THE FAILURE RATE IN RELIABILITY. NUMERICAL TREATMENT

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This paper contains approximations and bounds of failure rate functions of redundant systems. We illustrate properties of these bounds and their accuracy.

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AMS subject classifications: 60K10, 60K15, 60K20.

1. Introduction

In this paper, we consider various examples to gain a more precise understanding of the constructions and estimates developed in the first part of this work [3]. These examples are chosen because of their importance in reliability area. Our purpose is two-fold: (1) to illustrate real accuracy of the proposed bounds and depict their domain; (2) to show the simplicity of corresponding calculations for their immediate applications. We suppose (perhaps, self-sufficiently) that the reader is familiar with the results of [3]. Moreover, we will often refer to some relations and assertions from [3]. In this case, if we refer to Corollary 5.4 proved in [3], then we will write Corollary I.5.4. Similarly, if we refer to equation (5.1) from [3], then we will write (I.5.1). Of course, similar agreement remains true for other references from [3] as well. In addition, we preserve some basic notations from [3].

Recall that the principal object of our investigation is a random process \( (\eta_t) \) describing the dynamics of a system; its state space \( E \) is partitioned into two subsets: \( E = \mathcal{M} \cup \mathcal{P} \), where \( \mathcal{M} \) is treated as a subset of “good” (or operating) states and \( \mathcal{P} \) as

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a subset of “bad” (or failed) states. Let $\sigma = \inf\{s; \eta_s \in \mathcal{P}\}$ be the first break-down time of the system. The failure rate function for process $(\eta_s)$ is defined as

$$\lambda(t) = \lim_{\Delta \to 0} \frac{P(\sigma \leq t + \Delta \mid \sigma > t)}{\Delta} \tag{1.1}$$

and the Vesely failure rate as

$$\lambda_V(t) = \lim_{\Delta \to 0} \frac{P(\eta_t + \Delta \in \mathcal{P} \mid \eta_t \in \mathcal{M})}{\Delta}. \tag{1.2}$$

In Proposition 1.3.3 we introduced a quantity $\lambda^0$ which can be regarded as the failure intensity provided that the states of $(\eta_s)$ follow the stationary distribution of an auxiliary process $(\eta^0_s)$ (see (1.3.9)). We are interested in limiting values $\lambda(\infty)$ and $\lambda_V(\infty)$ of the above functions. More precisely, we are looking for accuracy estimates in approximations of $\lambda(\infty)$ by $\lambda_V(\infty)$ and by $\lambda^0$. It turned out that, in many cases $\lambda(\infty) \leq \lambda^0$ (see Proposition 1.3.3). Hence, the following accuracy estimates are of interest:

$$\rho_V = \frac{\lambda_V(\infty) - \lambda(\infty)}{\lambda(\infty)}, \quad \rho^0 = \frac{\lambda^0 - \lambda(\infty)}{\lambda(\infty)}. \tag{1.3}$$

For the case of special interest, when $(\eta_s)$ is a Markov process with a finite state space, Corollary 1.5.4 gives accuracy estimates in terms of the following quantities

$$\epsilon = \delta \max_{\eta \neq 0} \alpha(\eta), \tag{1.4}$$

$$\epsilon_0 = \frac{\alpha(0)}{|A^0(0,0)|} + \epsilon, \tag{1.5}$$

$$\delta_0 = \frac{1}{|A^0(0,0)|} + \sum_{\eta \neq 0} \frac{A^0(0,\eta)}{A^0(0,0)A^0(\eta,\eta)}, \tag{1.6}$$

$$\beta_D = \frac{1}{|A(0,0)|} + \sum_{\eta \in \mathcal{M}, \eta \neq 0} \frac{A(0,\eta)}{A(0,0)A(\eta,\eta)}, \tag{1.7}$$

where $A(i,j)$ is the intensity for Markov process $(\eta_s)$ to jump from state $i$ to state $j$ and $A^0(i,j) = A(i,j)$ for all $i, j \in \mathcal{M}$. Denote upper bounds of $\rho_V$ and $\lambda(\infty)$ by

$$r_V = \epsilon' (\epsilon + \delta \epsilon_0), \tag{1.8}$$

$$\lambda_{sup} = (1 + r_V) \lambda_V(\infty), \tag{1.9}$$

respectively (see Corollary 1.5.4). Following Section 1.5.1, let

$$\rho^0 = \frac{\delta \mathcal{E} \xi / \delta_0 + \delta \mathcal{E} \xi^2 / 2 \mathcal{E} \xi}{1 - \mathcal{E} \xi^2 / 2 \mathcal{E} \xi} \quad \text{if} \ 1 - \mathcal{E} \xi^2 / 2 \mathcal{E} \xi > 0 \tag{1.10}$$

be an upper bound of $\rho^0$ in the case where $\mathcal{E} \xi$ and $\mathcal{E} \xi^2$ can be calculated explicitly (see (1.5.7)), and let

$$\lambda^0_{sup} = \frac{\epsilon_0}{\delta_0} \tag{1.11}$$

be an upper bound of $\lambda^0$ (see (1.5.1)).

The paper is organized as follows. In Section 2, we consider a set of simple redundant systems consisting of three components under Markov assumptions and var-
ious types of redundancy. Section 3 is devoted to so-called $k$-out-of-$n$ systems (also under Markov assumptions) consisting of $n$ components such that the system is operating if and only if at least $k$ components are operating. A generalization of this type of systems is considered in Section 4 where Markov systems with independent components are studied and where we do not impose specific restrictions on set $\mathcal{M}$ except for the system coherence. In Section 5, we relax the Markov assumption and consider semi-Markov models. All considerations are demonstrated by corresponding numerical results.

2. Simple Examples

In this group of examples, we show how to estimate the parameters involved in Corollary 1.5.4 and to obtain the accuracies of the bounds derived for $\lambda(\infty)$. In all examples, we deal with a system consisting of three components, designated by $C_1$, $C_2$, and $C_3$. Further restrictions will be imposed in the upcoming subsections.

2.1 Passive redundancy

Assume that components $C_1$ and $C_3$ can be in two states: "operating" denoted by 1 (resp. 3) and "failed" denoted by 1 (resp. 3). Component $C_2$ is redundant with respect to $C_1$. Normally, $C_2$ is waiting (state 2w), but when $C_1$ fails, $C_2$ tries to replace the failed component. The replacement occurs with probability $1 - \gamma$, $0 \leq \gamma \leq 1$, and then $C_2$ enters the operating state denoted by 2; otherwise, $C_2$ enters the failed state 2. Upon repairing $C_1$, component $C_2$ (if it is not failed) returns to the waiting state. The failure rate of $Ci$ is $\lambda_i$ and its repair rate is $\mu_i$. Component $C_2$ cannot fail while it is waiting.

The dynamics of the system can be described by a Markov process with 8 states:

$e_0 = (1w23), e_1 = (123), e_2 = (12w3), e_3 = (1w23), e_4 = (123), e_5 = (123),
\quad e_6 = (123), e_7 = (123)$.

Transition rates of the first four states are given as follows:

$A(e_0, e_1) = (1 - \gamma)\lambda_1, A(e_0, e_3) = \lambda_3, A(e_0, e_6) = \gamma\lambda_1,$

$A(e_1, e_0) = \mu_1, A(e_1, e_5) = \lambda_3, A(e_1, e_6) = \lambda_2,$

$A(e_2, e_0) = \mu_2, A(e_2, e_4) = \lambda_3, A(e_2, e_6) = \lambda_1,$

$A(e_3, e_0) = \mu_3, A(e_3, e_5) = (1 - \gamma)\lambda_1, A(e_3, e_7) = \gamma\lambda_1.$

The rest rates can be written easily. Let us consider three different modes for the subset of operating states:

$\mathcal{M}_1 = \{e_0, e_1, e_2\}, \quad \mathcal{M}_2 = \{e_0, e_1, e_2, e_3, e_4\}, \quad \mathcal{M}_3 = \{e_0, e_1, e_2, e_3, e_5, e_6\}.$

Mode $\mathcal{M}_1$ corresponds to the situation where $C_3$ is in series with the other components $C_1$ and $C_2$, in the sense that the system is operating if $C_3$ is operating and either $C_1$ or $C_2$ is operating. In mode $\mathcal{M}_2$, the system is operating if $C_1$ is operating or if $C_2$ and $C_3$ are both operating. Mode $\mathcal{M}_3$ has no clear meaning, in general; it is introduced just to test our methods and to compare this case with the case of independent components.

