ON A MILD SOLUTION OF A SEMILINEAR FUNCTIONAL-DIFFERENTIAL EVOLUTION NONLOCAL PROBLEM

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The existence, uniqueness, and continuous dependence of a mild solution of a nonlocal Cauchy problem for a semilinear functional-differential evolution equation in a general Banach space are studied. Methods of a \( C_0 \) semigroup of operators and the Banach contraction theorem are applied. The result obtained herein is a generalization and continuation of those reported in references [2-8].

Key words: Abstract Cauchy Problem, Evolution Equation, Functional-Differential Equation, Nonlocal Condition, Mild Solution, Existence and Uniqueness of the Solution, Continuous Dependence of the Solution, a \( C_0 \) Semigroup, the Banach Contraction Theorem.

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1. Introduction

In this paper we study the existence, uniqueness, and continuous dependence of a mild solution of a nonlocal Cauchy problem for a semilinear functional-differential evolution equation. Methods of functional analysis concerning a \( C_0 \) semigroup of operators and the Banach theorem about the fixed point are applied. The nonlocal Cauchy problem considered here is of the form:

\[ u'(t) + Au(t) = f(t, u_t), \quad t \in [0, a], \quad (1.1) \]

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where \( 0 < t_1 < \ldots < t_p \leq a \) \((p \in \mathbb{N})\); \(-A\) is the infinitesimal generator of a \(C_0\) semigroup of operators on a general Banach space; \(f, g\) and \(\phi\) are given functions satisfying some assumptions, and \(u_t(s) = u(t + s)\) for \(t \in [0, a], s \in [-r, 0]\).

Theorems about the existence, uniqueness, and stability of solutions of differential and functional-differential abstract evolution Cauchy problems were studied previously by Byszewski and Lakshmikantham [2], by Byszewski [3-8], and by Lin and Liu [10]. The result obtained herein is a generalization and continuation of those reported in references [2-8].

If the case of the nonlocal condition considered in this paper is reduced to the classical initial condition, the result of the paper is reduced to previous results of Hale [9], Thompson [11], and Akca, Shakhmurow and Arslan [1] on the existence, uniqueness, and continuous dependence of the functional-differential evolution Cauchy problem.

2. Preliminaries

We assume that \(E\) is a Banach space with norm \(\| \cdot \|\); \(-A\) is the infinitesimal generator of a \(C_0\) semigroup \(\{T(t)\}_{t \geq 0}\) on \(E\), \(D(A)\) is the domain of \(A\); and

\[
0 < t_1 < \ldots < t_p \leq a \quad (p \in \mathbb{N})
\]

and

\[
M := \sup_{t \in [0, a]} \|T(t)\|_{BL(E,E)}.
\]

In the sequel the operator norm \(\| \cdot \|_{BL(E,E)}\) will be denoted by \(\| \cdot \|\).

For a continuous function \(w: [0, a] \times C([-r, 0], E) \to E\), we denote by \(w_t\) a function belonging to \(C([-r, 0], E)\) given by the formula

\[
w_t(s) = w(t + s) \text{ for } t \in [0, a], \quad s \in [-r, 0].
\]

Let \(f: [0, a] \times C([-r, 0], E) \to E\). We require the following assumptions:

**Assumption \((A_1)\):** For every \(w \in C([-r, a], E)\) and \(t \in [0, a]\),

\[
f(t, w_t) \in C([0, a], E).
\]

**Assumption \((A_2)\):** There exists a constant \(L > 0\) such that:

\[
\|f(t, w_t) - f(t, \bar{w}_t)\| \leq L \|w - \bar{w}\|_{C([-r, t], E)}
\]

for \(w, \bar{w} \in C([-r, a], E), \quad t \in [0, a]\).

