Journal of Applied Mathematics and Stochastic Analysis, **10**:3 (1997), 239-248.

CONTINUITY PROPERTIES OF SOLUTIONS OF MULTIVALUED EQUATIONS WITH WHITE NOISE PERTURBATION

MARIUSZ MICHTA
Technical University, Institute of Mathematics
Podgorna 50, 65-246 Zielona Gora, Poland

(Received April, 1996; Revised November, 1996)

In the paper, we consider a set-valued stochastic equation with stochastic perturbation in a Banach space. We prove first the existence theorem and then study continuity properties of solutions.

Key words: Set-valued Mappings, Aumann's Integral, Convergence in Probability of Random Elements.

AMS subject classifications: 54C65, 54C60.

1. Preliminaries

Problems of existence of solutions to set-valued differential equations were studied by many (see e.g., [3, 8, 9]). In particular, random cases were considered by the author in [11, 12].

In this paper we study the set-valued stochastic equation with white noise drift:

$$\begin{aligned} DX_t &= F(t, X_t)dt + \sigma_t dw_t, t \in I, \\ X_0 &= U \quad P.1, \end{aligned} \tag{I}$$

where F and U are given random set-valued mappings with values in the space $K_c(E)$, of all nonempty, compact and convex subsets of the separable Banach space $(E, \| \ \|)$, $I := [0, T]$; $T > 0$. We assume also that there is a predictable stochastic process σ with values in E. Finally, $(w_t)_{t \in I}$ denotes a real Wiener process. We interpret the above equation through its integral form as

$$X_t = U + \int_0^t F(s, X_s)ds + \int_0^t \sigma_s dw_s \quad P.1, \quad t \in I. \tag{II}$$

Integrals above are Aumann's integral of F and stochastic (Itô) integral of σ, respectively.

The aim of this work is to study continuity properties of set-valued solutions of

Printed in the U.S.A. ©1997 by North Atlantic Science Publishing Company

(I). First, we recall several notions needed in the sequel. In the space $K_c(E)$ we consider the Hausdorff metric H (see e.g., [5, 7]): $H(A,B) = \max(\bar{H}(A,B), \bar{H}(B,A))$ for $A, B \in K_c(E)$, where $\bar{H}(A,B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|$. By $\| A \|$ we denote the distance $H(A,0)$. It can be proved that $(K_c(E), H)$ is a Polish metric space.

By $C_I = C(I, K_c(E))$ we denote the space of all H-continuous set-valued mappings. In this space we consider metric ρ of uniform convergence:

$$\rho(X,Y) := \sup_{0 \le t \le T} H(X(t), Y(t)), \text{ for } X, Y \in C_I.$$

Then we have a Polish metric space.

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)_{t \in I}$ be a given complete filtered probability space satisfying the usual conditions. We recall the notion of a multivalued $\mathfrak{F}_t$-adapted stochastic process. The family of set-valued mappings $X = (X_t)_{t \in I}$ is said to be a *multivalued $\mathfrak{F}_t$-adapted stochastic process* if for every $t \in I$, the mapping $X_t: \Omega \to K_c(E)$ is $\mathfrak{F}_t$-measurable, i.e., $\{\omega: X_t \cap V \neq \emptyset\} \in \mathfrak{F}_t$, for every open set $V \subset E$ (see e.g., [7]). It can be noted that V can be chosen as a closed or Borel subset. If the mapping $t \to X_t(\omega)$ is H-continuous with probability one $(P.1)$ then we say it has *continuous paths.* In this case, the set-valued process X can be thought as random element $X: \Omega \to C_I$. Let (X_n) be a sequence of random elements with values in metric space (S, ρ). Then we say that X_n converges in probability to the random element $X: \Omega \to S$ $(X_n \xrightarrow{P} X)$, of for every $\epsilon > 0$, it holds true that $P(\rho(X_n, X) \ge \epsilon) \to 0$, as n tends to infinity. It is known (see e.g., [13]) that $X_n \xrightarrow{P} X$ if and only if every subsequence of (X_n) has a subsequence converging to X with probability one $(P.1)$.

In the theory of differential equations in Banach space the notion of measure of noncompactness plays one of the central roles (see e.g., [1]). Let $B(E)$ denote a family of all nonempty and bounded subsets in E.