Let the system be in the “perfect” state $e_0 = 0$ at time 0.
Mode $\mathcal{M}_1$. Matrix $A^0$ has the form

$$A^0 = \begin{pmatrix}
-(1-\gamma)\lambda_1 & (1-\gamma)\lambda_1 & 0 \\
\mu_1 & -\mu_1 & 0 \\
\mu_2 & 0 & -\mu_2
\end{pmatrix}$$

and non-zero values of the intensity function are listed below:

$$\alpha(e_0) = A(e_0; \mathcal{P}) = \gamma\lambda_1 + \lambda_3, \alpha(e_1) = A(e_1; \mathcal{P}) = \lambda_2 + \lambda_3, \alpha(e_2) = A(e_2; \mathcal{P}) = \lambda_1 + \lambda_3.$$

Let

$$\epsilon_\eta = \mathbf{E}_\eta \int_0^{\tau_0} \alpha(\eta^0_u)du, \quad \delta_\eta = \mathbf{E}_\eta \tau_0.$$

Then

$$\epsilon_1 = \frac{1}{\mu_1} \lambda_2 + \lambda_3, \quad \epsilon_2 = \frac{1}{\mu_2} \lambda_1 + \lambda_3, \quad \epsilon_0 = \epsilon_0 = \frac{\gamma\lambda_1 + \lambda_3}{(1-\gamma)\lambda_1} + \epsilon_1,$$

$$\delta_1 = \frac{1}{\mu_1}, \quad \delta_2 = \frac{1}{\mu_2}, \quad \delta_0 = \frac{1}{(1-\gamma)\lambda_1} + \frac{1}{\mu_1},$$

and

$$\mathbf{E}(\sigma \land \sigma_0) = \frac{1}{\lambda_1 + \lambda_3} + \frac{(1-\gamma)\lambda_1}{\lambda_1 + \lambda_3} + \frac{1}{\mu_1 + \lambda_2 + \lambda_3}.$$

So,

$$\epsilon = \max(\epsilon_1, \epsilon_2), \quad \epsilon_0 = \epsilon_0, \quad \delta = \max(\delta_1, \delta_2), \quad \delta_0 = \delta_0, \quad \beta_D = \mathbf{E}(\sigma \land \sigma_0),$$

and, by (1.5.1)

$$\lambda_0 = \frac{\epsilon_0}{\delta_0}.$$

In order to find the accuracy of relation (1.5.7), we have to calculate moments of the r.v. $\xi$. It can be done by using the equality

$$\xi = (\gamma\lambda_1 + \lambda_3)W_0 + (\lambda_2 + \lambda_3)W_1,$$

where $W_0$ and $W_1$ are two independent r.v.’s exponentially distributed with parameters $(1-\gamma)\lambda_1$ and $\mu_1$, respectively. This yields

$$\rho^0 = \frac{\delta\epsilon_0/\delta e_0 + \mathbf{E}\xi^2/2\epsilon e_0}{1 - \mathbf{E}\xi^2/2\epsilon e_0} \quad \text{if} \quad \mathbf{E}\xi^2 > 2\epsilon e_0,$$

$$\mathbf{E}\xi^2 = \frac{(\gamma\lambda_1 + \lambda_3)^2}{(1-\gamma)^2\lambda_1^2} + \frac{(\lambda_2 + \lambda_3)^2}{\mu_1^2} + \epsilon e_0^2.$$

Mode $\mathcal{M}_2$. Matrix $A^0$ has the form

$$A^0 = \begin{pmatrix}
-((1-\gamma)\lambda_1 + \lambda_3) & (1-\gamma)\lambda_1 & 0 & \lambda_3 & 0 \\
\mu_1 & -\mu_1 & 0 & 0 & 0 \\
\mu_2 & 0 & -(\mu_2 + \lambda_3) & 0 & \lambda_3 \\
\mu_3 & 0 & 0 & -\mu_3 & 0 \\
0 & 0 & \mu_3 & \mu_2 & -(\mu_2 + \mu_3)
\end{pmatrix}$$
and non-zero values of the intensity function are

\[ \alpha(e_0) = \gamma \lambda_1, \quad \alpha(e_1) = \lambda_2 + \lambda_3, \quad \alpha(e_2) = \alpha(e_3) = \alpha(e_4) = \lambda_1. \]

Let \( \epsilon_\eta \) and \( \delta_\eta \) be as in (2.1). In order to estimate them we can use (1.4) through (1.7) and Lemma 1.5.5. However, a better result can be achieved if we estimate \( \delta_e \) and \( \epsilon_e \) separately from \( \delta_e \) and \( \epsilon_e \) for \( i \geq 2 \). In fact,

\[ \delta_e = \frac{1}{\mu_1}, \quad \epsilon_e = (\lambda_2 + \lambda_3) \delta_e. \]

Quantities \( \delta_e, i \geq 2 \), can easily be found as the solution of three linear algebraic equations. But we prefer to use Lemma 1.5.5 to show how it works and to prove its accuracy.

Take the test function with

\[ V(e_0) = 0, \quad V(e_2) = V(e_3) = a, \quad V(e_4) = a + b. \]

Conditions (I.5.26) yield

\[ -\mu_2 a + \lambda_3 b \leq -1, \quad -\mu_3 a \leq -1, \quad -(\mu_2 + \mu_3) b \leq -1. \]

Let us choose

\[ b = \frac{1}{\mu_2 + \mu_3}, \quad a = \max \left( \frac{1}{\mu_3} \frac{1}{\mu_2} \left( 1 + \frac{\lambda_3}{\mu_2 + \mu_3} \right) \right), \]

\[ \delta = \max(\delta_e, a + b), \quad \epsilon = \max(\epsilon_e, \lambda_1 (a + b)), \]

and

\[ \epsilon_0 = \frac{\gamma \lambda_1}{(1 - \gamma) \lambda_1 + \lambda_3} \frac{(1 - \gamma) \lambda_1 \epsilon_e}{(1 - \gamma) \lambda_1 + \lambda_3} + \frac{\lambda_3}{(1 - \gamma) \lambda_1 + \lambda_3} \frac{\lambda_1}{\mu_3}. \]

Parameters \( \delta_0 \) and \( \beta_D \) are given in (1.6) and (1.7).

