 1)),$$

where

$$\nu = \nu(A, T) \sim \frac{2\mu |A|}{T},$$

as $T \rightarrow \infty$.

Introduce the following notations:

$$\begin{aligned}\mathbb{C}(R) &= \mathbb{C}(0, R), \\ \mathbb{C}_i &= \mathbb{C}_i(\Delta, T) = \mathbb{C}(0, (i+1)\Delta T^{1/2}) - \mathbb{C}(0, i\Delta T^{1/2}), \\ B(R) &= B(\mathbb{C}(R), T), \\ B_i &= B_i(\Delta, T) = B(\mathbb{C}_i, T).\end{aligned}$$

Then we have

$$\begin{aligned}|\mathbb{C}(R)| &= R^d \omega_d, \\ &= d\omega_d \Delta^d T^{d/2} i^{d-1} \leq |\mathbb{C}_i| = \Delta^d T^{d/2} ((i+1)^d - i^d) \omega_d \leq 2^d \omega_d \Delta^d T^{d/2} i^{d-1}, \\ \mathbb{E}B(R) &\sim \frac{2\mu\omega_d R^d}{T},\end{aligned}\tag{4.1}$$

$$\exp\left(- (1+\epsilon)\frac{2\mu\omega_d R^d}{T}\right) \leq \mathbf{P}\{B(R) = 0\} \leq \exp\left(- (1-\epsilon)\frac{2\mu\omega_d R^d}{T}\right),\tag{4.2}$$

$$\exp\left(- (1+\epsilon)\frac{2\mu\omega_d R^d}{T}(e^{-z} - 1)\right) \leq \mathbf{E}\exp(-zB(R)) \leq \exp\left(- (1-\epsilon)\frac{2\mu\omega_d R^d}{T}(e^{-z} - 1)\right),\tag{4.3}$$

$$(1-\epsilon)2\mu d\omega_d \Delta^d i^{d-1} T^{(d-2)/2} \leq \mathbb{E}B_i \sim \frac{2\mu|\mathbb{C}_i|}{T} \leq (1+\epsilon)2\mu 2^d \omega_d \Delta^d i^{d-1} T^{(d-2)/2},\tag{4.4}$$

$$\begin{aligned}\exp(-(1+\epsilon)2\mu 2^d \omega_d \Delta^d i^{d-1} T^{(d-2)/2}) &\leq \mathbf{P}\{B_i = 0\} \\ &\sim \exp\left(-\frac{2\mu|\mathbb{C}_i|}{T}\right) \leq \exp(-(1-\epsilon)2\mu d\omega_d \Delta^d i^{d-1} T^{(d-2)/2})\end{aligned}\tag{4.5}$$

$$\begin{aligned}\exp((1+\epsilon)2\mu 2^d \omega_d \Delta^d i^{d-1} T^{(d-2)/2}(e^{-z} - 1)) &\leq \mathbf{E}\exp(-zB_i) \sim \exp\left(\frac{2\mu|\mathbb{C}_i|}{T}(e^{-z} - 1)\right) \\ &\leq \exp((1-\epsilon)2\mu d\omega_d \Delta^d i^{d-1} T^{(d-2)/2}(e^{-z} - 1)).\end{aligned}\tag{4.6}$$

Now we present the proof of Theorem 1 in eight steps.

Step 1: Let $d = 2$ and

$$R_1 = R_1(T) = K(T \log T)^{1/2}.$$

Then by (4.2),

$$\mathbf{P}\{B(R_1) = 0\} \leq \exp(-(1-\epsilon)2\mu\pi K^2 \log T).\tag{4.7}$$

Consider a particle which is located in $\mathbb{C}(R_1)$ at time $t = 0$ and which has a living offspring at time T . Let $V(0)$ be the location of the considered particle at $t = 0$ and let $V(T)$ be the location of an arbitrary, fixed offspring of the considered particle at time T . Then,

$$\mathbf{P}\{|V(T) - V(0)| \geq R_1\} \leq \exp\left(-\frac{K^2}{2}\log T\right).$$

Consequently,

$$\mathbf{P}\{\Lambda(\mathbb{C}(2R_1), T) = 0\} \leq \exp(-(1-\epsilon)2\mu\pi K^2 \log T) + \exp\left(-\frac{K^2}{2}\log T\right).$$