Definition 1: The mapping $\mathcal{N}: B(E) \to [0, \infty)$, defined by $\mathcal{N}(A) = \inf\{\epsilon > 0: A$ can be covered with a finite number of balls of radii $\le \epsilon\}$, is called *Hausdorff (ball) measure of noncompactness.*

2. A Set-Valued Stochastic Equation and Stochastic Inclusion

We begin with the designation of restrictions imposed on F, U and σ. Let us assume that $F: I \times \Omega \times K_c(E) \to K_c(E)$, $U: \Omega \to K_c(E)$, and $\sigma: I \times \Omega \to E$ have the following properties:

1) F is an integrably bounded multifunction i.e. there exists a joint measurable function $m: I \times \Omega \to R_+$ such that $\int_0^T m(s,\omega) ds < \infty$ $P.1$ and $\| F(t, \omega, A) \| \le m(t, \omega)$ $P.1$, t-a.e. $A \in K_c(E)$.
2) $F(t, \omega, \ )$ is H-continuous with $P.1$, t-a.e.
3) $F(t, \ , A)$ is $\mathfrak{F}_t$-adapted for every $t \in I$, $A \in K_c(E)$.
4) $F(\ , \ , A)$ is measurable for every $A \in K_c(E)$.
5) U is an $\mathfrak{F}_0$-measurable multifunction.
6) $\sigma = (\sigma_t)$ is an $\mathfrak{F}_t$-adapted stochastic process for which $E \int_0^T \| \sigma_s \|^2 ds$ is finite.

Let us notice that under assumptions given above, for every $A \in K_c(E)$, the set-valued process

$$\Phi_t = U + \int_0^t F(s, A)ds + \int_0^t \sigma_s dw_s, t \in I,$$

is $\mathcal{F}_t$-adapted with values in $K_c(E)$. It is also clear that Φ has continuous "paths".

We also assume the so-called "Kamke condition" imposed on multifunction F: for every $A_1, A_2, \ldots \in K_c(E)$ one has

$$\mathcal{N}(\bigcup_{n \geq 1} F(t, A_n)) \leq k(t, \mathcal{N}(\bigcup_{n \geq 1} A_n)) \text{ with } P.1 \quad t \in I \text{ a.e.}, \tag{$*$}$$

where $k: I \times \Omega \times R \to R_+$ satisfies the following conditions:

a) $k(t, , x)$ is $\mathcal{F}$-measurable for every $(t, x) \in I \times R_+$,

b) $k(, \omega,)$ is a Kamke function (see e.g., [14]) with $P.1$.

Definition 2: A multivalued process $X = (X_t)_{t \in I}$ is said to be a *solution of* (I) if it satisfies multivalued stochastic equation (II).

Let us notice that without stochastic perturbation, equation (II) can be written as:

$$D_H X_t = F(t, X_t) \; P.1, \quad t\text{-a.e.}$$

$$X_0 = U \; P.1,$$

where D_H denotes the Hukuchara derivative operator ([6]) for multifunctions.

Before stating the existence theorem to equation (II) let us recall its special case.

Theorem 1: ([11]) *Let F and U be multivalued mappings satisfying conditions* 1)-4) *and* 5), *respectively. Let us also suppose that F satisfies the "Kamke condition." Then the multivalued random differential equation*

$$D_H X_t = F(t, X_t) \text{ with } P.1 \quad t \in I \text{ a.e.}$$

$$X_0 = U \quad \text{with } P.1$$

has at least one solution.

Remark: In fact, the existence of solutions to the above initial value problem is based on the fact that under these conditions there exists at least one solution to the multivalued equation $X_t = U + \int_0^t F(s, X_s)ds$ and on well-known connection between Aumann's integral of set-valued mapping and its Hukuchara derivative via Radström Embedding Theorem (see e.g. [14]).

Theorem 2: *Let E be a Banach space such that its dual E^* is separable. If F, U and σ have properties* 1)-6) *and F satisfies the "Kamke condition" then there exists at least one solution of the equation (II).*

Proof: Let $\xi_t = \int_0^t \sigma_s dw_s$. Let $\hat{X_t} := X_t - \xi_t$, where X_t is a solution of (II), and $\hat{X_t}(\omega) = \{\hat{x} - \xi_t(\omega); \hat{x} \in X_t(\omega)\}$. The process $\hat{X}$ satisfies the equation

$$\hat{X_t} = U + \int_0^t \hat{F}(s, \hat{X_s})ds \quad P.1, \; t \in I, \tag{$**$}$$

where $\hat{F}(s, \omega, A) = F(s, \omega, A + \xi_s(\omega))$. The set-valued mapping $\hat{F}$ meets properties

1)-4). By properties of measure of noncompactness it also satisfies $(*)$ (cf. [1]). Hence, equation (II) has at least one solution if and only if equation $(**)$ has one. By Theorem 1 (via Remark 1) the proof is completed.