**Mode \( \mathcal{A}_3 \)**. Matrix \( A^0 \) has the form

\[
A^0 = 
\begin{pmatrix}
-(\lambda_1 + \lambda_3) & (1 - \gamma) \lambda_1 & 0 & \lambda_3 & 0 & \gamma \lambda_1 \\
\mu_1 & - (\mu_1 + \lambda_2 + \lambda_3) & 0 & 0 & \lambda_3 & \lambda_2 \\
\mu_2 & 0 & -(\mu_2 + \lambda_1) & 0 & 0 & \lambda_1 \\
\mu_3 & 0 & 0 & -(\mu_3 + (1 - \gamma) \lambda_1) & (1 - \gamma) \lambda_1 & 0 \\
0 & \mu_3 & 0 & \mu_1 & -(\mu_1 + \mu_3) & 0 \\
0 & \mu_2 & \mu_1 & 0 & 0 & -(\mu_1 + \mu_2)
\end{pmatrix}
\]

and non-zero values of the intensity function are

\[ \alpha(e_2) = \lambda_3, \quad \alpha(e_3) = \gamma \lambda_1, \quad \alpha(e_5) = \lambda_2, \quad \alpha(e_6) = \lambda_3. \]

For estimating, let us employ (1.4) through (1.7) and Lemma 1.5.5. Take the test function with

\[ V(e_0) = 0, \quad V(e_1) = \frac{1}{\mu_1} \left( 1 + \frac{\lambda_2 + \lambda_3}{\mu} \right), \quad V(e_i) = V(e_1) + \frac{1}{\mu}(i \in \{2, 3, 5, 6\}). \]
where
\[ \mu = \min(\mu_2, \mu_3). \]

Then
\[ \delta = \left(1 + \frac{\lambda_2 + \lambda_3}{\mu} \right) \frac{1}{\mu_1} + \frac{1}{\mu}. \]

Parameters \( \delta_0 \) and \( \beta_D \) can be estimated with the help of (1.6) and (1.7).

2.2 Independent Components

Assume now that the three components are independent.

**Mode \( M_1 \)**. All the characteristics can be calculated explicitly:

\[ \delta = \max\left(\frac{1}{\mu_1}, \frac{1}{\mu_2} \right), \quad \epsilon = \max\left(\frac{\lambda_2 + \lambda_3}{\mu_1}, \frac{\lambda_1 + \lambda_3}{\mu_2} \right), \]

\[ \epsilon_0 = \frac{1}{\lambda_1 + \lambda_2} \left( \lambda_3 + \frac{\lambda_1 (\lambda_2 + \lambda_3)}{\mu_1} + \frac{\lambda_2 (\lambda_1 + \lambda_3)}{\mu_2} \right), \]

\[ E_0 r_0 = \frac{1}{\lambda_1 + \lambda_2} \left( 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} \right), \quad E_0 (\sigma \land \sigma_0) = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} \left( 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} \right). \]

**Mode \( M_2 \)**. We estimate \( \delta \) similarly for the passive redundancy case. Take \( \delta = \max(1/\mu_1, a + b) \) with

\[ b = \frac{1}{\mu_2 + \mu_3}, \quad a = \max\left(\frac{1}{\mu_2} \left(1 + \frac{\lambda_3}{\mu_2 + \mu_3}\right), \frac{1}{\mu_1} \left(1 + \frac{\lambda_2}{\mu_2 + \mu_3}\right) \right), \]

\[ \epsilon = \epsilon_0 = \max\left(\frac{\lambda_2 + \lambda_3}{\mu_1}, \delta \max(\lambda_1 + \lambda_3, \lambda_1 + \lambda_2)\right), \]

Formulas (1.6) and (1.7) yield

\[ \delta_0 = \beta_D = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} \left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_3}{\mu_3}\right). \]

**Mode \( M_3 \)**. The steps are the same as in the passive redundancy case. Let us take the following test function

\[ V(e_0) = 0, \quad V(e_1) = V(e_2) = V(e_3) = a, \quad V(e_6) = V(e_7) = a + b, \]

where

\[ a = \max\left(\frac{1}{\mu_1} \left(1 + \frac{\lambda_2 + \lambda_3}{\mu_1 + \mu}\right), \frac{1}{\mu} \left(1 + \frac{\lambda_1}{\mu_1 + \mu}\right) \right), \quad b = \frac{1}{\mu_1 + \mu}, \quad \mu = \min(\mu_1, \mu_2). \]

Then

\[ \delta = a + b, \quad \epsilon = \epsilon_0 = \delta \max(\lambda_2, \lambda_3). \]

Quantities \( \delta_0 \) and \( \beta_D \) can be estimated with the help of formulas (1.6) and (1.7). The result is the same as in the case \( M_2 \).

2.3 Numerical Results

The Vesely failure rate is supposed to approximate the true failure rate to a good accuracy if “failure rates of components are small in comparison with their repair
This is examined in the numerical calculations. Consider the following five groups of figures.

<table>
<thead>
<tr>
<th></th>
<th>$\lambda_1$</th>
<th>$\mu_1$</th>
<th>$\lambda_2$</th>
<th>$\mu_2$</th>
<th>$\lambda_3$</th>
<th>$\mu_3$</th>
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<td>$10^4$</td>
<td>10</td>
<td>$10^4$</td>
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<td>0</td>
</tr>
<tr>
<td>1.b</td>
<td>10</td>
<td>$10^4$</td>
<td>10</td>
<td>$10^4$</td>
<td>1</td>
<td>$5 \cdot 10^3$</td>
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<tr>
<td>2.b</td>
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<td>$10^4$</td>
<td>100</td>
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<td>1</td>
<td>$5 \cdot 10^3$</td>
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<td>$10^3$</td>
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<td>1</td>
<td>$5 \cdot 10^2$</td>
<td>0.05</td>
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<tr>
<td>3.b</td>
<td>10</td>
<td>$10^3$</td>
<td>1</td>
<td>$5 \cdot 10^2$</td>
<td>10</td>
<td>$10^3$</td>
<td>0.05</td>
</tr>
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<td>4.a</td>
<td>10</td>
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<td>$2 \cdot 10^5$</td>
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<td>0.05</td>
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<tr>
<td>4.b</td>
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<td>100</td>
<td>100</td>
<td>$2 \cdot 10^5$</td>
<td>10</td>
<td>$10^4$</td>
<td>0.05</td>
</tr>
<tr>
<td>5.a</td>
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<td>$10^{-3}$</td>
<td>1</td>
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<td>0.05</td>
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<tr>
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<td>10</td>
<td>$10^4$</td>
<td>$10^{-3}$</td>
<td>1</td>
<td>0.05</td>
</tr>
</tbody>
</table>

In groups 1, 2, 4, and 5 $\lambda_i/\mu_i \leq 10^{-3}$ for all $i$; $\lambda_i/\mu_j \leq 10^{-2}$ in group 3. In addition, $\lambda_i/\mu_j \leq 2 \cdot 10^{-3}$ in group 1 and $\lambda_i/\mu_j \leq 2 \cdot 10^{-2}$ in groups 2 and 3 for all $i$ and $j$; in group 4 and 5, there exists a pair $i, j$ such that $\lambda_i = \mu_j$ and $\lambda_i = 10\mu_j$ correspondingly. The results are contained in the following table where $\text{pas}.\text{red.}$ means passive redundancy, and $\text{ind.}$ means independent case. See (1.3) to (1.11) for other notations.