Let \(g: C([-r, 0], E)^p \to C([-r, 0], E)\). We apply the assumption:

**Assumption \((A_3)\):** There exists a constant \(K > 0\) such that:

\[
\|(g(w_{t_1}, \ldots, w_{t_p}))(s) - (g(\bar{w}_{t_1}, \ldots, \bar{w}_{t_p}))(s)\| \leq K \|w - \bar{w}\|_{C([-r, a], E)}
\]

for \(w, \bar{w} \in C([-r, a], E), \quad s \in [-r, 0]\).
Moreover, we require the assumption:

**Assumption (A4):** $\phi \in C([-r,0], E)$.

A function $u \in C([-r,a], E)$ satisfying the conditions:

(i) \[ u(t) = T(t)\phi(0) - T(t) \left[ (g(u_{t_1}, \ldots, u_{t_p}))(0) \right] \]

\[ + \int_0^t T(t-s)f(s, u_s)ds, \quad t \in [0,a], \]

(ii) \[ u(s) + (g(u_{t_1}, \ldots, u_{t_p}))(s) = \phi(s), \quad s \in [-r,0), \]

is said to be a **mild solution** of the nonlocal Cauchy problem (1.1)-(1.2).

### 3. Existence and Uniqueness of a Mild Solution

**Theorem 3.1:** Assume that the functions $f, g, \phi$ satisfy Assumptions (A1)-(A4). Additionally, suppose that:

\[ M(aL + K) < 1. \quad (3.1) \]

Then the nonlocal Cauchy problem (1.1)-(1.2) has a unique mild solution.

**Proof:** Introduce an operator $F$ on the Banach space $C([-r,a], E)$ by the formula:

\[ (Fw)(t) = \begin{cases} 
\phi(t) - (g(w_{t_1}, \ldots, w_{t_p}))(t), & t \in [-r,0), \\
T(t)\phi(0) - T(t) \left[ (g(w_{t_1}, \ldots, w_{t_p}))(0) \right] \\
+ \int_0^t T(t-s)f(s, w_s)ds, & t \in [0,a], 
\end{cases} \quad (3.2) \]

where $w \in C([-r,a], E)$.

It is easy to see that

\[ F:C([-r,a], E)\rightarrow C([-r,a], E). \quad (3.2) \]

Now, we will show that $F$ is a contraction on $C([-r,a], E)$. For this purpose consider two differences:

\[ (Fw)(t) - (F\tilde{w})(t) = (g(\tilde{w}_{t_1}, \ldots, \tilde{w}_{t_p}))(t) - (g(w_{t_1}, \ldots, w_{t_p}))(t) \quad (3.3) \]

for $w, \tilde{w} \in C([-r,a], E), \quad t \in [-r,0)$

and

\[ (Fw)(t) - (F\tilde{w})(t) = T(t) \left[ (g(\tilde{w}_{t_1}, \ldots, \tilde{w}_{t_p}))(0) - (g(w_{t_1}, \ldots, w_{t_p}))(0) \right] \\
+ \int_0^t T(t-s)f(s, w_s)ds - f(s, \tilde{w}_s)ds \quad (3.4) \]

for $w, \tilde{w} \in C([-r,a], E), \quad t \in [0,a]$.

From (3.3) and Assumption (A3):
\[(Fw)(t) - (\tilde{F}\tilde{w})(t) \leq K \| w - \tilde{w} \|_{C([-r,a], E)} \]

for \( w, \tilde{w} \in C([-r,a], E), \ t \in [-r,0). \)

Moreover, by (3.4), (2.1), Assumption (A2), and Assumption (A3):

\[
(Fw)(t) - (F\tilde{w})(t) \\ \leq T(t) \| (g(w_{t_1}, \ldots, w_{t_p}))(0) - (g(\tilde{w}_{t_1}, \ldots, \tilde{w}_{t_p}))(0) \| \\ + \int_0^t \| T(t - s) \| f(s, w_s) - f(s, \tilde{w}_s) \| \, ds \\ \\ \leq M \| w - \tilde{w} \|_{C([-r,a], E)} + ML \int_0^t \| w - \tilde{w} \|_{C([-r,s], E)} ds \\ \leq M(aL + K) \| w - \tilde{w} \|_{C([-r,a], E)}
\]

for \( w, \tilde{w} \in C([-r,a], E), \ t \in [0,a]. \)