Let us suppose now that $\Gamma: I \times \Omega \times E \to K_c(E)$ is a given set-valued mapping. Let us set $F(t,\omega,A) := \overline{\text{co}}\Gamma(t,\omega,A)$, $A \in K_c(E)$, where $\overline{\text{co}}B$ denotes the closed convex hull of the set B. It is noteworthy to observe the connections between solutions of equation (II), with $F = \overline{\text{co}}\Gamma$ and solutions of stochastic inclusion

$$x_t - x_s \in \int_s^t \Gamma(u,x_u)du + \int_s^t \sigma_u dw_u \text{ with } P.1,\ 0 \le s \le t \le T \qquad (II')$$

$$x_0 \in U \text{ with } P.1.$$

We suppose that Γ is an integrable bounded multifunction such that:

1′) $\Gamma(t,\omega,\ )$ is H-continuous with $P.1$, t-a.e.,

2′) $\Gamma(t,\ ,x)$ is $\mathcal{F}_t$-adapted for every $t \in I$, $x \in E$,

3′) $\Gamma(\ ,\ ,x)$ is measurable for every $x \in E$,

4′) $\forall A \subset S_r(U): \mathcal{N}(\Gamma(t,A)) \le k(t,\mathcal{N}(A))$ $P.1$, $t \in I$,

where $S_r(U) = U + rB(0,1)$ and $B(0,1)$ is a closed unit ball in Banach space E, centered at zero.

Theorem 3: *Suppose that Γ satisfies conditions 1′-4′. If a multivalued stochastic process $X = (X_t)_{t \in I}$ is a solution of equation (II) with $F = \overline{co}\Gamma$ then there exists stochastic process $x = (x_t)$ being both a solution to stochastic inclusion (II') and the selection of X.*

Proof: Similarly, as above, let $\xi_t = \int_0^t \sigma_s dw_s$, $\hat{\Gamma}(t,\omega,x) := \Gamma(t,\omega,x+\xi_t(\omega))$ and $\hat{F}(t,\omega,A) := F(t,\omega,A+\xi_t(\omega))$. Then $\hat{F} = \overline{\text{co}}\hat{\Gamma}$. Let us notice than $\hat{F}$ also satisfies 1′-4′. Hence, by Corollary 1 [11], there exists at least one solution of equation

$$\hat{X}_t = U + \int_0^t \hat{F}(s,\hat{X}_s)ds \quad P.1, \quad t \in I.$$

Taking $X = \hat{X} + \xi$ we get a solution of equation (II), where $F = \overline{\text{co}}\Gamma$. Moreover, by Theorem 4 [11] there exists stochastic process, say $\hat{x} = (\hat{x}_t)$, being a selection of $\hat{X}$ such that: $\hat{x}_t - \hat{x}_s \in \int_s^t \hat{\Gamma}(u,\hat{x}_u)du$ with $P.1$, $0 \le s \le t \le T$, and $\hat{x}_0 \in U$ $P.1$.

Consequently, there exists stochastic process $x = (x_t)$, as a selection of X, such that: $\hat{x}_t = x_t - \xi_t$ with $P.1$. It remains to observe that x is a desired solution of inclusion (II').

3. Continuity Properties of Solutions

By $S(I \times \Omega)$ we denote the class of "simple" multivalued processes that can be expressed by: $X = \sum_{i=1}^{n} I_{D_i} C_i$, where the sets D_i, $i = 1,2,\ldots,n$ form a measurable partition of $I \times \Omega$ and $C_i \in K_c(E)$, $i = 1,2,\ldots,n$.

Lemma 1: *If $X = (X_t)_{t \in T}$ is a multivalued stochastic process with continuous*

"paths" then there exists a sequence $\{X_n\} \subseteq S(I \times \Omega)$ *such that* $\forall (t,\omega) \in I \times \Omega$: $\lim_{n\to\infty} H(X(t,\omega), X_n(t,\omega)) = 0$.

Proof: It follows directly from the fact that $K_c(E)$ is a separable metric space and Proposition 1.9 [15].