<table>
<thead>
<tr>
<th>data</th>
<th>mode</th>
<th>$\lambda(\infty)$</th>
<th>$\lambda_V(\infty)$</th>
<th>$\rho_V$</th>
<th>$\tau_V$</th>
<th>$\lambda_{sup}$</th>
<th>$\lambda_{sup}^0$</th>
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<tr>
<td></td>
<td></td>
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<td>$\lambda_0$</td>
<td>$\rho_0$</td>
<td>$\rho_0$</td>
<td>$\lambda_0$</td>
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</tr>
<tr>
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<td>pas.</td>
<td>$\lambda_0$</td>
<td>$\lambda_0$</td>
<td>$\rho_0$</td>
<td>$\rho_0$</td>
<td>$\lambda_0$</td>
<td>$\lambda_0$</td>
</tr>
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<td>1.0100</td>
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<td>9.86 $\cdot 10^{-6}$</td>
<td>1.21 $\cdot 10^{-3}$</td>
<td>1.11 $\cdot 10^{-1}$</td>
<td>1.0112</td>
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<td>1.2984 $\cdot 10^{-2}$</td>
<td>1.20 $\cdot 10^{-3}$</td>
<td>1.23 $\cdot 10^{-3}$</td>
<td>2.68 $\cdot 10^{-3}$</td>
<td>1.24 $\cdot 10^{-3}$</td>
<td>1.3019 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>$\mathcal{M}_3$</td>
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<td>2.9949 $\cdot 10^{-6}$</td>
<td>2.56 $\cdot 10^{-4}$</td>
<td>4.78 $\cdot 10^{-4}$</td>
<td>3.02 $\cdot 10^{-3}$</td>
<td>3.0040 $\cdot 10^{-6}$</td>
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</tr>
<tr>
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<td>ind.</td>
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<td>$\lambda_0$</td>
<td>$\rho_0$</td>
<td>$\rho_0$</td>
<td>$\lambda_0$</td>
<td>$\lambda_0$</td>
</tr>
<tr>
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<td>1.0199</td>
<td>1.0200</td>
<td>1.95 $\cdot 10^{-5}$</td>
<td>1.21 $\cdot 10^{-3}$</td>
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<tr>
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<td>2.2952 $\cdot 10^{-2}$</td>
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In the above table, some values of $r^0$ are absent. This means that we could not estimate them for different reasons. It is seen that estimate $\lambda^0$ is always better than $\lambda_V(\infty)$ and, since it is an upper bound of $\lambda(\infty)$, it is worthy to be used instead of $\lambda(\infty)$ when it can be calculated easily. The results on the Vesely failure rate accuracy are quite expected. The accuracy is very good for group 1, good for groups 2 and 3, bad for group 4, and very bad for group 5. The difference between passive redundancy and independent case is negligible. Parameter $\gamma$ does not affect the accuracy. We never have $r_V < 10^{-3}$, even if $P_V$ is small. Nevertheless, estimate $\lambda_{sup}$ is good in the first three groups, very bad in group 4 and awful in group 5. Generally, $\delta_0$ is not a good estimate for $ES$, except for the case where $ES$ can be calculated explicitly (passive redundancy, variants $M_1$ and $M_2$). Consequently, $\lambda_{sup}^0$ is not a good approximation of $\lambda(\infty)$.

3. The $k$-out-of-$n$ Systems

Let us consider a system with $n$ independent components $C_1$ to $C_n$. The system is operating if and only if at least $k$ components are operating ($1 \leq k < n$). The failure rate of component $C_i$ is $\lambda_i$, its repair rate is $\mu_i$. If all components are identical, then we can lump the states with the same number of operating components resulting in a birth-and-death process describing the dynamics of such a system. In this case, the system can be examined easily. If the components are not identical, it is still possible to find a pessimistic bound of the reliability in the following way. Let
\( \lambda_{(1)} \geq \lambda_{(2)} \geq \ldots \geq \lambda_{(n)} \) (resp. \( \mu_{(1)} \leq \mu_{(2)} \leq \ldots \leq \mu_{(n)} \)) be failure rates (resp. repair rates) arranged in a decreasing order (resp. ascending order), and let

\[
\begin{align*}
    a_i &= \sum_{j=1}^{n-i} \lambda_{(j)}, \quad 0 \leq i \leq n - 1, \\
    b_i &= \sum_{j=1}^{i} \mu_{(j)}, \quad 1 \leq i \leq n.
\end{align*}
\]

Consider a birth-and-death process with transition rates

\[
\begin{align*}
    \lambda(i, i + 1) &= a_i, \quad 0 \leq i \leq n - 1, \\
    \lambda(i, i - 1) &= b_i, \quad 1 \leq i \leq n.
\end{align*}
\]

For this process, state \( i \) means that exactly \( i \) components failed. It can be shown (see Cocozza-Thivent and Roussignol [2]) that the system described by the birth-and-death process (with the subset of “good” states \( \mathcal{M} = \{0, 1, \ldots, n - k\} \)) has a worse reliability performance than the initial one. But the pessimistic estimations using the birth-and-death process may be not very accurate. We will see that, in this case, it is better to use the Vesely failure rate approximation and to give bounds for the relative error using the birth-and-death process.

3.1 Principle

Let \((\eta^0_t)\) be a birth-and-death process with transition rates

\[
\begin{align*}
    \lambda^0(i, i + 1) &= a_i, \quad 0 \leq i < n - k, \quad (3.1) \\
    \lambda^0(i, i - 1) &= b_i, \quad 1 \leq i \leq n - k, \quad (3.2)
\end{align*}
\]

be its first return time to state 0. It can be shown, using test functions or Cocozza-Thivent and Roussignol [2], that \( E\eta^0_t \leq E_{n-k}\bar{\eta}_0^0(\eta \in \mathcal{M}) \) and \( ES \leq E_0\bar{\eta}_0 \). Let us take

\[
\delta = E_{n-k}\bar{\eta}_0, \quad \bar{\delta}_0 = E_0\bar{\eta}_0.
\]

These quantities can easily be found as solutions of the following linear system:

\[
\begin{align*}
    -(a_1 + b_1)x_1 + a_1x_2 &= -1, \\
    b_ix_{i-1} - (a_i + b_i)x_i + a_{i+1}x_{i+1} &= -1, \quad 2 \leq i \leq n - k - 1, \\
    b_{n-k}x_{n-k-1} - b_{n-k}x_{n-k} &= -1,
\end{align*}
\]

where \( x_i = E_i\bar{\eta}_0 \). This yields

\[
\delta = x_{n-k}, \quad \bar{\delta}_0 = \frac{1}{a_0} + x_1.
\]
Let $\alpha$ be the intensity function of the process. Then $\alpha(\eta) \neq 0$ if $\eta$ is a state with exactly $k$ operating components. In this case,

$$\alpha(\eta) \leq \lambda_1 + \ldots + \lambda_k = \alpha_{n-k}.$$ 

Therefore,

$$E_\eta \int_0^{\tau_0} \alpha(\eta_u^0)du \leq a_{n-k}E_\eta \int_0^{\tau_0} I(\eta_u^0 = n-k)du \leq a_{n-k}E \int_0^{\tau_0} I(\eta_u^0 = n-k)du.$$ 

The last inequality is very intuitive and can be proved using Cocozza-Thivent and Roussignol [2]. In fact,

$$E_\eta \int_0^{\tau_0} \alpha(\eta_u^0)du \leq \frac{a_{n-k}}{q\alpha_{n-k}}, \quad (3.4)$$

where $q$ is the probability that the process $(\eta_s^0)$ starting from state $n-k-1$ reaches state 0 earlier than state $n-k$. If $y_i$ is the probability that this process starting from state $i$ reaches state 0 earlier than state $n-k$, then

$$-(a_1 + b_1)y_1 + a_1y_2 = -b_1,$$

$$b_iy_{i-1} - (a_i + b_i)y_i + a_iy_{i+1} = 0, \quad 2 \leq i \leq n-k-2 \quad (3.5)$$

$$b_{n-k-1}y_{n-k-2} - (a_{n-k-1} + b_{n-k-1})y_{n-k-1} = 0,$$

and $q = y_{n-k-1}$. Using (3.4), we obtain

$$\epsilon = \frac{a_{n-k}}{b_{n-k}} \frac{1}{q} = \frac{a_{n-k}}{y_{n-k-1}b_{n-k}}$$

(if $k = n-1$ take $y_1 = 0$). Since $\alpha(\eta) = 0$ if $\eta \neq n-k$,

$$E_0 \int_0^{\tau_0} \alpha(\eta_u^0)du = (1 - y_1)E_{n-k} \int_0^{\tau_0} \alpha(\eta_u^0)du \leq (1 - y_1)\epsilon,$$

and hence

$$\epsilon_0 = (1 - y_1)\epsilon.$$

Now, let us find $\beta_D = E_0(\sigma \wedge \sigma_0)$ by solving the linear system

$$-(a_1 + b_1)z_1 + a_1z_2 = -1,$$

$$b_i(z_{i-1} - (a_i + b_i)z_i + a_iz_{i+1} = -1, \quad 2 \leq i \leq n-k-1 \quad (3.6)$$

$$b_{n-k}z_{n-k-1} - (a_{n-k} + b_{n-k})z_{n-k} = -1,$$

where $z_i$ is the mean first passage time for the (initial) birth-and-death process.
(starting from state $i$) to subset $\{0, n-k+1\}$. It follows that

$$\beta_D = \frac{1}{a_0} + z_1.$$ 

### 3.2 Numerical results

In all these examples $n = 6$. Consider the following four groups of data.