Hence, by Borel-Cantelli lemma,

$$\Lambda(\mathbb{C}(2R_1), T) > 0, \text{ a.s.}$$

for all but finitely many T provided that

$$K > \max\{2^{1/2}, (2\mu\pi)^{-1/2}\},$$

Step 2: Let $d = 2$ and

$$R_2 = R_2(T) = \frac{T^{1/2}}{f(T)},$$

where

$$f(T) \uparrow \infty, \quad R_2(T) \uparrow \infty.$$

Consider B_i particles located in the ring $\mathcal{C}_i$ at time $t = 0$ having offspring living at time T . Let $\lambda_j^{(i)}(A, T)$ ($j = 1, 2, \dots, B_i$) be the number of those offspring of the j th particle which are located in A at time T . Then, by (1.3),

$$\mathbf{P}\{\lambda_j^{(i)}(\mathcal{C}(R_2), T) > 0\} \leq \frac{\exp\left(-\frac{i^2 \Delta^2}{2}\right)}{\log f(T)} \quad (4.8)$$

and by (4.6)

$$\begin{aligned} \mathbf{P}\left\{\prod_{j=1}^{B_i} \{\lambda_j^{(i)}(\mathcal{C}(R_2), T) = 0\}\right\} &= \mathbf{E}\mathbf{P}\left\{\prod_{j=1}^{B_i} \{\lambda_j^{(i)}(\mathcal{C}(R_2), T) = 0\} \mid B_i\right\} \\ &\geq \mathbf{E}\left(-\frac{\exp\left(-\frac{i^2 \Delta^2}{2}\right)}{\log f(T)}\right)^{B_i} = \exp(\nu(e^{-z} - 1)) \end{aligned}$$

where

$$\nu = \nu_i = \frac{2\mu |\mathcal{C}_i|}{T} \leq 8\mu\pi\Delta^2 i \quad (4.9)$$

and

$$e^{-z} = e^{-z_i} = 1 - \frac{\exp\left(-\frac{i^2 \Delta^2}{2}\right)}{\log f(T)}. \quad (4.10)$$

Hence,

$$\begin{aligned} \mathbf{P}\{\Lambda(\mathcal{C}(R_2), T) = 0\} &\geq \prod_{i=0}^{\infty} \exp(\nu_i(e^{-z_i} - 1)) \geq \exp\left(-\sum_{i=0}^{\infty} 8\mu\pi i \Delta^2 \frac{\exp\left(-\frac{i^2 \Delta^2}{2}\right)}{\log f(T)}\right) \\ &\geq \exp\left(-\frac{8\mu\pi}{\log f(T)} \sum_{i=0}^{\infty} i \Delta^2 \exp\left(-\frac{i^2 \Delta^2}{2}\right)\right). \end{aligned} \quad (4.11)$$

Since, as $\Delta \rightarrow \infty$,

$$\sum_{i=0}^{\infty} i \Delta^2 \exp\left(-\frac{i^2 \Delta^2}{2}\right) \sim \int_0^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx = 1,$$

we have

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_2), T) = 0\} \geq 1 - \frac{8\mu\pi}{\log f(T)} \rightarrow 1.$$

Which, in turn, implies

$$\Lambda(\mathcal{C}(R_2), T) = 0 \quad \text{i.o. a.s.}$$

Step 3: Let $d = 2$,

$$R_3 = R_3(T) = \log T)^{-1/2 + \epsilon}, \quad (\epsilon > 0)$$

$$T_k = e^k, \quad \rho_k = T_k^{1/2},$$

$$R_3(k) = R_3(T_k).$$

Then by Lemma 8,

$$\begin{aligned} \mathbf{P}\{B(\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k), T_{k+1}) = 0\} &\leq \exp\left(-(1-\epsilon) \frac{2\mu\pi}{T_{k+1}} (\rho_{k+1}^2 - \rho_k^2)\right) \\ &= \exp\left(-(1-\epsilon) \frac{2\mu\pi(e-1)}{e}\right) < 1. \end{aligned} \quad (4.12)$$

