CONTINUITY PROPERTIES OF SOLUTIONS OF MULTIVALUED EQUATIONS WITH WHITE NOISE PERTURBATION

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In the paper, we consider a set-valued stochastic equation with stochastic perturbation in a Banach space. We prove first the existence theorem and then study continuity properties of solutions.

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1. Preliminaries

Problems of existence of solutions to set-valued differential equations were studied by many (see e.g., [3, 8, 9]). In particular, random cases were considered by the author in [11, 12].

In this paper we study the set-valued stochastic equation with white noise drift:

\[ DX_t = F(t, X_t)dt + \sigma_t dw_t, t \in I, \]

\[ X_0 = U \quad P.1, \tag{I} \]

where \( F \) and \( U \) are given random set-valued mappings with values in the space \( K_c(E) \), of all nonempty, compact and convex subsets of the separable Banach space \( (E, \| \cdot \|) \), \( I = [0, T]; \ T > 0 \). We assume also that there is a predictable stochastic process \( \sigma \) with values in \( E \). Finally, \((w_t)_{t \in I}\) denotes a real Wiener process. We interpret the above equation through its integral form as

\[ X_t = U + \int_0^t F(s, X_s)ds + \int_0^t \sigma_s dw_s \quad P.1, \ t \in I. \tag{II} \]

Integrals above are Aumann’s integral of \( F \) and stochastic (Itô) integral of \( \sigma \), respectively.

The aim of this work is to study continuity properties of set-valued solutions of
First, we recall several notions needed in the sequel. In the space $K_c(E)$ we consider the Hausdorff metric $H$ (see e.g., [5, 7]): $H(A, B) = \max(\overline{H}(A, B), \overline{H}(B, A))$ for $A, B \in K_c(E)$, where $\overline{H}(A, B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|$. By $\| A \|$ we denote the distance $H(A, 0)$. It can be proved that $(K_c(E), H)$ is a Polish metric space.

By $C_I = C(I, K_c(E))$ we denote the space of all $H$-continuous set-valued mappings. In this space we consider metric $p$ of uniform convergence:

$$p(X, Y) = \sup_{0 \leq t \leq T} H(X(t), Y(t)),$$

for $X, Y \in C_I$.

Then we have a Polish metric space.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)_{t \in I}$ be a given complete filtered probability space satisfying the usual conditions. We recall the notion of a multivalued $\mathcal{F}_t$-adapted stochastic process. The family of set-valued mappings $X = (X_t)_{t \in I}$ is said to be a multivalued $\mathcal{F}_t$-adapted stochastic process if for every $t \in I$, the mapping $X_t: \Omega \rightarrow K_c(E)$ is $\mathcal{F}_t$-measurable, i.e., $\{ \omega: X_t(\omega) \in V \}$ is $\mathcal{F}_t$, for every open set $V \subset E$ (see e.g., [7]). It can be noted that $V$ can be chosen as a closed or Borel subset. If the mapping $t \mapsto X_t(\omega)$ is $H$-continuous with probability one ($P.1$) then we say it has continuous paths. In this case, the set-valued process $X$ can be thought as random element $X: \Omega \rightarrow C_I$. Let $(X_n)$ be a sequence of random elements with values in metric space $(S, \rho)$. Then we say that $X_n$ converges in probability to the random element $X: \Omega \rightarrow C_I$. Let $\sigma$, $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\zeta$, $\eta$, $\theta$, $\varphi$, $\chi$, $\psi$, $\omega$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi$, $\Omega$, $\Phi$, $\Delta$, $\Gamma$, $\Lambda$, $\Xi$, $\Psi
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\[ \Phi_t = U + \int_0^t F(s, A)ds + \int_0^t \sigma_s dw_s, \quad t \in I, \]

is \( \mathcal{F}_t \)-adapted with values in \( K_c(E) \). It is also clear that \( \Phi \) has continuous "paths".