<table>
<thead>
<tr>
<th>case</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$\mu_1$</th>
<th>$\lambda_2$</th>
<th>$\mu_2$</th>
<th>$\lambda_3$</th>
<th>$\mu_3$</th>
<th>$\lambda_4$</th>
<th>$\mu_4$</th>
<th>$\lambda_5$</th>
<th>$\mu_5$</th>
<th>$\lambda_6$</th>
<th>$\mu_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.a</td>
<td>3</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>1.b</td>
<td>3</td>
<td>$10^{-3}$</td>
<td>$5 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>1.c</td>
<td>2</td>
<td>$10^{-3}$</td>
<td>$5 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
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<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>1.d</td>
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<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$10^{-2}$</td>
<td>$2 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>2.a</td>
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<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
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<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>2.b</td>
<td>2</td>
<td>$10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
<td>$10^{-2}$</td>
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<td>$10^{-2}$</td>
</tr>
<tr>
<td>2.c</td>
<td>3</td>
<td>$10^{-2}$</td>
<td>$5 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>2.d</td>
<td>3</td>
<td>$10^{-2}$</td>
<td>$5 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
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<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>3.a</td>
<td>3</td>
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<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>3.b</td>
<td>2</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
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<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
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<td>$5 \cdot 10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-1}$</td>
<td>$8 \cdot 10^{-2}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
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<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
</tr>
</tbody>
</table>

In group 1, $\lambda_i/\mu_i \leq 1.25 \cdot 10^{-3}$ and $\lambda_j/\mu_j \leq 1.25 \cdot 10^{-2}$ for all $i$ and $j$. In group 2, $\lambda_i/\mu_i \leq 1.25 \cdot 10^{-2}$ and $\lambda_j/\mu_j \leq 1.25 \cdot 10^{-1}$ for all $i$ and $j$. In group 3, $\lambda_i/\mu_j = 10^{-1}$ for all $i$ and $j$. In group 4, there exists $i$ such that $\lambda_i/\mu_i = 1$.

Since the components are independent, the underlying Markov process is reversible and $\lambda_V(\infty) = \lambda^0$ is an upper bound for $\lambda(\infty)$; therefore, the calculation of $\lambda_{sup}$ is useless. Let us introduce quantity $\lambda^1(\infty)$ which is the asymptotic failure rate of the birth-and-death process with transition rates $a_i$ and $b_i$. This coincides with the true asymptotic failure rate if $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$ for all $i, j$; in all other cases this is an upper bound of $\lambda(\infty)$. Notice that if $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$ for all $i$ and $j$, then $\epsilon_0$ and $\delta_0$ are the exact values of $\mathbb{E}X$ and $\mathbb{E}S$, respectively, and, therefore, $\lambda^0 = \lambda_{sup}^0$. The results are listed in the following table.
The upper bound $\lambda_V(\infty)$ is better than $\lambda'(\infty)$ and $\lambda_V(\infty)$ is a good approximation of $\lambda(\infty)$ in groups 1 to 3 (very good in groups 1 and 2). In group 4, the approximation is not tight.

If $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$ for all $i$ and $j$, then a good way to find $\lambda_V(\infty)$ is to compute $\lambda^0 = \epsilon_0/\delta_0$ with

$$\epsilon_0 = \frac{1 - y_1}{y_{n-k-1}} \frac{a_{n-k}}{b_{n-k}}, \quad \delta_0 = \frac{1}{a_0} + x_1.$$ 

### 4. Systems with Independent Components

#### 4.1 Principle

Consider a system with independent components and general working space which means that we do not impose specific restrictions on $\mathcal{M}$ (such as in the case of $k$-out-of-$n$ systems) except for the coherence of the system. But the dynamics of the system is still described by a Markov process with finite states space and transition rates $A$. 

<table>
<thead>
<tr>
<th>data</th>
<th>$\lambda(\infty)$</th>
<th>$\lambda(V)(\infty) = \lambda^0$</th>
<th>$\rho_V$</th>
<th>$\lambda'(\infty)$</th>
<th>$\lambda^0_{sup}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.a</td>
<td>$5.9582 \cdot 10^{-11}$</td>
<td>$5.9641 \cdot 10^{-11}$</td>
<td>$1.00 \cdot 10^{-3}$</td>
<td>$5.9582 \cdot 10^{-11}$</td>
<td>$5.9641 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>1.b</td>
<td>$1.9636 \cdot 10^{-10}$</td>
<td>$1.9682 \cdot 10^{-10}$</td>
<td>$2.31 \cdot 10^{-3}$</td>
<td>$2.7684 \cdot 10^{-8}$</td>
<td>$2.7854 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>1.c</td>
<td>$9.8144 \cdot 10^{-14}$</td>
<td>$9.8409 \cdot 10^{-14}$</td>
<td>$2.70 \cdot 10^{-3}$</td>
<td>$8.6771 \cdot 10^{-11}$</td>
<td>$8.7044 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>1.d</td>
<td>$1.9623 \cdot 10^{-9}$</td>
<td>$1.9672 \cdot 10^{-9}$</td>
<td>$2.47 \cdot 10^{-3}$</td>
<td>$1.8703 \cdot 10^{-8}$</td>
<td>$1.8810 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>2.a</td>
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<td>$5.6523 \cdot 10^{-7}$</td>
<td>$1.02 \cdot 10^{-2}$</td>
<td>$5.5952 \cdot 10^{-7}$</td>
<td>$5.6523 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>2.b</td>
<td>$2.8119 \cdot 10^{-9}$</td>
<td>$2.8261 \cdot 10^{-9}$</td>
<td>$5.05 \cdot 10^{-3}$</td>
<td>$2.8261 \cdot 10^{-9}$</td>
<td>$2.8261 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>2.c</td>
<td>$1.6147 \cdot 10^{-5}$</td>
<td>$1.6563 \cdot 10^{-5}$</td>
<td>$2.58 \cdot 10^{-2}$</td>
<td>$1.4831 \cdot 10^{-4}$</td>
<td>$1.5753 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>2.d</td>
<td>$1.8219 \cdot 10^{-6}$</td>
<td>$1.8652 \cdot 10^{-6}$</td>
<td>$2.38 \cdot 10^{-2}$</td>
<td>$2.0977 \cdot 10^{-4}$</td>
<td>$2.2388 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>3.a</td>
<td>$3.0335 \cdot 10^{-3}$</td>
<td>$3.3898 \cdot 10^{-3}$</td>
<td>$1.17 \cdot 10^{-1}$</td>
<td>$3.0335 \cdot 10^{-3}$</td>
<td>$3.3898 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>3.b</td>
<td>$1.6040 \cdot 10^{-4}$</td>
<td>$1.6935 \cdot 10^{-4}$</td>
<td>$5.58 \cdot 10^{-2}$</td>
<td>$1.6935 \cdot 10^{-4}$</td>
<td>$1.6935 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>$3.3541 \cdot 10^{-2}$</td>
<td>$4.3139 \cdot 10^{-2}$</td>
<td>$2.86 \cdot 10^{-1}$</td>
<td>$2.0175 \cdot 10^{-1}$</td>
<td>$3.2033 \cdot 10^{-1}$</td>
</tr>
</tbody>
</table>
Let all the components be operating at time 0.

