MONOTONE ITERATIONS FOR DIFFERENTIAL EQUATIONS WITH A PARAMETER

TADEUSZ JANKOWSKI
Technical University of Gdansk, Department of Numerical Analysis
Gdansk, POLAND

V. LAKSHMIKANTHAM
Florida Institute of Technology, Applied Mathematics Program
Melbourne, FL 32901 USA

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Consider the problem
\[ \begin{align*}
  y'(t) &= f(t, y(t), \lambda), \\
  y(0) &= k_0, \\
  G(y, \lambda) &= 0.
\end{align*} \]

Employing the method of upper and lower solutions and the monotone iterative technique, existence of extremal solutions for the above equation are proved.

**Key words:** Monotone Iterations, Differential Equations, Monotone Iterative Technique.

**AMS subject classifications:** 34A45, 34B99.

1. Preliminaries

Consider the following differential equation
\[ x'(t) = f(t, x(t), \lambda), \quad t \in J = [0, b] \]
with the boundary conditions
\[ x(0) = k_0, \quad x(b) = k_1, \]
where \( f \in C(J \times R \times R, R) \) and \( k_0, k_1 \in R \) are given. The corresponding solution of (1) yields a pair of \((x, \lambda) \in C^1(J, R) \times R\) for which problem (1) is satisfied. Problem (1) is called a problem with a parameter.

Conditions on \( f \) which guarantee the existence of solutions to (1) are important analysis theorems. Such theorems can be formulated under the assumption that \( f \) satisfies the Lipschitz condition with respect to the last two variables with suitable
Lipschitz constants or Lipschitz functions [1-3, 5].

This paper applies the method of lower and upper solutions for proving existence results [4]. Using this technique, we construct monotone sequences, giving sufficient conditions under which they are convergent. Moreover, this method gives a problem solution in a closed set.

Note that \( x(b) \) in condition (1b) may appear in a nonlinear way, so it is a reason that we consider the following problem in the place of (1):

\[
\begin{aligned}
  y'(t) &= f(t, y(t), \lambda), \\
  y(0) &= k_0, \\
  G(y, \lambda) &= 0.
\end{aligned}
\]  

where \( f \in C(J \times R \times R, R), G \in C(R \times R, R) \).

2. Main Results

A pair \((v, \alpha) \in C^1(J, R) \times R\) is said to be a lower solution of (2) if:

\[
\begin{aligned}
  v'(t) &\leq f(t, v(t), \alpha), \\
  v(0) &\leq k_0, \\
  0 &\leq G(v, \alpha),
\end{aligned}
\]

and an upper solution of (2) if the inequalities are reversed.

**Theorem 1**: Assume that \( f \in C(J \times R \times R, R), G \in C(R \times R, R), \) and:

1° \( y_0, z_0 \in C^1(J, R), \lambda_0, \gamma_0 \in R, \) such that \((y_0, \lambda_0), (z_0, \gamma_0)\) are lower and upper solutions of problem (2) such that \( y_0(t) \leq z_0(t), t \in J \) and, \( \lambda_0 \leq \gamma_0 \);

2° \( f \) is nondecreasing with respect to the last two variables;

3° \( G \) is nondecreasing with respect to the first variable;

4° \( G(y, \lambda) - G(y, \beta) \leq N(\beta - \lambda) \) for \( y_0(t) \leq y(t) \leq z_0(t), t \in J, \lambda_0 \leq \lambda \leq \beta \leq \gamma_0 \) with \( N \geq 0 \).

Then there exist monotone sequences \( \{y_n, \lambda_n\}, \{z_n, \gamma_n\} \) such that \( y_n(t) \to y(t), z_n(t) \to z(t), t \in J; \lambda_n \to \lambda, \gamma_n \to \gamma \) as \( n \to \infty \); and this convergence is uniformly and monotonically on \( J \). Moreover, \((y, \lambda), (z, \gamma)\) are minimal and maximal solutions of problem (2), respectively.

**Proof**: From the above assumptions, it is known that:

\[
\begin{aligned}
  y_0(t) &\leq f(t, y_0(t), \lambda_0), \\
  y_0(0) &\leq k_0, \\
  0 &\leq G(y_0, \lambda_0),
\end{aligned}
\]

and \( y_0(t) \leq z_0(t), t \in J, \lambda_0 \leq \gamma_0 \). Let \((y_1, \lambda_1), (z_1, \gamma_1)\) be the solutions of:

\[
\begin{aligned}
  y'_1(t) &= f(t, y_0(t), \lambda_0), \\
  y_1(0) &= k_0, \\
  0 &= G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0),
\end{aligned}
\]

and

\[
\begin{aligned}
  z'_1(t) &= f(t, z_0(t), \gamma_0), \\
  z_1(0) &= k_0, \\
  0 &= G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0),
\end{aligned}
\]
respectively.