We also assume the so-called "Kamke condition" imposed on multifunction \( F \): for every \( A_1, A_2, \ldots \in K_c(E) \) one has

\[ N\left( \bigcup_{n \geq 1} F(t, A_n) \right) \leq k(t, N\left( \bigcup_{n \geq 1} A_n \right)) \text{ with } P.1 \quad t \in I \text{ a.e.}, \]

where \( k: I \times \Omega \times R \rightarrow R_+ \) satisfies the following conditions:

\[ \begin{align*}
  a) & \quad k(t, x) \text{ is } \mathcal{F}-\text{measurable for every } (t, x) \in I \times R_+, \\
  b) & \quad k(., \omega, ) \text{ is a Kamke function (see e.g., [14]) with } P.1.
\end{align*} \]

**Definition 2:** A multivalued process \( X = (X_t)_{t \in I} \) is said to be a solution of \((I)\) if it satisfies multivalued stochastic equation \((II)\).

Let us notice that without stochastic perturbation, equation \((II)\) can be written as:

\[ D_H X_t = F(t, X_t) \quad P.1, \quad t \text{-a.e.} \]

\[ X_0 = U \quad P.1, \]

where \( D_H \) denotes the Hukuchara derivative operator ([6]) for multifunctions.

Before stating the existence theorem to equation \((II)\) let us recall its special case.

**Theorem 1:** ([11]) Let \( F \) and \( U \) be multivalued mappings satisfying conditions 1)-4) and 5), respectively. Let us also suppose that \( F \) satisfies the "Kamke condition." Then the multivalued random differential equation

\[ D_H X_t = F(t, X_t) \quad P.1, \quad t \in I \text{ a.e.} \]

\[ X_0 = U \quad P.1, \]

has at least one solution.

**Remark:** In fact, the existence of solutions to the above initial value problem is based on the fact that under these conditions there exists at least one solution to the multivalued equation \( X_t = U + \int_0^t F(s, X_s)ds \) and on well-known connection between Aumann's integral of set-valued mapping and its Hukuchara derivative via Radström Embedding Theorem (see e.g. [14]).

**Theorem 2:** Let \( E \) be a Banach space such that its dual \( E^* \) is separable. If \( F, U \) and \( \sigma \) have properties 1)-6) and \( F \) satisfies the "Kamke condition" then there exists at least one solution of the equation \((II)\).

**Proof:** Let \( \xi_t = \int_0^t \sigma_s dw_s \). Let \( X_t^\sim = X_t - \xi_t, \) where \( X_t \) is a solution of \((II)\), and \( X_t^\sim(\omega) = \{ x^\sim - \xi_t(\omega); x^\sim \in X_t(\omega) \} \). The process \( X^\sim \) satisfies the equation

\[ X_t^\sim = U + \int_0^t F^\sim(s, X_s^\sim)ds \quad P.1, \quad t \in I, \]

where \( F^\sim(s, \omega, A) = F(s, \omega, A + \xi_s(\omega)) \). The set-valued mapping \( F^\sim \) meets properties

\[ (*) \]

\[ \text{where } k: I \times \Omega \times R \rightarrow R_+ \text{ satisfies the following conditions:} \]

\[ \begin{align*}
  a) & \quad k(t, x) \text{ is } \mathcal{F}-\text{measurable for every } (t, x) \in I \times R_+, \\
  b) & \quad k(., \omega, ) \text{ is a Kamke function (see e.g., [14]) with } P.1.
\end{align*} \]
1)-4). By properties of measure of noncompactness it also satisfies (*) (cf. [1]). Hence, equation (II) has at least one solution if and only if equation (**) has one. By Theorem 1 (via Remark 1) the proof is completed.

Let us suppose now that \( \Gamma: I \times \Omega \times E \to K_c(E) \) is a given set-valued mapping. Let us set \( F(t, \omega, A) = \overline{\text{co}} \Gamma(t, \omega, A), \ A \in K_c(E) \), where \( \overline{\text{co}}B \) denotes the closed convex hull of the set \( B \). It is noteworthy to observe the connections between solutions of equation (II), with \( F = \overline{\text{co}} \Gamma \) and solutions of stochastic inclusion

\[
x_t - x_s \in \int_s^t \Gamma(u, x_u)du + \int_s^t \sigma_u dw_u \quad \text{with } P.1, \ 0 \leq s \leq t \leq T \quad (II')
\]

\( x_0 \in U \) with \( P.1 \).