Denote by \( C_i \) the set of the states with exactly \( n - 1 \) operating components. Let

\[
a_i = \max_{\eta \in C_i} \sum_{\xi \in C_{i+1} \cap M_b} A(\eta, \xi), \quad b_i = \min_{\eta \in C_i \cap M_b} \sum_{\xi \in C_{i-1}} A(\eta, \xi),
\]

\( n_c = \max\{i : C_i \subseteq M_b\}, \quad \bar{n}_c = \max\{i : C_i \cap M_b \neq \emptyset\}, \)

where the number \( n_c + 1 \) is the order of the smallest minimal cutset.

In this section, when referring to relations (3.1) to (3.6), the coefficients \( a_i \) and \( b_i \) are assumed to be those given in (4.1).

Let

\[
\delta = x_{\bar{n}_c},
\]

where \( x_i \)'s are solutions of system (3.3) with \( n - k = \bar{n}_c \) and

\[
\bar{\alpha} = \max_{\eta \notin C_0} \alpha(\eta) = \max_{\eta \in M_b, \eta \notin C_0} A(\eta; \mathcal{F}).
\]

Let \( \eta_t^0 \) be a birth-and-death process with transition rates given by relations (3.1) and (3.2). The same arguments as for the \( k \)-out-of-\( n \) system give

\[
E_t \int_0^{\tau_0} \alpha(\eta_u^0) du \leq \bar{\alpha} E_t \int_0^{\tau_0} I(\eta_u^0 \in \cup_{n_c \leq i \leq \bar{n}_c} C_i \cap M_b) du
\]

\[
\leq \bar{\alpha} E_{\bar{n}_c} \int_0^{\tau_0} I(n_c \leq \eta_u^0 \leq \bar{n}_c) du.
\]

Let \( t_i, 1 \leq i \leq \bar{n}_c - n_c + 1, \) be the mean time for process \( \eta_t^0 \) starting at \( n_c + i - 1 \) to leave interval \([n_c, \bar{n}_c] \). Then

\[
E_{\bar{n}_c} \int_0^{\tau_0} I(n_c \leq \eta_u^0 \leq \bar{n}_c) du = t_{\bar{n}_c - n_c + 1} + \frac{1 - q'}{q'} t_1,
\]

where

\[
q' = y_{\bar{n}_c - 1}, \quad (q' = 1 \text{ if } \bar{n}_c = 1),
\]

and \( y_i \)'s satisfy linear system (3.5) with \( n - k = n_c \). Quantities \( t_i \)'s are solutions of the following system

\[
-(a_{n_c} + b_{n_c})t_1 + a_{n_c} t_2 = -1,
\]

\[
b_{n_c} + i - t_{i+1} = \frac{(a_{n_c} + i - 1 + b_{n_c} + i - 1) t_i + a_{n_c} + i - 1 - t_{i+1}}{2} = -1, \quad 2 \leq i \leq \bar{n}_c - n_c,
\]

\[
b_{\bar{n}_c} t_{\bar{n}_c - n_c} - b_{\bar{n}_c} t_{\bar{n}_c} = -1.
\]

This yields

\[
\epsilon = \bar{\alpha} \left( t_{\bar{n}_c - n_c + 1} + \frac{1 - q'}{q'} t_1 \right), \quad \epsilon_0 = \frac{A(C_0; \mathcal{F})}{a_0} + \frac{1}{q'} t_1 \bar{\alpha} (1 - y_1)
\]

\( (y_1 = 0 \text{ if } n_c = 1) \). Let us notice that \( A(C_0; \mathcal{F}) \neq 0 \) if and only if \( n_c = 1 \). Finally, we take

\[
\delta_0 = \frac{1}{a_0} + x_1,
\]

and

\[
\beta_D = \frac{1}{a_0} + z_1
\]

where \( x_i \) are solutions of the system (3.6) with \( n - k = n_c \) and \( a_{n_c} = \max_{\eta \in C_{n_c}} \sum_{x \in C_{n_c}} A(\eta, \xi). \)
4.2 Systems with common mode

Let us return to the previous example but add there a so-called common mode: the system behavior is just the same as described in Subsection 4.1 but it is subjected to additional events called the common mode. If a common mode occurs, then the ith component (if it is operating) can fail with probability $p_i$ independently of other components. Let the occurrence rate of the common mode be $\Lambda$. Define $p = \max_i p_i$, and denote by $\overline{\alpha}$ the maximum value of the intensity process calculated without the common mode factor. Let $n_c$ and $\overline{n}_c$ be the quantities defined in (4.2).

Process $(\overline{n}_t^0)$ is no longer a birth-and-death process. Let $\overline{A}^0$ be the transition rates matrix of the process $(\overline{n}_t^0)$ without the common mode. The true transition rates matrix $\overline{B}^0$ of process $(\overline{n}_t^0)$ can be expressed in terms of $\overline{A}^0$ as follows:

$$
\overline{B}^0(i, i - 1) = \overline{A}^0(i, i - 1),
$$
$$
\overline{B}^0(i, i + 1) = \overline{A}^0(i, i + 1) + \Lambda(n - i)p(1 - p)^{n - i - 1},
$$
$$
\overline{B}^0(i, i + m) = \Lambda C_{n-i}^m p^m(1 - p)^{n - i - m}, \quad 2 \leq m \leq n_c - i.
$$

Define $\overline{B}^0(i, i) = -\sum_{j: j \neq i} \overline{B}^0(i, j)$ and take $\delta = x_{n_c}$, where $x_i$'s satisfy the linear equations

$$
x_0 = 0, \quad \sum_j \overline{B}^0(i, j)x_j = -1 \quad (i \geq 1).
$$

Let

$$
q'' = y_{n_c - 1},
$$

where $y_i$'s are the solutions of the system

$$
y_0 = 1, \quad \sum_j \overline{B}^0(i, j)y_j = 0 \quad (1 \leq i \leq n_c - 1), \quad y_i = 0(n_c \leq i \leq \overline{n}_c).
$$

We now define $t_i$'s as the solutions of the system

$$
t_0 = 0, \quad \sum_j \overline{B}^0(i + n_c - 1, j)t_j = -1 \quad (1 \leq i \leq \overline{n}_c - n_c + 1).
$$

Then

$$
\epsilon = \Lambda p \delta + \frac{\overline{\alpha} t_{\overline{n}_c - n_c + 1}}{q''}.
$$

Quantities $\epsilon_0$, $\delta_0$, and $\beta_D$ can be estimated with the help (1.5), (1.6), and (1.7).

4.3 Numerical results

In practice, the set of failed states is often expressed in terms of so-called minimal cutsets. A cutset is a collection of components such that their simultaneous failures imply the failure of the system. A cutset is called minimal if it does not contain smaller cutsets. Clearly, the knowledge of all minimal cutsets enables us to construct all failed states.

In the following examples, we consider a system with $n = 5$ components $C_1, \ldots, C_5$, and four families of minimal cutsets:

- $\mathcal{C}_1 = \{(C_1, C_2), (C_2, C_4, C_5), (C_1, C_3, C_5)\}$,
- $\mathcal{C}_2 = \{(C_2, C_4, C_5), (C_1, C_3, C_5)\}$,
- $\mathcal{C}_3 = \{(C_1, C_3, C_5), (C_2, C_3, C_4, C_5)\}$,
- $\mathcal{C}_4 = \{(C_2, C_3, C_4, C_5)\}$. 
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The following table contains the results of our calculations. If a system consists of independent components (see lines 1 to 8 in the table above and below), then we estimate \( r_V \) and \( \lambda^0_{\text{sup}} \) by formulas from paragraph 4.1. In the absence of common mode, \( \lambda_V(\infty) = \lambda^0_{\text{sup}} \) is an upper bound for \( \lambda(\infty) \) and therefore, no need to fill the column \( \lambda_{\text{sup}} \). In the case of common mode, we use formulas from paragraph 4.2.