Consider a particle which is located in $\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k)$ at time $t = 0$ and which has a living offspring at time T_{k+1} . Note that by (4.12) with positive probability there exists such a particle. Let $\lambda_3(k)$ be the number of those offspring of the considered particle which are located

in $\mathbb{C}(R_3(k+1))$ at time T_{k+1} . Then by (1.3),

$$\mathbf{P}\{\lambda_3(k) > 0\} \geq \frac{1-\epsilon}{k+1} e^{-1/2}. \quad (4.13)$$

Since the events $\{\lambda_3(k) > 0\}$ are independent we have

$$\Lambda(\mathbb{C}(R_3), T) > 0 \quad \text{i.o. a.s.}$$

Step 4: Let $d = 2$,

$$R_4 = R_4(T) = T^{-1/2}(\log T)^{-1/2-\epsilon}.$$

Then by (4.4),

$$\mathbf{E}B_i \leq (1+\epsilon)8\mu\pi\Delta^2 i. \quad (4.14)$$

Consider the particles located in the ring $\mathbb{C}_i$ at time $t = 0$ having offspring living at time T . Let $\lambda_j^{(i)}(A, T)$ ($j = 1, 2, \dots, B_i$) be the number of those offspring of the j th particle which are located in A at time T . Then by (1.4),

$$\mathbf{P}\{\lambda_j^{(i)}(\mathbb{C}(R_4), T) > 0\} \leq \frac{R_4^2}{4} \exp\left(-\frac{i^2\Delta^2}{2}\right) \quad (4.15)$$

and

$$\begin{aligned} \mathbf{P}\left\{\sum_{j=1}^{B_i} \lambda_j^{(i)}(\mathbb{C}(R_4), T) > 0\right\} &\leq \mathbf{E}\mathbf{P}\left\{\sum_{j=1}^{B_i} \lambda_j^{(i)}(\mathbb{C}(R_4), R) > 0 \mid B_i\right\} \\ &\leq \mathbf{E}B_i \frac{R_4^2}{4} \exp\left(-\frac{i^2\Delta^2}{2}\right) \leq 2\mu\pi R_4^2 \Delta^2 i \exp\left(-\frac{i^2\Delta^2}{2}\right). \end{aligned} \quad (4.16)$$

Hence,

$$\begin{aligned} \mathbf{P}\{\Lambda(\mathbb{C}(R_4), T) > 0\} &\leq \sum_{i=0}^{\infty} \mathbf{P}\left\{\sum_{j=1}^{B_i} \lambda_j^{(i)}(\mathbb{C}(R_4), T) > 0\right\} \\ &\leq 2\mu\pi R_4^2 \sum_{i=0}^{\infty} \Delta^2 i \exp\left(-\frac{i^2\Delta^2}{2}\right). \end{aligned} \quad (4.17)$$

Consequently,

$$\Lambda(\mathbb{C}(R_4), T) = 0 \quad \text{a.s.}$$

for all but finitely many T .

Step 5: Let $d \geq 3$ and

$$R_1 = R_1(T) = K(\log T)^{1/(d-2)}.$$

Define $\lambda_j^{(i)}(\cdot, \cdot)$ as in Step 2. Then by (1.6),

$$\mathbf{P}\{\lambda_j^{(i)}(\mathbb{C}(R_1), T) > 0\} \geq \frac{2^d \omega_d (d-2)}{8\omega_d + 4(8\pi)^{d/2}(d-2)} \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right) \left(\frac{R_1}{T^{1/2}}\right)^{d-2}$$

and

$$\begin{aligned} \mathbf{P}\left\{\prod_{j=1}^{B_i} \{\lambda_j^{(i)}(\mathbb{C}(R_1), T) = 0\}\right\} &= \mathbf{E}\mathbf{P}\left\{\prod_{j=1}^{B_i} \{\lambda_j^{(i)}(\mathbb{C}(R_1), T) = 0\} \mid B_i\right\} \\ &\leq \mathbf{E}\left(1 - M \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right) \left(\frac{R_1}{T^{1/2}}\right)^{d-2}\right)^{B_i} = \exp(\nu(e^{-z} - 1)), \end{aligned}$$

where

$$M = \frac{2^d \omega_d (d-2)}{8\omega_d + 4(8\pi)^{d/2}(d-2)},$$

$$\nu = \nu_i = \frac{2\mu |\mathbb{C}_i|}{T} = \frac{2\mu\omega_d}{T}((i+1)^d - i^d)T^{d/2}\Delta^d \geq \frac{2\mu d\omega_d}{T}i^{d-1}T^{d/2}\Delta^d,$$

$$e^{-z} = e^{-z_i} = 1 - M \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right) \left(\frac{R_1}{T^{1/2}}\right)^{d-2}.$$