Formulas (3.5) and (3.6) imply the inequality

\[
\| Fw - F\tilde{w} \|_{C([-r,a], E)} \leq q \| w - \tilde{w} \|_{C([-r,a], E)}
\]

for \( w, \tilde{w} \in C([-r,a], E), \)

where \( q = M(aL + K). \)

Since, from (3.1), \( q \in (0,1), \) then (3.7) shows that \( F \) is a contraction on \( C([-r,a], E). \) Consequently, by (3.2) and (3.7), operator \( F \) satisfies all the assumptions of the Banach contraction theorem. Therefore, in space \( C([-r,a], E) \) there is only one fixed point of \( F \) and this point is the mild solution of the nonlocal Cauchy problem (1.1)-(1.2).

The proof of Theorem 3.1 is complete.

4. Continuous Dependence of a Mild Solution

**Theorem 4.1:** Suppose that the functions \( f \) and \( g \) satisfy Assumptions (A1)-(A3) and \( M(aL + K) < 1. \) Then, for each \( \phi_1, \phi_2 \in C([-r,0], E), \) and for the corresponding mild solutions \( u_1, u_2 \) of the problems

\[
\begin{align*}
&u'(t) + Au(t) = f(t, u_t), \quad t \in [0,a], \\
&u(s) + (g(u_{t_1}, \ldots, u_{t_p}))(s) = \phi_i(s), \quad s \in [-r,0] \quad (i = 1,2),
\end{align*}
\]

the inequality

\[
\| u_1 - u_2 \|_{C([-r,a], E)} \leq Me^{aML} \left( \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + K \| u_1 - u_2 \|_{C([-r,a], E)} \right)
\]

(4.2)
is true.

Additionally, if \( K < \frac{1}{Me^{aML}} \), then
\[
\| u_1 - u_2 \|_{C([-r,a], E)} \leq \frac{Me^{aML}}{1 - KMe^{aML}} \| \phi_1 - \phi_2 \|_{C([-r,0], E)}
\]  
(4.3)

**Proof:** Let \( \phi_i \) \( (i = 1, 2) \) be arbitrary functions belonging to \( C([-r,0], E) \), and let \( u_i \) \( (i = 1, 2) \) be the mild solutions of problems (4.1). Consequently,
\[
 u_1(t) - u_2(t) = T(t)[\phi_1(0) - \phi_2(0)]
\]
\[
 - T(t) \left[ (g((u_1)_{t_1}, \ldots, (u_1)_{t_p}))(0) - (g((u_2)_{t_1}, \ldots, (u_2)_{t_p}))(0) \right]
\]
\[
 + \int_0^t T(t-s)[f(s, (u_1)_s) - f(s, (u_2)_s)]ds \text{ for } t \in [0,a],
\]
and
\[
u_1(t) - u_2(t) = \phi_1(t) - \phi_2(t)
\]
\[
 + (g((u_2)_{t_1}, \ldots, (u_2)_{t_p}))(t) - (g((u_1)_{t_1}, \ldots, (u_1)_{t_p}))(t)
\]
for \( t \in [-r,0) \).

From (4.4), (2.1), Assumption (A2) and Assumption (A3):
\[
\| u_1(\tau) - u_2(\tau) \| \leq M \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + MK \| u_1 - u_2 \|_{C([-r,a], E)}
\]
\[
+ ML \int_0^\tau \| u_1 - u_2 \|_{C([-r,s], E)} ds
\]
\[
\leq M \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + MK \| u_1 - u_2 \|_{C([-r,a], E)}
\]
\[
+ ML \int_0^t \| u_1 - u_2 \|_{C([-r,s], E)} ds \text{ for } 0 \leq \tau \leq t \leq a.
\]

Therefore,
\[
\sup_{\tau \in [0,t]} \| u_1(\tau) - u_2(\tau) \|
\leq M \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + MK \| u_1 - u_2 \|_{C([-r,a], E)}
\]
\[
+ ML \int_0^t \| u_1 - u_2 \|_{C([-r,s], E)} ds \text{ for } t \in [0,a].
\]
(4.6)

Simultaneously, by (4.5) and Assumption (A3):
\[ \| u_1(t) - u_2(t) \| \leq \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + K \| u_1 - u_2 \|_{C([-r,a], E)} \]
for \( t \in [-r,0) \).