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<th>( \lambda_V(\infty) )</th>
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<th>( \lambda_{\text{sup}} )</th>
<th>( \frac{\lambda^0}{\lambda^0_{\text{sup}}} )</th>
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One can see that $\lambda_V(\infty)$ approximates $\lambda(\infty)$ to a good accuracy and that $r_V$ is an accurate estimate of $\rho_V$ in all cases except for 7, 22, and 23. In all cases, $\lambda^0$ is a better bound than the Vesely failure rate $\lambda_V(\infty)$. The smaller ratios of failure rates to repair rates are and the larger the set $\mathcal{M}$ of good states is, the better are the approximations. When failure rates are small in comparison with repair rates, the approximation is good, since the first regeneration time is small and processes $(\eta^0_1), (\eta^1_1)$ and $(\eta^2_1)$ are close to each other. The reason of the influence of the size of set $\mathcal{M}$ will be explained in the following section (see Remark 5.1).
Remark 4.1: All systems considered in previous examples are NBU (see Cocozza-Thivent and Roussignol [1]) and therefore,
\[ R(t) \geq e^{-\lambda(\infty)t}. \]
Because of this, our estimates of \( \lambda(\infty) \) lead to pessimistic bounds of the system reliability.

5. Semi-Markov Process

5.1 An alternative renewal process

We have already seen (Remark 1.3.13) that the Vesely failure rate of a semi-Markov process is equal to \( 1/MUT \). We now consider an alternative renewal process, which can be regarded as a semi-Markov process with two states: 1 (operating state) and 0 (failed state). Denote by \( F \) the distribution function of the sojourn time in state 1, by \( m = \int_0^\infty (1-F(t))dt \) its mean value, and by \( h(t) = \frac{F'(t)}{1-F(t)} \) its hazard rate function. Assume that there exists the limit \( h(\infty) = \lim_{t \to \infty} h(t) < \infty \). Then the asymptotic failure rate and the asymptotic Vesely failure rate functions for such a system are equal to
\[ \lambda(\infty) = h(\infty), \quad \text{and} \quad \lambda_V(\infty) = \frac{1}{m}, \]
respectively. (5.1)

If \( \lambda_V(\infty) \) is regarded as an estimate of \( \lambda(\infty) \), then the accuracy of this estimate depends (in general) on distribution function \( F \). Indeed, in the exponential case, \( h \) is constant and \( \lambda(\infty) = \lambda_V(\infty) \equiv \lambda(t) \). But if, say, \( F \) is a gamma-distribution with the density
\[ F'(t) = \frac{1}{\Gamma(a)b^a}t^{a-1}e^{-x/b}, \]
then
\[ \lambda(\infty) = \frac{1}{b}, \quad \lambda_V(\infty) = \frac{1}{ab}, \quad \frac{\lambda_V(\infty) - \lambda(\infty)}{\lambda(\infty)} = 1 - \frac{1}{a}, \]
and the accuracy depends on how far is the gamma-distribution from an exponential distribution.

Although we cannot use our method because process \( (\eta_t^0) \) is not regenerative, this does not matter, since we obtain accuracy estimates directly from (5.1).

Remark 5.1: Let us return for a while to the Markovian case. When failure rates are small in comparison with repair rates and when set \( \mathcal{M} \) is large, then the probability for Markov process \( (\eta_t) \) to return to the “perfect” state before entering into \( \mathcal{P} \) is high. Because of this, the sojourn times distributions in set \( \mathcal{M} \) are close to exponential distributions (see Keilson [4]), and it is intuitively reasonable to consider that the sequence of successive sojourn times of process \( (\eta_t) \) in sets \( \mathcal{M} \) and \( \mathcal{P} \) is approximately an alternative renewal process with an exponential distribution function \( F \) of interrenewal times. This can explain why the approximation of \( \lambda(\infty) \) by \( \lambda_V(\infty) \) (and even of \( \lambda(t) \) by \( \lambda_V(t) \)) is good in this case.

5.2 A simple example of a semi-Markov process

Let a facility have four operating states \( e_1, e_2, e_3, e_4 \) and two failed states \( e_5 \) and \( e_6 \). The initial state is \( e_1 \). When leaving state \( e_1 \), the facility can reach one of the three following states with probabilities \( p_1, q_1 \), and \( r_1 \), respectively, \( (p_1 + q_1 + r_1 = 1) \):
state $e_2$; this can be treated as if a minor failure would have occurred which was detected and the facility had been repaired;

- state $e_3$; this means that a minor failure has occurred but it was not detected;

- state $e_5$; the system fails.

After this, the system behaves as follows. Upon repairing a minor failure, the facility comes back to state $e_1$. From state $e_3$, the facility can go either to operating state $e_4$ (if safety system has worked) with probability $p_2$ or to failed state $e_6$ with probability $1 - p_2$. State $e_4$ leads to state $e_2$.

Let us describe the facility’s behavior by a semi-Markov process with the transition rate functions including only non-zero terms as follows

\[
A(e_1, e_2, s) = p_1 \lambda_1(s), \quad A(e_1, e_3, s) = q_1 \lambda_1(s), \quad A(e_1, e_5, s) = r_1 \lambda_1(s),
\]

\[
A(e_3, e_4, s) = p_2 \lambda_2(s), \quad A(e_3, e_6, s) = (1 - p_2) \lambda_2(s),
\]

\[
A(e_5, e_1, s) = \mu_1(s), \quad A(e_2, e_1, s) = \mu_2(s), \quad A(e_4, e_2, s) = \mu_3(s), \quad A(e_6, e_2) = \mu_4(s),
\]

and apply Proposition 1.5.2 to Markov process $\eta_t = (\eta_t, y_t)$, where $y_t$ is the elapsed time of the process in its current state at time $t$. Using Proposition 1.2.5, one can see that non-zero terms of transition rate functions of semi-Markov process $(\eta_s^0)$ have the form

\[
A^0(e_1, e_2, s) = p_1 \lambda_1(s), \quad A^0(e_1, e_3, s) = q_1 \lambda_1(s),
\]

\[
A^0(e_3, e_4, s) = p_2 \lambda_2(s), \quad A^0(e_2, e_1, s) = (1 - p_2) \lambda_2(s),
\]

\[
A^0(e_4, e_2, s) = \mu_2(s), \quad A^0(e_4, e_2, s) = \mu_3(s).
\]

Process $(\eta_s^0) = ((\eta_s^0, y_s^0))$ is Markovian and

\[
\alpha(\eta_t^0, y_t^0) = A^0(\eta_t^0, \varphi, y_t^0) = \begin{cases} 
 r_1 \lambda_1(y_t^0), & \text{if } \eta_t^0 = e_1, \\
(1 - p_2) \lambda_2(y_t^0), & \text{if } \eta_t^0 = e_3.
\end{cases}
\]

Let $W_1, W_2, W_3$ and $W_4$ be r.v.’s with hazard rates $(p_1 + q_1)\lambda_1$, $\mu_2$, $p_2\lambda_2$, and $\mu_3$ respectively. Evidently the sojourn time of the process $(\eta_s^0)$ in state $e_i$ has the same distribution as $W_i$. Process $(\eta_s^0)$ is regenerative with regeneration state $e_1$ and

\[
E \xi = E \left( \int_0^1 r_1 \lambda_1(s) ds + \frac{q_1}{p_1 + q_1} \int_0^{W_3} (1 - p_2) \lambda_2(s) ds \right)
\]

\[
= \int_0^\infty r_1 \lambda_1(s) P(W_1 > s) ds + \frac{q_1}{p_1 + q_1} (1 - p_2) \int_0^\infty \lambda_2(s) P(W_3 > s) ds
\]

\[
= \frac{r_1}{p_1 + q_1} + \frac{q_1 (1 - p_2)}{(p_1 + q_1) p_2}.
\]