Hence,

$$\begin{aligned} \mathbf{P}\{\Lambda(\mathbb{C}(R_1), T) = 0\} &\leq \prod_{i=0}^{\infty} \exp(\nu_i(e^{-z_i} - 1)) \\ &\leq \exp\left(-\sum_{i=0}^{\infty} \frac{2\mu d\omega_d}{T} T^{d/2} i^{d-1} M \left(\frac{R_L}{T^{1/2}}\right)^{d-2} \Delta^d \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\right) \\ &\leq \exp\left(-2\mu d\omega_d M K^{d-2} \log T \sum_{i=0}^{\infty} i^{d-1} \Delta^d \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right)\right). \end{aligned}$$

Choose K such that

$$2\mu d\omega_d M K^{d-2} \sum_{i=0}^{\infty} i^{d-1} \Delta^d \exp\left(-\frac{(i+1)^2\Delta^2}{2}\right) > 1.$$

Then, we have

$$\Lambda(\mathbb{C}(R_1), T) > 0 \quad \text{a.s.}$$

for all but finitely many T .

Step 6: Let $d \geq 3$ and

$$R_2 = R_2(T) = K(\log \log \log T)^{1/(d-2)}.$$

Now follow the proof of Step 2, with the following modifications: instead of (4.8) by (1.6), we have

$$\mathbf{P}\{\lambda_j^{(i)}(\mathbb{C}(R_2), T) > 0\} \leq 2^{d-2}(d-2) \exp\left(-\frac{i^2\Delta^2}{2}\right) R_2^{d-2} T^{-(d-2)/2},$$

instead of (4.9), we have

$$\nu = \nu_i = \frac{2\mu |\mathbb{C}_i|}{T} \leq 2^{d+1} \mu \omega_d \Delta^d T^{(d-2)/2} i^{d-1},$$

instead of (4.10), we have

$$e^{-z} = e^{-z_i} = 1 - 2^{d-2}(d-2) \exp\left(-\frac{i^2\Delta^2}{2}\right) R_2^{d-2} T^{-(d-2)/2},$$

instead of (4.11), we have

$$\mathbf{P}\{\Lambda(\mathbb{C}(R_2), T) = 0\} \geq \exp\left(-2^{2d-1} \mu \omega_d (d-2) R_2^{d-2} \sum_{i=0}^{\infty} i^{d-1} \Delta^d \exp\left(-\frac{i^2\Delta^2}{2}\right)\right).$$

Hence, if K is small enough, then

$$\mathbf{P}\{\Lambda(\mathbb{C}(R_2(T_k)), T_k) = 0\} \geq \frac{1}{k},$$

where

$$T_k = \exp(\exp k^2).$$

Observe that the probability that at least one particle among the ones who are located in $\mathbb{C}(T_k)$ at time $t = 0$ would live at time T_{k+1} , is equal to

$$1 - \exp\left(-2\omega_d \mu \frac{T_k^d}{T_{k+1}}\right) \sim 2\omega_d \mu \frac{T_k^d}{T_{k+1}} = 2\omega_d \mu \exp(-e^{k^2}(e^{2k+1} - d)).$$

Hence, there is no particle in $\mathbb{C}(T_k)$ living up to time T_{k+1} .

Consequently, by Borel-Cantelli lemma we have

$$\Lambda(\mathbb{C}(R_2), T) = 0 \quad \text{i.o. a.s.}$$

Step 7: Let $d \geq 3$,

$$R_3 = R_3(T) = M(\log T)^{-1/d}.$$

Now follow the proof of Step 3 with the following modifications: instead of (4.12), we have

$$\mathbf{P}\{B(\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k), T_{k+1}) = 0\} \leq \left(-(1-\epsilon) \frac{2\mu\omega_d}{T_{k+1}} (T_{k+1}^{d/2} - T_k^{d/2}) \right) \rightarrow 0.$$