Let Λ be a metric space. Let us consider the multivalued mapping $F: I \times \Omega \times K_c(E) \times \Lambda \to K_c(E)$ such that:

A1. For every fixed $A \in K_c(E)$ and $\lambda \in \Lambda$, $F(\,,\,,A,\lambda)$ is a measurable and integrably bounded multifunction.

A2. The mapping $F(t,\omega,\,,\lambda)$ is with P.1 uniformly continuous with respect to $t \in I$ and $\lambda \in \Lambda$.

Definition 2: A multifunction F (with properties A1 and A2) is said to be *integrably continuous in probability (icp) at* $\lambda_0 \in \Lambda$ *with respect to a family* $\mathcal{C} \subset K_c(E)$ $(\mathcal{C} \neq \emptyset)$ if

$$\forall C \in \mathcal{C}, \forall t \in I: \int_0^t F(s,C,\lambda)ds \xrightarrow{P} \int_0^t F(s,C,\lambda_0)ds$$

for $\lambda \to \lambda_0$.

The results presented below give characterizations of icp multifunctions. We use them to obtain the main theorem.

Lemma 2: *If F is an icp multifunction at λ_0 with respect to $\mathcal{C}$ then for every $C \in \mathcal{C}$ one has:* $\int_0^t F(s,C,\lambda_n)ds \to \int_0^t F(s,C,\lambda_0)ds$ *P.1 uniformly in $t \in I$, for some sequence (λ_n) convergent to λ_0.*

Proof: Let D be a set of rationals in I, $D = \{t_1, t_2, \ldots\}$ and let (λ_n) be an arbitrary sequence of elements of Λ that converges to λ_0. Fix $C \in \mathcal{C}$. Then for $t_1 \in D$, there exist a sequence $(\lambda_n(t_1))_n$, convergent to λ_0 and set $\Omega(t_1) \subset \Omega$, $P(\Omega(t_1)) = 1$, such that

$$\forall \omega \in \Omega(t_1): H\Big(\int_0^{t_1} F(s,\omega,C,\lambda_n(t_1))ds, \int_0^{t_1} F(s,\omega,C,\lambda_0)ds\Big) \to 0, \text{ for } n \to \infty.$$

Similarly, for $t_2 \in D$ we can find a sequence $(\lambda_n(t_2))_n$ being a subsequence of $(\lambda_n(t_1))_n$ and $\Omega(t_2) \subset \Omega$, $P(\Omega(t_2)) = 1$ for which a similar convergence holds. Continuing this selection process we obtain the infinite table

$$\begin{array}{ccccc} \lambda_1(t_1) & \lambda_2(t_1) & \ldots & \lambda_n(t_1) & \ldots \\ \lambda_1(t_2) & \lambda_2(t_2) & \ldots & \lambda_n(t_2) & \ldots \\ \vdots & \vdots & & \vdots & \\ \lambda_1(t_n) & \lambda_2(t_n) & \ldots & \lambda_n(t_n) & \ldots \\ \vdots & \vdots & & \ldots & \end{array} \tag{1}$$

By diagonal selection we can find a sequence $(\lambda_n)_n$ being a subsequence of each row of table (1) that converges to λ_0. Let $\Omega_0 = \bigcap \{\Omega(t_n); n \geq 1\}$. Then $P(\Omega_0) = 1$. Moreover,

$$\forall \omega \in \Omega_0, \forall t \in D: H\Big(\int_0^t F(s,\omega,C,\lambda_n)ds, \int_0^t F(s,\omega,C,\lambda_0)ds\Big) \to 0, \; n \to \infty.$$

Since the set-valued process $J_t = \int_0^t F(s,C,\lambda)ds$, $t \in I$ has with $P.1$ uniformly continuous "paths", we can find Ω_0', $P(\Omega_0') = 1$ such that

$$\forall \omega \in \Omega_0' \colon \ \sup_{t \in I} H\Big(\int_0^t F(s,\omega,C,\lambda_n)ds, \int_0^t F(s,\omega,C,\lambda_0)ds\Big) \to 0, \text{ if } n \to \infty.$$

This completes the proof.

By $\mathbb{B}_I$ we denote the σ-field of Borel subsets of I.