We suppose that \( \Gamma \) is an integrable bounded multifunction such that:
1') \( \Gamma(t, \omega, \cdot) \) is \( H \)-continuous with \( P.1, \ t \text{-a.e.} \),
2') \( \Gamma(t, \cdot, x) \) is \( \mathcal{F}_t \)-adapted for every \( t \in I, \ x \in E \),
3') \( \Gamma(\cdot, \cdot, x) \) is measurable for every \( x \in E \),
4') \( \forall A \subset S_t(U) : \mathcal{N}(\Gamma(t, A)) \leq k(t, \mathcal{N}(A)) \ P.1, \ t \in I \),

where \( S_t(U) = U + rB(0,1) \) and \( B(0,1) \) is a closed unit ball in Banach space \( E \), centered at zero.

**Theorem 3:** Suppose that \( F \) satisfies conditions 1'-4'. If a multivalued stochastic process \( X = (X_t)_{t \in I} \) is a solution of equation (II) with \( F = \overline{\text{co}} \Gamma \) then there exists stochastic process \( x = (x_t) \) being both a solution to stochastic inclusion (II') and the selection of \( X \).

**Proof:** Similarly, as above, let \( \xi_t = \int_t^s \sigma_u dw_u, \ \Gamma^\sim(t, \omega, x) = \Gamma(t, \omega, x + \xi_t(\omega)) \) and \( F^\sim(t, \omega, A) = F(t, \omega, A + \xi_t(\omega)) \). Then \( F^\sim = \overline{\text{co}} \Gamma^\sim \). Let us notice than \( F^\sim \) also satisfies 1'-4'. Hence, by Corollary 1 [11], there exists at least one solution of equation

\[
X_t^\sim = U + \int_0^t F^\sim(s, X_s^\sim)ds \quad P.1, \ t \in I.
\]

Taking \( X = X_t + \xi \) we get a solution of equation (II'), where \( F = \overline{\text{co}} \Gamma \). Moreover, by Theorem 4 [11] there exists stochastic process, say \( x = (x_t^\sim) \), being a selection of \( X^\sim \) such that: \( x_t^\sim - x_s^\sim \in \int_s^t \Gamma^\sim(u, x_u^\sim)du \) with \( P.1, \ 0 \leq s \leq t \leq T \), and \( x_0^\sim \in U \ P.1 \).

Consequently, there exists stochastic process \( x = (x_t) \), as a selection of \( X \), such that: \( x_t = x_t^\sim - \xi_t \) with \( P.1 \). It remains to observe that \( x \) is a desired solution of inclusion (II').

3. Continuity Properties of Solutions

By \( S(I \times \Omega) \) we denote the class of “simple” multivalued processes that can be expressed by: \( X = \sum_{i=1}^n I_{D_i} C_i \), where the sets \( D_i, \ i = 1, 2, \ldots, n \) form a measurable partition of \( I \times \Omega \) and \( C_i \in K_c(E), \ i = 1, 2, \ldots, n \).

**Lemma 1:** If \( X = (X_t)_{t \in T} \) is a multivalued stochastic process with continuous
"paths" then there exists a sequence \( \{X_n\} \subseteq S(I \times \Omega) \) such that \( \forall (t, \omega) \in I \times \Omega: \lim_{n \to \infty} H(X(t, \omega), X_n(t, \omega)) = 0. \)

**Proof:** It follows directly from the fact that \( K_c(E) \) is a separable metric space and Proposition 1.9 [15].

Let \( \Lambda \) be a metric space. Let us consider the multivalued mapping \( F: I \times \Omega \times K_c(E) \times \Lambda \to K_c(E) \) such that:

A1. For every fixed \( A \in K_c(E) \) and \( \lambda \in \Lambda \), \( F(, , A, \lambda) \) is a measurable and integrably bounded multifunction.

A2. The mapping \( F(t, \omega, \lambda) \) is with P.1 uniformly continuous with respect to \( t \in I \) and \( \lambda \in \Lambda \).

**Definition 2:** A multifunction \( F \) (with properties A1 and A2) is said to be integrably continuous in probability (icp) at \( \lambda_0 \in \Lambda \) with respect to a family \( \mathcal{C} \subseteq K_c(E) (\mathcal{C} \neq \emptyset) \) if

\[
\forall C \in \mathcal{C}, \forall t \in I: \int_0^t F(s, C, \lambda) ds \to \int_0^t F(s, C, \lambda_0) ds
\]

for \( \lambda \to \lambda_0 \).

The results presented below give characterizations of icp multifunctions. We use them to obtain the main theorem.