Quite similarly,

\[
\max(E\xi_1, E\xi_2) \leq E \int_0^{W_3} (1 - p_2) \lambda_2(s) ds = \frac{1 - p_2}{p_2}.
\]

It can be easily seen that

\[
\max(E\xi_0^1, E\xi_2^0) \leq E(W_2 + W_3 + W_4),
\]

\[
ES = EW_1 + EW_2 + \frac{q_1}{p_1 + q_1} E(W_3 + W_4).
\]
Recall that $\sigma$ is the first entrance time of process $(\eta_s^0)$ into $\mathcal{F} = \{\epsilon_5, \epsilon_6\}$. Let $\sigma_0$ be the recurrence time to state $\epsilon_1$. Then
\[
\beta_D = \mathbb{E} \int_0^S \exp \left( - \int_0^t \alpha(\eta_u^0) du \right) dt = \mathbb{E}(\sigma \wedge \sigma_0)
\]
\[
= \mathbb{E}W_1 + \frac{q_1}{p_1 + q_1} \mathbb{E}W_3 + \frac{q_1 p_2}{p_1 + q_1} \mathbb{E}(W_4 + W_2).
\]

Proposition 1.5.2 yields
\[
\rho_V \leq \epsilon \left( \frac{\epsilon + \delta_{\epsilon_0}}{\beta_D} \right), \quad \lambda_0 \leq \frac{\epsilon_0}{\beta_D},
\]
where
\[
\epsilon = \frac{r_1}{p_1 + q_1} + \frac{q_1(1 - p_2)}{(p_1 + q_1)p_2}, \quad \epsilon_0 = \frac{1 - p_2}{p_2}, \quad (5.2)
\]
\[
\delta = \mathbb{E}(W_2 + W_3 + W_4), \quad \beta_D = \mathbb{E}W_1 + \frac{q_1}{p_1 + q_1} \mathbb{E}W_3 + \frac{q_1 p_2}{p_1 + q_1} \mathbb{E}(W_4 + W_2),
\]
\[
\beta'_D = \mathbb{E}W_1 + \mathbb{E}W_2 + \frac{q_1}{p_1 + q_1} \mathbb{E}(W_3 + W_4).
\]

5.3 Numerical results

Let us consider the example described in Subsection 5.2. With $m_i$ denoting the mean sojourn time in state $\epsilon_i$ ($1 \leq i \leq 6$), we set
\[
m_1 = 1, \quad m_2 = \frac{1}{200}, \quad m_3 = 1, \quad m_4 = \frac{1}{100}, \quad m_5 = \frac{1}{10}, \quad m_6 = 1
\]
and suppose that $W_i$ (the sojourn time of the process $(\eta_i)$ in state $\epsilon_i$) has the Erlang distribution with parameter $k_i$ (and mean $m_i$) (that is, $W_i$ can be treated as a sum of $k_i$ i.i.d.r.v.'s with common exponential distribution having parameter $k_i/m_i$).

Case 1. Let
\[
p_1 = 0.5, \quad q_1 = 0.4, \quad r_1 = 0.1, \quad p_2 = 0.99.
\]
Then
\[
\lambda_V(\infty) = 7.3840 \cdot 10^{-2}, \quad r_V = 9.19 \cdot 10^{-2},
\]
\[
\lambda_{sup} = 8.0624 \cdot 10^{-2}, \quad \lambda_0 = 7.9511 \cdot 10^{-2},
\]
and
- for $k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1, k_5 = 1, k_6 = 1$
  \[
  \lambda(\infty) = 7.2432 \cdot 10^{-2}, \quad \rho_V = 1.94 \cdot 10^{-2};
  \]
- for $k_1 = 1, k_2 = 5, k_3 = 1, k_4 = 10, k_5 = 7, k_6 = 6$
  \[
  \lambda(\infty) = 7.2432 \cdot 10^{-2}, \quad \rho_V = 1.94 \cdot 10^{-2};
  \]
- for $k_1 = 3, k_2 = 10, k_3 = 12, k_4 = 50, k_5 = 10, k_6 = 15$
  \[
  \lambda(\infty) = 7.4459 \cdot 10^{-2}, \quad \rho_V = 8.32 \cdot 10^{-3};
  \]

Case 2. Let
\[
p_1 = 0.1, \quad q_1 = 0.5, \quad r_1 = 0.4, \quad p_2 = 0.99.
\]
Then
\[
\lambda_V(\infty) = 2.6858 \cdot 10^{-1}, \quad r_V = 3.85 \cdot 10^{-1},
\]
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\[ \lambda_{sup} = 3.7204 \cdot 10^{-1}, \quad \lambda_0 = 3.6557 \cdot 10^{-1}, \]

and

- for \( k_1 = 1, \ k_2 = 1, k_3 = 1, k_4 = 1, k_5 = 1, k_6 = 1 \)
  \[ \lambda(\infty) = 2.4331 \cdot 10^{-1}, \quad \rho_V = 1.04 \cdot 10^{-1}; \]

- for \( k_1 = 1, k_2 = 5, k_3 = 1, k_4 = 10, k_5 = 7, k_6 = 6 \)
  \[ \lambda(\infty) = 2.4332 \cdot 10^{-1}, \quad \rho_V = 1.04 \cdot 10^{-1}; \]

- for \( k_1 = 3, k_2 = 10, k_3 = 12, k_4 = 50, k_5 = 10, k_6 = 15 \)
  \[ \lambda(\infty) = 2.7024 \cdot 10^{-1}, \quad \rho_V = 6.15 \cdot 10^{-1}. \]

**Case 3.** Let

\[ p_1 = 0.1, \ q_1 = 0.5, \ r_1 = 0.4, \ p_2 = 0.99. \]

Then

\[ \lambda_V(\infty) = 2.9851 \cdot 10^{-1}, \quad \rho_V = 5.91 \cdot 10^{-1}, \]

\[ \lambda_{sup} = 4.7494 \cdot 10^{-1}, \quad \lambda_0 = 4.1115 \cdot 10^{-1}, \]

and

- for \( k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1, k_5 = 1, k_6 = 1 \)
  \[ \lambda(\infty) = 2.7586 \cdot 10^{-1}, \quad \rho_V = 8.21 \cdot 10^{-2}; \]

- for \( k_1 = 1, k_2 = 5, k_3 = 1, k_4 = 10, k_5 = 7, k_6 = 6 \)
  \[ \lambda(\infty) = 2.7586 \cdot 10^{-1}, \quad \rho_V = 8.21 \cdot 10^{-2}; \]

- for \( k_1 = 3, k_2 = 10, k_3 = 12, k_4 = 50, k_5 = 10, k_6 = 15 \)
  \[ \lambda(\infty) = 3.1128 \cdot 10^{-1}, \quad \rho_V = 4.10 \cdot 10^{-2}. \]

**Comments:** The results displayed above show that the influence of the order of the Erlang distributions on reliability characteristics is minor. But our estimate is very sensitive to the value of \( p_2. \) From other examples, one can see that the sensitivity with respect to \( p_1, q_1, \) and \( r_1 \) is also high. This can be seen from equation (5.2). In our examples, \( \lambda(\infty) \leq \lambda_V(\infty) \) for small values of \( k_i \)'s, but not for large values. In addition, the bounding by \( \lambda_0 \) gives better results than by \( \lambda_{sup}. \)

**References**


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