Lemma 3: *A multifunction F is icp at λ_0 with respect to family $\mathcal{C}$ if and only if:* $\forall C \in \mathcal{C}$, $\forall (\lambda_n) \subset \Lambda \colon \lambda_n \to \lambda_0, \exists (\lambda_n') \subset (\lambda_n)$:

$$H\Big(\int_B F(s,C,\lambda_n')ds, \int_B F(s,C,\lambda_0 ds\Big) \to 0 \quad P.1 \tag{2}$$

as $n \to \infty$, for every $B \in \mathbb{B}_I$.

Proof: Fix $C \in \mathcal{C}$ and let (λ_n) be an arbitrary sequence convergent to λ_0. Then by Lemma 2, we can find its subsequence (λ_n') and $\Omega_0 \colon P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$ and $0 \le s < t \le T$,

$$\int_s^t F(u,\omega,C,\lambda_n')du \to \int_s^t F(u,\omega,C,\lambda_0)du, \text{ as } n \to \infty. \tag{3}$$

Let $\mathcal{I} \colon = \{[s,t) \colon 0 \le s < t \le T\}$ and

$$\mathcal{A} \colon = \{\bigcup_{i=1}^n R_i \colon R_i \in \mathcal{I}, R_i \cap R_j = \emptyset, i \ne j, i,j = 1,2,\ldots,n, n \ge 1\}.$$

Since $\sigma(\mathcal{I}) = \sigma(\mathcal{A}) = \mathbb{B}_I$ and $\mathcal{A}$ is a ring of subsets of I, then for every $\epsilon > 0$ and $B \in \mathbb{B}_I$, there exists $A \in \mathcal{A}$ such that $|B \Delta A| < \epsilon$ (c.f. e.g., Th. 11.4 [2]), where $|\ |$ is Lebesgue measure and $B \Delta A \colon = (B \backslash A) \cup (A \backslash B)$. By integrably boundness of F we get:

$$H\Big(\int_B F(s,C,\lambda_n')ds, \int_B F(s,C,\lambda_0)ds\Big) \le H\Big(\int_A F(s,C,\lambda_n')ds, \int_A F(s,C,\lambda_0)ds\Big)$$
$$+ \int_{B \Delta A} m(s,\omega)ds, \text{ for every } A \in \mathcal{A}.$$

Then by (3), $\limsup_{n \to \infty} H\big(\int_B F(s,C,\lambda_n')ds, \int_B F(s,C,\lambda_0)ds\big) \le \int_{B \Delta A} m(s,\omega)ds$.

Taking A sufficiently close to B we claim (2). The converse is obvious.

Lemma 4: *A multifunction F is icp at λ_0 with respect to $K_c(E)$ if and only if F is icp at λ_0 with respect to $S(I \times \Omega)$.*

Proof: Let us assume that F is icp with respect to $K_c(E)$. Let $X \in S(I \times \Omega)$. Then there exist $C_1, C_2, \ldots, C_r \in K_c(E)$ and a measurable partition $\{D_1, D_2, \ldots, D_r\}$ of space $I \times \Omega$ such that $X = \sum_{i=1}^r I_{D_i} C_i$. Take C_1 and (λ_n) to be an arbitrary sequence convergent to λ_0. Next let (λ_{n_k}) be any subsequence of (λ_n). By Lemma 3 there exists a sequence $(\lambda_{n,1}')$ being a subsequence of (λ_{n_k}) and a subset $\Omega_{0,1} \subseteq \Omega$;

$P(\Omega_{0,1}) = 1$ such that:

$$\forall \omega \in \Omega_{0,1}, \forall B \in \mathbb{B}_I: \lim_{n\to\infty} H\Big(\int_B F(s,\omega,C_1,\lambda'_{n,1})ds, \int_B F(s,\omega,C_1,\lambda_0)ds\Big) = 0.$$

Similarly, for C_2 we can extract a subsequence $(\lambda'_{n,2})$ from $(\lambda'_{n,1})$ and $\Omega_{0,2} \subseteq \Omega$ $P(\Omega_{0,2}) = 1$, with the desired property, and so on. Thus we obtain a sequence $(\lambda'_{n,r})$ which is a subsequence of $(\lambda'_{n,i})$, $i = 1,2,\ldots,r-1$ and $\Omega_{0,r}$, $P(\Omega_{0,r}) = 1$, such that

$$\forall \omega \in \Omega_{0,r}, \forall B \in \mathbb{B}_I: \lim_{n\to\infty}\Big(\int_B F(s,\omega,C_r,\lambda'_{n,r})ds, \int_B F(s,\omega,C_r,\lambda_0)ds\Big) = 0.$$