(4.7)

Since \( M \geq 1 \), formulas (4.6) and (4.7) imply:

\[ \| u_1 - u_2 \|_{C((-r,t], E)} \leq M \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + MK \| u_1 - u_2 \|_{C([-r,a], E)} \]
\[ + ML \int_0^t \| u_1 - u_2 \|_{C([-r,s], E)} ds \text{ for } t \in [0,a]. \]

(4.8)

From (4.8) and Gronwall's inequality:

\[ \| u_1 - u_2 \|_{C([-r,a], E)} \leq \left[ M \| \phi_1 - \phi_2 \|_{C([-r,0], E)} + MK \| u_1 - u_2 \|_{C([-r,a], E)} \right] e^{aML}. \]

Therefore, (4.2) holds. Finally, inequality (4.3) is a consequence of inequality (4.2). The proof of Theorem 4.1 is complete.

**Remark 4.1:** If \( K = 0 \), inequality (4.2) is reduced to the classical inequality

\[ \| u_1 - u_2 \|_{C([-r,a], E)} \leq Me^{aML} \| \phi_1 - \phi_2 \|_{C([-r,0], E)}, \]

which is characteristic for the continuous dependence of the semilinear functional-differential evolution Cauchy problem with the classical initial condition.

5. Remarks

1. Let

\[ 0 < t_1 < \ldots < t_p \leq a \quad (p \in \mathbb{N}). \]

Theorems 3.1 and 4.1 can be applied for \( g \) defined by the formula:

\[ (g(w_{t_1}, \ldots, w_{t_p}))(s) = \sum_{k=1}^p c_k w(t_k + s) \text{ for } w \in C([-r,a], E), \quad s \in [-r,0], \]

where \( c_k(k = 1, \ldots, p) \) are given constants such that

\[ M \left( aL + \sum_{k=1}^p |c_k| \right) < 1. \]

(5.1)

2. Let

\[ 0 < t_1 < \ldots < t_p \leq a \quad (p \in \mathbb{N}) \]

and let \( \epsilon_k(k = 1, \ldots, p) \) be given positive constants such that:

\[ 0 < t_1 - \epsilon_1 \text{ and } t_{k-1} < t_k - \epsilon_k \quad (k = 2, \ldots, p). \]
Theorems 3.1 and 4.1 can be applied for $g$ defined by the formula:

\[
(g(w_{t_1}, \ldots, w_{t_p}))(s) = \sum_{k=1}^{p} \frac{c_k}{\epsilon_k} \int_{t_k - \epsilon_k}^{t_k} w(\tau + s) d\tau
\]

for $w \in C([-r,a], E)$, $s \in [-r,0]$, where $c_k$ $(k = 1, \ldots, p)$ are given constants satisfying condition (5.1). Indeed,

\[
\| (g(w_{t_1}, \ldots, w_{t_p}))(s) - (g(\tilde{w}_{t_1}, \ldots, \tilde{w}_{t_p}))(s) \|
\]

\[
= \| \sum_{k=1}^{p} \frac{c_k}{\epsilon_k} \int_{t_k - \epsilon_k}^{t_k} [w(\tau + s) - \tilde{w}(\tau + s)] d\tau \|
\]

\[
\leq \left( \sum_{k=1}^{p} |c_k| \right) \| w - \tilde{w} \| C([-r,a], E) \text{ for } s \in [-r,0].
\]

References


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