**Lemma 2:** If \( F \) is an icp multifunction at \( \lambda_0 \) with respect to \( \mathcal{C} \) then for every \( C \in \mathcal{C} \) one has: \( \int_0^t F(s, C, \lambda_n) ds \to \int_0^t F(s, C, \lambda_0) ds \) P.1 uniformly in \( t \in I \), for some sequence \( (\lambda_n) \) convergent to \( \lambda_0 \).

**Proof:** Let \( D \) be a set of rationals in \( I \), \( D = \{t_1, t_2, \ldots\} \) and let \( (\lambda_n) \) be an arbitrary sequence of elements of \( \Lambda \) that converges to \( \lambda_0 \). Fix \( C \in \mathcal{C} \). Then for \( t_1 \in D \), there exist a sequence \( (\lambda_n(t_1))_n \), convergent to \( \lambda_0 \) and set \( \Omega(t_1) \subseteq \Omega \), \( P(\Omega(t_1)) = 1 \), such that

\[
\forall \omega \in \Omega(t_1): H(\int_0^{t_1} F(s, \omega, C, \lambda_n(t_1)) ds, \int_0^{t_1} F(s, \omega, C, \lambda_0) ds) \to 0, \text{ for } n \to \infty.
\]

Similarly, for \( t_2 \in D \) we can find a sequence \( (\lambda_n(t_2))_n \) being a subsequence of \( (\lambda_n(t_1))_n \) and \( \Omega(t_2) \subseteq \Omega \), \( P(\Omega(t_2)) = 1 \) for which a similar convergence holds. Continuing this selection process we obtain the infinite table

\[
\begin{array}{cccccccc}
\lambda_1(t_1) & \lambda_2(t_1) & \cdots & \lambda_n(t_1) & \cdots \\
\lambda_1(t_2) & \lambda_2(t_2) & \cdots & \lambda_n(t_2) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
\lambda_1(t_n) & \lambda_2(t_n) & \cdots & \lambda_n(t_n) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
\end{array}
\]

By diagonal selection we can find a sequence \( (\lambda_n) \) being a subsequence of each row of table (1) that converges to \( \lambda_0 \). Let \( \Omega_0 = \bigcap \{\Omega(n); n \geq 1\} \). Then \( P(\Omega_0) = 1 \). Moreover,

\[
\forall \omega \in \Omega_0, \forall t \in D: H(\int_0^t F(s, \omega, C, \lambda_n) ds, \int_0^t F(s, \omega, C, \lambda_0) ds) \to 0, \text{ for } n \to \infty.
\]
Since the set-valued process \( J_t = \int_0^t F(s, C, \lambda) ds, \quad t \in I \) has with \( P.1 \) uniformly continuous “paths”, we can find \( \Omega_0, P(\Omega_0) = 1 \) such that

\[
\forall \omega \in \Omega_0: \sup_{t \in I} H(\int_0^t F(s, \omega, C, \lambda_n) ds, \int_0^t F(s, \omega, C, \lambda_0) ds) \to 0, \text{ if } n \to \infty.
\]

This completes the proof.

By \( \mathcal{B}_I \) we denote the \( \sigma \)-field of Borel subsets of \( I \).

**Lemma 3:** A multifunction \( F \) is icp at \( \lambda_0 \) with respect to family \( \mathcal{C} \) if and only if:

\[
H(\int_B F(s, C, \lambda_n') ds, \int_B F(s, C, \lambda_0) ds) \to 0 \quad P.1
\]

as \( n \to \infty \), for every \( B \in \mathcal{B}_I \).

**Proof:** Fix \( C \in \mathcal{C} \) and let \( (\lambda_n) \) be an arbitrary sequence convergent to \( \lambda_0 \). Then by Lemma 2, we can find its subsequence \( (\lambda_{n_k}') \) and \( \mathcal{F} \in \mathcal{B}_I \) such that for every \( \omega \in \Omega_0 \) and \( 0 \leq s < t \leq T \),

\[
\int_s^t F(u, \omega, C, \lambda_{n_k}') du \to \int_s^t F(u, \omega, C, \lambda_0) du, \text{ as } n \to \infty.
\]