Let $\Omega_0 = \bigcap_{1\le i\le r} \Omega_{0,1}$. For any $A \in \mathbb{B}_I \otimes \mathcal{F}$ and $\omega \in \Omega$, we define the set $(A)_\omega := \{t \in I: (t,\omega) \in A\}$. Then $(A)_\omega \in \mathbb{B}_I$. Let $\omega \in \Omega_0$. Then $X(\cdot,\omega) = \sum_{i=1}^{r} I_{(D_i)_\omega}(\cdot)\, C_1$ and $\{(D_i)_\omega: i = 1,2,\ldots,r\}$ is measurable partition of I. Hence, the following inequality holds:

$$H\Big(\int_0^t F(s,\omega,X_s,\lambda'_{n,r})ds, \int_0^t F(s,\omega,X_s,\lambda_0)ds\Big)$$

$$\le \sum_{i=1}^{r} H\Big(\int_{(D_i)_\omega \cap [0,t]} F(s,\omega,C_i,\lambda'_{n,r})ds, \int_{(D_i)_\omega \cap [0,t]} F(s,\omega,C_i,\lambda_0 ds)\Big).$$

It remains to observe that each term of the above sum converges to zero as n tends to infinity.

The converse statement is obvious. It is enough to take $X := I_{I\times\Omega}C$, for $C \in K_c(E)$. This completes the proof.

By X^λ we denote a multivalued process being the solution of the equation

$$X_t = U + \int_0^t F(s,X_s,\lambda)ds + \int_0^t \sigma_s dw_s \; P.1,\; t \in I,\; \lambda \in A. \qquad (III)$$

Theorem 3: *Let us assume that F is an icp set-valued mapping at $\lambda_0 \in A$ with respect to $K_c(E)$. Then,*

i) *if $X^\lambda \xrightarrow{P} X^{\lambda_0}$ then $\forall t \in I$: $\int_0^t F(s,X_s^\lambda,\lambda)ds \xrightarrow{P} \int_0^t F(s,X_s^{\lambda_0},\lambda_0)ds$, $\lambda \to \lambda_0$,*

ii) *if for every $A_1, A_2, \ldots \in K_c(E)$ and $(\lambda_n); \lambda_n \to \lambda_0$ we have*

$$\mathcal{N}\Big\{\bigcup_{n\ge 1} F(t,A_n,\lambda_n)\Big\} \le k\Big(t, \mathcal{N}\Big\{\bigcup_{n\ge 1} A_n\Big\}\Big) \text{ with } P.1\ t \in I \text{ a.e., then } X^\lambda \xrightarrow{P} X^{\lambda_0}.$$

Proof: (i) Let (λ_n) be an arbitrary sequence convergent to λ_0. Then its every subsequence contains a further subsequence, say, (λ'_n), such that $X^{\lambda'_n} \to X^{\lambda_0}$ with $P.1$ in C_I. Take ω from an appropriate set (for which this convergence holds). By condition A2, for any $\epsilon > 0$, there exists $\delta > 0$ such that $H(F(t,C,\lambda'_n), F(t,D,\lambda'_n)) < \epsilon/4T$, for $n \in N$, $C, D \in K_c(E)$ whenever $H(C,D) < \delta$.

Let V_0 be an open neighborhood for λ_0 such that

$$\text{if } \lambda'_n \in V_0 \text{ then } \sup_{t \in I} H(X_t^{\lambda'_n}, X^{\lambda_0}) < \delta. \tag{4}$$

Let $(X_k^{\lambda_0})_k$ be a sequence of simple multifunctions (Lemma 1) convergent to X^{λ_0} for every $t \in I$ and $\omega \in \Omega$. Then for every $t \in I$ and $\lambda \in \Lambda$, we have:

$$\lim_k H(F(t,\omega,X_k^{\lambda_0}(t,\omega),\lambda), F(t,\omega,X_t^{\lambda_0}(\omega),\lambda) = 0 \quad P.1.$$

Next by the Lebesgue Dominated Convergence Theorem (via integrably boundedness of F) we obtain that

$$\int_0^T H(F(s,X_k^{\lambda_0}(s),\lambda), F(s,X_s^{\lambda_0},\lambda))ds \to 0 \ P.1$$

for every $\lambda \in \Lambda$. Hence by (4), after standard calculation we see that

$$H(\int_0^t F(s,\omega,X_s^{\lambda_n}(\omega),\lambda'_n)ds, \int_0^t F(s,\omega,X_s^{\lambda_0}(\omega),\lambda_0)ds) \le (3/4)\epsilon$$
$$+ H(\int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega),\lambda'_n)ds, \int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega)\lambda_0)ds),$$

for $t \in I$, k sufficiently large and ω taken from an appropriate set of probability one.