Put $p = \lambda_0 - \lambda_1$, so:

$$0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) \geq - N(\lambda_1 - \lambda_0) = N_p,$$

thus $p \leq 0$ and $\lambda_0 \leq \lambda_1$. Now let $p = \lambda_1 - \gamma_1$. In view of $3^0$ and $4^0$, we have:

$$0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) = G(y_0, \lambda_0) - G(z_0, \gamma_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0)
\leq G(z_0, \lambda_0) - G(z_0, \gamma_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0)
\leq N(\gamma_0 - \lambda_0) - N(\lambda_1 - \lambda_0) + N(\gamma_1 - \gamma_0) = - Np.$$

Hence $\lambda_1 \leq \gamma_1$. Set $p = \gamma_1 - \gamma_0$, so that:

$$0 = G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0) \leq - N(\gamma_1 - \gamma_0) = - Np,$$

and thus $\gamma_1 \leq \gamma_0$. As a result, we have:

$$\lambda_0 \leq \lambda_1 \leq \gamma_1 \leq \gamma_0.$$

We shall show that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J. \quad (3)$$

Let $p(t) = y_0(t) - y_1(t), \quad t \in J$, so:

$$p'(t) = y_0'(t) - y_1'(t) \leq f(t, y_0(t), \lambda_0) - f(t, y_0(t), \lambda_0) = 0,$$

and $p(0) = y_0(0) - y_1(0) \leq 0$. This shows that $p(t) \leq 0, \quad t \in J$. Therefore $y_0(t) \leq y_1(t), \quad t \in J$. Put $p(t) = y_1(t) - z_1(t), \quad t \in J$. In view of $2^0$, we have

$$p'(t) = y_1'(t) - z_1'(t) = f(t, y_0(t), \lambda_0) - f(t, z_0(t), \gamma_0)
\leq f(t, z_0(t), \gamma_0) - f(t, z_0(t), \gamma_0) = 0,$$

and $p(0) = 0$, so $p(t) \leq 0, \quad t \in J$, and $y_1(t) \leq z_1(t), \quad t \in J$. Put $p(t) = z_1(t) - z_0(t), \quad t \in J$. We obtain:

$$p'(t) = z_1'(t) - z_0'(t) \leq f(t, z_0(t), \gamma_0) - f(t, z_0(t), \gamma_0) = 0,$$

so $p(t) \leq 0, \quad t \in J$, and hence $z_1(t) \leq z_0(t), \quad t \in J$. This shows that $(3)$ is satisfied.

Note that:

$$y_1'(t) - f(t, y_0(t), \lambda_0) \leq f(t, y_1(t), \lambda_1), y_1(0) = k_0,$$

and

$$z_1'(t) - f(t, z_0(t), \gamma_0) \geq f(t, z_1(t), \gamma_1), z_1(0) = k_0.$$

Moreover, in view of $3^0$ and $4^0$, we have:

$$0 = G(y_0, \lambda_0) - N(\lambda_1 - \lambda_0) \leq G(y_1, \lambda_0) - N(\lambda_1 - \lambda_0)
= G(y_1, \lambda_0) - G(y_1, \lambda_1) + G(y_1, \lambda_1) - N(\lambda_1 - \lambda_0)
\leq N(\lambda_1 - \lambda_0) + G(y_1, \lambda_1) - N(\lambda_1 - \lambda_0) = G(y_1, \lambda_1),$$

and

$$0 = G(z_0, \gamma_0) - N(\gamma_1 - \gamma_0) \geq G(z_1, \gamma_0) - N(\gamma_1 - \gamma_0)
= G(z_1, \gamma_0) - G(z_1, \gamma_1) + G(z_1, \gamma_1) - N(\gamma_1 - \gamma_0).$$
Consequently, \((y_1, \lambda_1), (z_1, \gamma_1)\) are lower and upper solutions of problem (2).