Let \( \mathcal{A} = \{ [s,t): 0 < s < t \leq T \} \) and

\[
\mathcal{A} = \left\{ \bigcup_{i=1}^n R_i : R_i \in \mathcal{A}, R_i \cap R_j = \emptyset, i \neq j, i, j = 1,2,\ldots,n, n \geq 1 \right\}.
\]

Since \( \sigma(\mathcal{A}) = \sigma(\mathcal{F}) = \mathcal{B}_I \) and \( \mathcal{A} \) is a ring of subsets of \( I \), then for every \( \epsilon > 0 \) and \( B \in \mathcal{B}_I \), there exists \( A \in \mathcal{A} \) such that \( |B \Delta A| < \epsilon \) (c.f. e.g., Th. 11.4 [2]), where \( |\cdot| \) is Lebesgue measure and \( B \Delta A = (B \setminus A) \cup (A \setminus B) \). By integrably boundedness of \( F \) we get:

\[
H(\int_B F(s, C, \lambda_{n_k}') ds, \int_B F(s, C, \lambda_0) ds) \leq H(\int_A F(s, C, \lambda_{n_k}') ds, \int_A F(s, C, \lambda_0) ds) + \int_{B \Delta A} m(s, \omega) ds, \text{ for every } A \in \mathcal{A}.
\]

Then by (3), \( \limsup_{n \to \infty} H(\int_B F(s, C, \lambda_{n_k}') ds, \int_B F(s, C, \lambda_0) ds) \leq \int_{B \Delta A} m(s, \omega) ds \).

Taking \( A \) sufficiently close to \( B \) we claim (2). The converse is obvious.

**Lemma 4:** A multifunction \( F \) is icp at \( \lambda_0 \) with respect to \( K_c(E) \) if and only if \( F \) is icp at \( \lambda_0 \) with respect to \( S(I \times \Omega) \).

**Proof:** Let us assume that \( F \) is icp with respect to \( K_c(E) \). Let \( X \in S(I \times \Omega) \). Then there exist \( C_1, C_2, \ldots, C_r \in K_c(E) \) and a measurable partition \( \{D_1, D_2, \ldots, D_r\} \) of space \( I \times \Omega \) such that \( X = \sum_{i=1}^r D_i C_i \). Take \( C_1 \) and \( \lambda_n \) to be an arbitrary sequence convergent to \( \lambda_0 \). Next let \( (\lambda_{n_k}) \) be any subsequence of \( (\lambda_n) \). By Lemma 3 there exists a sequence \( (\lambda_{n_k}) \) being a subsequence of \( (\lambda_{n_k}) \) and a subset \( \Omega_{0,1} \subseteq \Omega; \)
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Let \( \Omega \subseteq \Omega_{0,1} \) be such that:

\[
\forall \omega \in \Omega_{0,1}, \forall B \in B_f: \lim_{n \to \infty} H\left( \int_B F(s, \omega, C_1, \lambda_{n,1}) ds, \int_B F(s, \omega, C_1, \lambda_0) ds \right) = 0.
\]

Similarly, for \( C_2 \) we can extract a subsequence \( (\lambda'_{n,2}) \) from \( (\lambda'_{n,1}) \) and \( \Omega_{0,2} \subseteq \Omega \) \( P(\Omega_{0,2}) = 1 \), with the desired property, and so on. Thus we obtain a sequence \( (\lambda'_{n,r}) \) which is a subsequence of \( (\lambda'_{n,i}), i = 1, 2, \ldots, r - 1 \) and \( \Omega_{0,r} \), \( P(\Omega_{0,r}) = 1 \), such that

\[
\forall \omega \in \Omega_{0,r}, \forall B \in B_f: \lim_{n \to \infty} \left( \int_B F(s, \omega, C_r, \lambda'_{n,r}) ds, \int_B F(s, \omega, C_r, \lambda_0) ds \right) = 0.
\]