By Lemma 4, multifunction F is icp at λ_0 with respect to $S(I \times \Omega)$. Hence there exists a sequence (λ''_n) being a subsequence of (λ'_n), a subset of Ω of measure one such that for every $\epsilon > 0$ and appropriate ω we can find an open neighborhood V_1 of λ_0 with

$$H(\int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega),\lambda''_n)ds, \int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega),\lambda_0)ds) < \epsilon/4,$$

for $t \in I$ and $\lambda''_n \in V_1$. Therefore, taking n'' sufficiently large and $\lambda''_n \in V_0 \cap V_1$ we have:

$$H(\int_0^t F(s,\omega,X_s^{\lambda''_n}(\omega),\lambda''_n)ds, \int_0^t F(s,\omega,X_s^{\lambda_0}(\omega),\lambda_0)ds) < \epsilon$$

for $t \in I$. This completes the proof of part (i).

Proof of part (ii).

Let (λ_n) be a sequence convergent to λ_0. Consider its arbitrary subsequence, denoted for simplicity by the same symbol. We define the multivalued mapping $\Pi: \Omega \to 2^{C_I}$ by

$$\Pi(\omega) = \{X \in C_I : X^{\lambda'_n} \to X \text{ in } C_I \text{ for some sequence } (\lambda'_n), (\lambda'_n) \subseteq (\lambda_n)\}.$$

By the assumption of the integrably boundness of F it follows that

$$\forall n \in N, \forall 0 \le s \le t \le T: H(X_t^{\lambda_n}, X_s^{\lambda_n}) \le \int_s^t m(u)du \text{ with } P.1.$$

Thus the sequence (X^{λ_n}) is equicontinuous in C_I with $P.1$. Similarly, (compare [14]) by assumption (iii), it can be proved that $\bigcup_{n \geq 1} \{X_t^{\lambda_n}\}$ is a relatively compact subset of E, for every $t \in I$ with $P.1$. Thus, by Ascoli Theorem we claim that the sequence (X^{λ_n}) is relatively compact (with $P.1$). Hence the multifunction $\Pi \neq \emptyset$ $P.1$ and has closed values. Moreover, we claim that Π is measurable. To see this, let $\Omega_0 := \{\omega: \Pi(\omega)$ is closed subset of $C_I\}$. For $X \in C_I$ we consider a mapping $\Omega_0 \ni \omega \to \mathrm{Dist}(X, \Pi(\omega))$, where $\mathrm{Dist}(X, \Pi(\omega)) = \inf_{Y \in \Pi(\omega)} \rho(X, Y)$. Fix $r > 0$.

Then $\{\omega: \mathrm{Dist}(X, \Pi(\omega)) < r\} = \{\omega: \exists Y \in \Pi(\omega): Y \in B_r(X)\}$, where $B_r(X) := \{Y \in C_I: \rho(X, Y) < r\}$. Let $\{t_k\}$ be a sequence of rationals in I. Then we get:

$$\{\omega: \mathrm{dist}(X, \Pi(\omega)) < r\} = \{\omega: \Pi(\omega) \cap B_r(X) \neq \emptyset\}$$

$$= \bigcup_{m \geq 1} \bigcap_{l \geq 1} \bigcup_{j \geq l} \bigcap_{k \geq 1} \{\omega: X_{t_k}^{\lambda_{n_j}}(\omega) \cap B_{r-1/m}(X) \neq \emptyset\}.$$

Since $X_{t_k}^{\lambda_{n_j}}$ is an $\mathcal{F}_{t_k}$-measurable multifunction then the last set above belongs to σ-field $\mathcal{F}$, which yields the $\mathcal{F}$-measurability of Π (see, e.g. [4]). Thus, by Kuratowski and Ryll-Nardzewski Selection Theorem [10], there exists a measurable selection $\hat{X}$ of Π; $\hat{X} \in \Pi$ $P.1$. The definition of Π implies then that $X^{\lambda'_n} \to \hat{X}$ $P.1$ in C_I, for some sequence (λ'_n) tending to λ_0 and this yields convergence in probability in C_I. Finally, we claim that $\hat{X}$ is a solution of (III). Indeed, let us notice that

$$H(\hat{X}_t, U + \int_0^t F(s, \hat{X}_s, \lambda_0) ds + \int \sigma_s dw_s)$$

$$\leq H(\hat{X}_t, X_t^{\lambda'_n}) + H(\int_0^t F(s, X_s^{\lambda'_n}, \lambda'_n) ds, \int_0^t F(s, \hat{X}_s, \lambda_0) ds),$$

with $P.1$ and for $t \in I$.