Let us assume that
\[
\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{k-1} \leq \lambda_k \leq \gamma_{k-1} \leq \ldots \leq \gamma_1 \leq \gamma_0,
\]
\[
y_0(t) \leq y_1(t) \leq \ldots \leq y_{k-1}(t) \leq y_k(t) \leq z_k(t) \leq z_{k-1}(t) \leq \ldots \leq z_1(t) \leq z_0(t),
\]
\[t \in J\]

and
\[
\begin{cases}
  y_k'(t) \leq f(t, y_k(t), \lambda_k), & y_k(0) = k_0, \\
  0 \leq G(y_k, \lambda_k),
\end{cases}
\]
\[
\begin{cases}
  z_k'(t) \geq f(t, z_k(t), \gamma_k), & z_k(0) = k_0, \\
  0 \geq G(z_k, \gamma_k)
\end{cases}
\]

for some \(k > 1\). We shall prove that:

\[
\begin{cases}
  \lambda_k \leq \lambda_{k+1} \leq \gamma_{k+1} \leq \gamma_k, \\
  y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J,
\end{cases}
\]

(4)

and
\[
\begin{cases}
  y_{k+1}'(t) \leq f(t, y_{k+1}(t), \lambda_{k+1}), & y_{k+1}(0) = k_0, \\
  0 \leq G(y_{k+1}, \lambda_{k+1}),
\end{cases}
\]
\[
\begin{cases}
  z_{k+1}'(t) \geq f(t, z_{k+1}(t), \gamma_{k+1}), & z_{k+1}(0) = k_0, \\
  0 \geq G(z_{k+1}, \gamma_{k+1}),
\end{cases}
\]

where
\[
\begin{cases}
  y_{k+1}'(t) = f(t, y_k(t), \lambda_k), & y_{k+1}(0) = k_0, \\
  0 = G(y_k, \lambda_k) - N(\lambda_{k+1} - \lambda_k),
\end{cases}
\]
\[
\begin{cases}
  z_{k+1}'(t) = f(t, z_k(t), \gamma_k), & z_{k+1}(0) = k_0, \\
  0 = G(z_k, \gamma_k) - N(\gamma_{k+1} - \gamma_k).
\end{cases}
\]

Put \(p = \lambda_k - \lambda_{k+1}\), so:
\[
0 = G(y_k, \lambda_k) - N(\lambda_{k+1} - \lambda_k) \geq - N(\lambda_{k+1} - \lambda_k) = Np,
\]
and hence \(\lambda_k \leq \lambda_{k+1}\). Let \(p = \lambda_{k+1} - \gamma_{k+1}\). In view of 3° and 4°, we see that:
\[
0 = G(y_k, \lambda_k) - N(\lambda_{k+1} - \lambda_k)
\]
\[
= G(y_k, \lambda_k) - G(z_k, \gamma_k) - N(\lambda_{k+1} - \lambda_k) + N(\gamma_{k+1}, \gamma_k)
\]
\[
\leq G(z_k, \lambda_k) - G(z_k, \gamma_k) - N(\lambda_{k+1} - \lambda_k) + N(\gamma_{k+1}, \gamma_k)
\]
\[
\leq N(\gamma_k - \lambda_k) - N(\lambda_{k+1} - \lambda_k) + N(\gamma_{k+1} - \gamma_k) = - Np.
\]
Hence we have \( \lambda_{k+1} \leq \gamma_{k+1} \). Now, let \( p = \gamma_{k+1} - \gamma_k \). Then:

\[
0 = G(z_k, \gamma_k) - N(\gamma_{k+1} - \gamma_k) \leq -Np,
\]
so \( \gamma_{k+1} \leq \gamma_k \), which shows that the first inequality of (4) is satisfied.

As before, we set \( p(t) = y_k(t) - y_{k+1}(t) \), \( t \in J \). Then:

\[
p'(t) = y'_k(t) - y'_{k+1}(t) \leq f(t, y_k(t), \lambda_k) - f(t, y_{k+1}(t), \lambda_k) = 0,
\]
and \( p(0) = 0 \), so \( y_k(t) \leq y_{k+1}(t) \), \( t \in J \). We observe that for \( p(t) = y_{k+1}(t) - z_{k+1}(t) \), \( t \in J \), we have

\[
p'(t) = y'_{k+1}(t) - z'_{k+1}(t) - f(t, y_k(t), \lambda_k) + f(t, z_{k+1}(t), \gamma_k) \\
\leq f(t, z_k(t), \gamma_k) - f(t, z_{k+1}(t), \gamma_k) = 0
\]
which proves that \( y_{k+1}(t) \leq z_{k+1}(t) \), \( t \in J \). Put \( p(t) = z_{k+1}(t) - z_k(t) \), \( t \in J \). Then we have:

\[
p'(t) = z'_{k+1}(t) - z'_k(t) \leq f(t, z_k(t), \gamma_k) - f(t, z_{k+1}(t), \gamma_k) = 0,
\]
so \( z_{k+1}(t) \leq z_k(t) \), \( t \in J \). Therefore:

\[
y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t) \quad t \in J.
\]

It is simple to show that \((y_{k+1}, \lambda_{k+1}), (z_{k+1}, \gamma_{k+1})\) are lower and upper solutions of problem (2).