Let \( \Omega_0 = \bigcap_{1 \leq i \leq r} \Omega_{0,i} \). For any \( A \in B_f \oplus \mathcal{F} \) and \( \omega \in \Omega \), we define the set \( (A)_\omega = \{ t \in I: (t, \omega) \in A \} \). Then \( (A)_\omega \in B_f \). Let \( \omega \in \Omega_0 \). Then \( X(\cdot, \omega) = \sum_{i=1}^r I_{(D_i)_\omega}(\cdot) C_1 \) and \( \{(D_i)_\omega; i = 1, 2, \ldots, r\} \) is measurable partition of \( I \). Hence, the following inequality holds:

\[
H\left( \int_0^t F(s, \omega, X_s, \lambda'_{n,r}) ds, \int_0^t F(s, \omega, X_s, \lambda_0) ds \right) 
\leq \sum_{i=1}^r H\left( \int_{(D_i)_\omega \cap [0,t]} F(s, \omega, C_i, \lambda'_{n,r}) ds, \int_{(D_i)_\omega \cap [0,t]} F(s, \omega, C_i, \lambda_0) ds \right).
\]

It remains to observe that each term of the above sum converges to zero as \( n \) tends to infinity.

The converse statement is obvious. It is enough to take \( X: = I_{I \times \Omega} C \), for \( C \in K_c(E) \). This completes the proof.

By \( X^\lambda \) we denote a multivalued process being the solution of the equation

\[
X_t = U + \int_0^t F(s, X_s, \lambda) ds + \int_0^t \sigma_s dw_s \text{ P.a.s., } t \in I, \lambda \in A.
\]  \( (III) \)

**Theorem 3:** Let us assume that \( F \) is an icp set-valued mapping at \( \lambda_0 \in A \) with respect to \( K_c(E) \). Then,

i) if \( X^{\lambda P} \to X^{\lambda_0} \) then \( \forall t \in I: \int_0^t F(s, X^\lambda_s, \lambda) ds \to \int_0^t F(s, X^{\lambda_0}_s, \lambda_0) ds, \lambda \to \lambda_0 \),

ii) if for every \( A_1, A_2, \ldots \in K_c(E) \) and \( (\lambda_n); \lambda_n \to \lambda_0 \) we have

\[
\mathcal{N}\left( \bigcup_{n \geq 1} F(t, A_n, \lambda_n) \right) \leq k\left( t, \mathcal{N}\left( \bigcup_{n \geq 1} A_n \right) \right) \text{ with P.a.s., then } X^{\lambda P} \to X^{\lambda_0}.
\]

**Proof:** (i) Let \( (\lambda_n) \) be an arbitrary sequence convergent to \( \lambda_0 \). Then its every subsequence contains a further subsequence, say, \( (\lambda'_{n,i}) \), such that \( X^{\lambda'_{n,i}} \to X^{\lambda_0} \) with P.a.s. in \( C_f \). Take \( \omega \) from an appropriate set (for which this convergence holds). By condition A2, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( H(F(t, C, \lambda'_{n,i}), F(t, D, \lambda'_{n,i})) < \epsilon/4T \), for \( n \in N, C, D \in K_c(E) \) whenever \( H(C, D) < \delta \).

Let \( V_0 \) be an open neighborhood for \( \lambda_0 \) such that
Let \((X_k^0)_k\) be a sequence of simple multifunctions (Lemma 1) convergent to \(X^0\) for every \(t \in I\) and \(\omega \in \Omega\). Then for every \(t \in I\) and \(\lambda \in \Lambda\), we have:

\[
\lim_k H(F(t, \omega, X_k^0(t, \omega), \lambda), F(t, \omega, X^0(t, \omega), \lambda)) = 0 \quad P.1.
\]

Next by the Lebesgue Dominated Convergence Theorem (via integrably boundedness of \(F\)) we obtain that

\[
\int_0^T H(F(s, X_k^0(s), \lambda), F(s, X_s^0, \lambda))ds \to 0 \quad P.1
\]

for every \(\lambda \in \Lambda\). Hence by (4), after standard calculation we see that

\[
H\left(\int_0^t F(s, \omega, X_s^0(\omega), \lambda_n')ds, \int_0^t F(s, \omega, X_s^0(\omega), \lambda_0)ds\right) \leq \left(\frac{3}{4}\right)\epsilon
\]

for \(t \in I\), \(k\) sufficiently large and \(\omega\) taken from an appropriate set of probability one.