Consider the $B_k = B(\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k), T_{k+1})$ particles located in $\mathcal{C}(\rho_{k+1}) - \mathcal{C}(\rho_k)$ at time $t = 0$ having offspring living at time T_{k+1} . Let $\lambda_j^{(k)}(A, T)$ ($j = 1, 2, \dots, B_k$) be the number of those offspring of the j th particle which are located in A at time T . Then by (1.7) (with $\beta = 0$), we have

$$\begin{aligned} \mathbf{P}\{\lambda_j^{(k)}(\mathcal{C}(R_3), T_{k+1}) > 0\} &\geq (1-\epsilon) \frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2} M^d} \frac{T_{k+1}}{(T_{k+1})^{d/2} \log T_{k+1}} \\ &= (1-\epsilon) \frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2} M^d} \frac{R_3^d(T_{k+1})}{T_{k+1}^{(d-2)/2}} \end{aligned}$$

and

$$\begin{aligned} \mathbf{P}\{\Lambda(\mathcal{C}(R_3(T_{k+1}))), T_{k+1} = 0\} &= \mathbf{E} \mathbf{P}\{\Lambda(\mathcal{C}(R_3(T_{k+1}))), T_{k+1} = 0 \mid B_k\} \\ &\leq \mathbf{E} \exp(-z B_k) = \exp(\nu(e^{-z} - 1)), \end{aligned}$$

where

$$\nu = \nu_k = \frac{2\mu}{T_{k+1}} \omega_d (\rho_{k+1}^d - \rho_k^d)$$

and

$$e^{-z} = e^{-z_k} = 1 - (1-\epsilon) \frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2} M^d} \frac{R_3^d(T_{k+1})}{T_{k+1}^{(d-2)/2}}.$$

Hence,

$$\begin{aligned} &\mathbf{P}\{\Lambda(\mathcal{C}(R_3(T_{k+1}))), T_{k+1} > 0\} \\ &\geq 1 - \exp \left(- \frac{2\mu}{T_{k+1}} \omega_d (\rho_{k+1}^d - \rho_k^d) (1-\epsilon) \frac{\omega_d e^{-1/2}}{2(2\pi)^{d/2} M^d} \frac{R_3^d(T_{k+1})}{T_{k+1}^{(d-2)/2}} \right) \geq \frac{1}{k} \end{aligned}$$

if M is small enough. Consequently,

$$\Lambda(\mathcal{C}(R_3(T)), T) > 0 \text{ i.o. a.s.}$$

Step 8: Let $d \geq 3$ and

$$R_4 = R_4(T) = T^{-1/d} (\log T)^{-1/d - \epsilon}.$$

Now follow the proof of Step 4 with the following modifications: instead of (4.14) we have

$$\mathbf{E} B_i \leq 2^{d+1} \mu \omega_d \Delta^d i^{d-1} T^{(d-2)/2},$$

instead of (4.15), by (1.7) we have

$$\mathbf{P}\{\lambda_j^{(i)}(\mathcal{C}(R_4), T) > 0\} \leq \frac{\omega_d \exp \left(-\frac{i^2 \Delta^2}{2} \right)}{2(2\pi)^{d/2}} R_4^d T^{-(d-2)/2},$$

instead of (4.16), we have

$$\mathbf{P} \left\{ \sum_{j=1}^{B_i} \lambda_j^{(i)}(\mathcal{C}(R_4), T) > 0 \right\} \leq \mu \omega_d^2 \left(\frac{2}{\pi} \right)^{d/2} i^{d-1} \Delta^d \exp \left(-\frac{i^2 \Delta^2}{2} \right) R_4^d,$$

instead of (4.17), we have

$$\mathbf{P}\{\Lambda(\mathcal{C}(R_4), T) > 0\} \leq \mu \omega_d^2 \left(\frac{2}{\pi}\right)^{d/2} R_4^d \sum_{i=0}^{\infty} i^{d-1} \Delta^d \exp\left(-\frac{i^2 \Delta^2}{2}\right).$$

Consequently,

$$\Lambda(\mathcal{C}(R_4), T) = 0 \quad \text{a.s.}$$

for all but finitely many T .

References

- [1] Révész, P., *Random Walks of Infinitely Many Particles*, World Scientific, Singapore 1994.