Hence, by induction, we have:

\[
\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \gamma_n \leq \ldots \leq \gamma_1 \leq \gamma_0,
\]

\[
y_0(t) \leq y_1(t) \leq \ldots \leq y_n(t) \leq z_n(t) \leq \ldots \leq z_1(t) \leq z_0(t) \quad t \in J
\]
for all \( n \). Employing standard techniques [4], it can be shown that the sequences \( \{y_n, \lambda_n\}, \{z_n, \gamma_n\} \) converge uniformly and monotonically to \((y, \lambda), (z, \gamma)\), respectively. Indeed, \((y, \lambda)\) and \((z, \gamma)\) are solutions of problem (2) in view of the continuity of \( f \) and \( G \), and the definitions of the above sequences.

We have to show that if \((u, \beta)\) is any solution of problem (2) such that:

\[
y_0(t) \leq u(t) \leq z_0(t) \quad t \in J, \quad \lambda_0 \leq \beta \leq \gamma_0,
\]
then:

\[
y_0(t) \leq y(t) \leq u(t) \leq z(t) \leq z_0(t) \quad t \in J, \quad \lambda_0 \leq \lambda \leq \beta \leq \gamma \leq \gamma_0.
\]

To show this, we suppose that:

\[
y_k(t) \leq u(t) \leq z_k(t) \quad t \in J, \quad \lambda_k \leq \beta \leq \gamma_k
\]
for some \( k \). Put \( \beta = \lambda_{k+1} - \beta \). Then, in view of 3° and 4°, we have

\[
0 = G(y_k, \lambda_k) - N(\lambda_{k+1} - \lambda_k) \leq G(u, \lambda_k) - N(\lambda_{k+1} - \lambda_k) \\
= G(u, \lambda_k) - G(u, \beta) - N(\lambda_{k+1} - \lambda_k) \\
\leq N(\beta - \lambda_k) - N(\lambda_{k+1} - \lambda_k) = -Np,
\]
so \( p \leq 0 \), and hence \( \lambda_{k+1} \leq \beta \). Let \( p = \beta - \gamma_{k+1} \). Then we obtain:

\[
0 = G(u, \beta) \leq G(z_k, \beta) = G(z_k, \beta) - G(z_k, \gamma_k) + N(\gamma_{k+1} - \gamma_k)
\]
\[ N(T_k) + N(T_k + 1) \geq N_p, \]
and hence \( p \leq 0 \), so \( \beta \leq \gamma_{k+1} \). This shows that:
\[ \lambda_{k+1} \leq \beta \leq \gamma_{k+1}. \]

As before, we set \( p(t) = y_{k+1}(t) - u(t), t \in J \). In view of 2°, we obtain:
\begin{align*}
p'(t) &= y'_{k+1} - u'(t) = f(t, y_k(t), \lambda_k) - f(t, u(t), \beta) \\
&\leq f(t, u(t), \beta) - f(t, u(t), \beta) = 0;
\end{align*}

hence \( p(t) \leq 0, t \in J \), and \( y_{k+1}(t) \leq u(t), t \in J \). Now let \( p(t) = u(t) - z_{k+1}(t), t \in J \). We see that:
\begin{align*}
p'(t) &= u'(t) - z'_{k+1}(t) = f(t, u(t), \beta) - f(t, z_k(t), \gamma_k) \\
&\leq f(t, z_k(t), \gamma_k) - f(t, z_k(t), \gamma_k) = 0,
\end{align*}

and \( p(t) \leq 0, t \in J \), so \( u(t) \leq z_{k+1}(t), t \in J \). This shows that:
\[ y_{k+1}(t) \leq u(t) \leq z_{k+1}(t), t \in J. \]

By induction, this proves that the inequalities:
\[ y_n(t) \leq u(t) \leq z_n(t), t \in J, \text{ and } \lambda_n \leq \beta \leq \gamma_n \]
are satisfied for all \( n \). Taking the limit as \( n \to \infty \), we conclude that:
\[ y(t) \leq u(t) \leq z(t), t \in J, \text{ and } \lambda \leq \beta \leq \gamma. \]

Therefore, \( (y, \lambda), (z, \gamma) \) are minimal and maximal solutions of (2). The proof is complete.

References

[5] Pomentale, T., A constructive theorem of existence and uniqueness for the problem \( y' = f(x, y, \lambda), y(\alpha), \alpha, y(\beta) = \beta \), *Z. Angew. Math. Mech.* 56 (1976), 387-388.
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