By Lemma 4, multifunction \(F\) is icp at \(\lambda_0\) with respect to \(S(I \times \Omega)\). Hence there exists a sequence \((\lambda_n'')\) being a subsequence of \((\lambda_n')\), a subset of \(\Omega\) of measure one such that for every \(\epsilon > 0\) and appropriate \(\omega\) we can find an open neighborhood \(V_1\) of \(\lambda_0\) with

\[
H\left(\int_0^t F(s, \omega, X_s^0(s, \omega), \lambda_n'')ds, \int_0^t F(s, \omega, X_s^0(s, \omega), \lambda_0)ds\right) < \epsilon/4,
\]

for \(t \in I\) and \(\lambda_n'' \in V_1\). Therefore, taking \(n''\) sufficiently large and \(\lambda_n'' \in V_0 \cap V_1\) we have:

\[
H\left(\int_0^t F(s, \omega, X_s^0(\omega), \lambda_n'')ds, \int_0^t F(s, \omega, X_s^0(\omega), \lambda_0)ds\right) < \epsilon
\]

for \(t \in I\). This completes the proof of part (i).

Proof of part (ii).

Let \((\lambda_n)\) be a sequence convergent to \(\lambda_0\). Consider its arbitrary subsequence, denoted for simplicity by the same symbol. We define the multivalued mapping \(\Pi: \Omega \to \mathbb{2}^I\) by

\[
\Pi(\omega) = \{X \in C_I: X^\lambda_n \to X \text{ in } C_I \text{ for some sequence } (\lambda_n'), (\lambda_n') \subseteq (\lambda_n)\}.
\]

By the assumption of the integrably boundedness of \(F\) it follows that

\[
\forall n \in N, \forall 0 \leq s \leq t \leq T:\ H(X_t^\lambda_n, X_s^\lambda_n) \leq \int_s^t m(u)du \text{ with } P.1.
\]
Thus the sequence \((X^n)\) is equicontinuous in \(C_I\) with \(P.1\). Similarly, (compare [14]) by assumption \((iii)\), it can be proved that \(\bigcup_{n \geq 1} \{X^n\} \) is a relatively compact subset of \(E\), for every \(t \in I\) with \(P.1\). Thus, by Ascoli Theorem we claim that the sequence \((X^n)\) is relatively compact (with \(P.1\)). Hence the multifunction \(\Pi \neq \emptyset P.1\) and has closed values. Moreover, we claim that \(\Pi\) is measurable. To see this, let \(\Omega_0 = \{w: \Pi(w)\text{ is closed subset of } C_I\}\). For \(X \in C_I\) we consider a mapping \(\Omega_0 \ni w \rightarrow \text{Dist}(X, \Pi(w))\), where \(\text{Dist}(X, \Pi(w)) = \inf_{Y \in \Pi(w)} \rho(X, Y)\). Fix \(r > 0\).

Then \(\{w: \text{Dist}(X, \Pi(w)) < r\} = \{w: \exists Y \in \Pi(w): Y \in B_r(X)\}\), where \(B_r(X) = \{Y \in C_I: \rho(X, Y) < r\}\). Let \(\{t_k\}\) be a sequence of rationals in \(I\). Then we get:

\[
\{w: \text{dist}(X, \Pi(w)) < r\} = \{w: \Pi(w) \cap B_r(X) \neq \emptyset\}
\]

Since \(X_{t_k}^{n,j}\) is an \(\sigma\)-measurable multifunction then the last set above belongs to \(\sigma\)-field \(\mathcal{F}\), which yields the \(\mathcal{F}\)-measurability of \(\Pi\) (see, e.g. [4]). Thus, by Kuratowski and Ryll-Nardzewski Selection Theorem [10], there exists a measurable selection \(\hat{X}\) of \(\Pi; \hat{X} \in \Pi P.1\). The definition of \(\Pi\) implies then that \(X^{n} \rightarrow \hat{X} P.1\) in \(C_I\), for some sequence \((\lambda^i_n)\) tending to \(\lambda_0\) and this yields convergence in probability in \(C_I\). Finally, we claim that \(\hat{X}\) is a solution of \((III)\). Indeed, let us notice that

\[
H(X_{t_k}, U + \int_0^t F(s, X_s^{n_j}, \lambda_0)ds + \int \sigma_s dw_s) \\
\leq H(X_{t_k}^{n_j}, X_t^{n_j}) + H(\int_0^t F(s, X_s^{n_j}, \lambda_0)ds, \int_0^t F(s, X_s^{n_j}, \lambda_0)ds),
\]

with \(P.1\) and for \(t \in I\).

Since the first term above converges to zero then by \((i)\) the second term converges to zero as well. This completes the proof.

References