Since the first term above converges to zero then by (i) the second term converges to zero as well. This completes the proof.

References

[1] Banaś, J. and Goebel, K., *Measures of Noncompactness in Banach Spaces*, Marcel Dekker 1980.

[2] Billingsley, P., *Probability and Measure*, John Wiley, New York 1979.

[3] De Blasi, F.S. and Iervolino, F., Euler method for differential equation with compact, convex valued solutions, *Boll. U.M.I* **4**:4 (1971), 941-949.

[4] Himmelberg, C.J., Measurable relations, *Fund. Math.* **87** (1975), 53-72.

[5] Himmelberg, C.J. and Van Vleck, F.S., The Hausdorff metric and measurable selections, *Topology and its Appl.* **20** (1985), 121-133.

[6] Hukuchara, M., Sur l application semicontinue dont la valeur est un compact convexe, *Funkcial. Ekwac.* **10** (1967), 43-66.

[7] Kisielewicz, M., *Differential Inclusions and Optimal Control*, Kluwer 1991.

[8] Kisielewicz, M., Method of averaging for differential equation with compact convex valued solutions, *Rend. di Matem.* **9**:3 (1976), 1-12.

[9] Kisielewicz, M., Serafin, B. and Sosulski, W., Existence theorem for functional-differential equation with compact convex valued solutions, *Demonstratio Math.* **IX**:2 (1976), 229-237.

[10] Kuratowski, K. and Ryll-Nardzewski, Cz., A general theorem on selectors, *Bull. Acad. Polon. Sci. Ser, Sci Math. Astronom. Phys.* **13** (1965), 397-403.

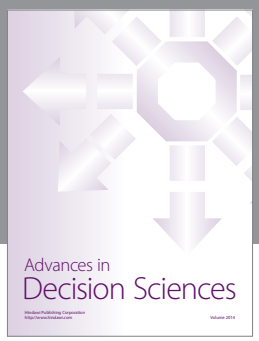
[11] Michta, M., Set-valued random differential equations in Banach space, *Discus. Math. Diff. Inclusions* **15**:2 (1995), 191-200.

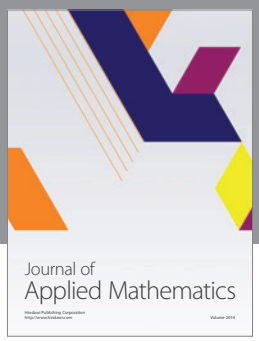
[12] Michta, M., On weak solutions of set-valued random differential equations, *Demonstratio Math.* **29**:3 (1996), 523-528.

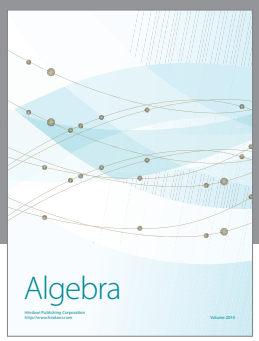
[13] Parthasarathy, K.R., *Probability Measures on Metric Spaces*, Acad. Press, New York 1967.

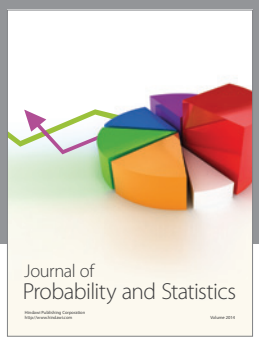
[14] Tolstonogov, A., *Differencjalny je Wkluczenija w Banachowych Prostranstwach*, Nauka, Moscow 1986 (Russian).

[15] Wachanija, N.N., Tarieladze, W.I. and Tshobanian, S.A., *Werojatnostnyje Raspredelenija w Banachowych Prostranstwach*, Nauka, Moscow 1985 (Russian).









Hindawi

Submit your manuscripts at
http://www.hindawi.com