Textures and Microstructures, Vol. 30, pp. 1–42
Reprints available directly from the publisher
Photocopying permitted by license only

GEOMETRICAL FOUNDATIONS OF TEXTURE ANALYSIS. GEODESIC CURVES AND MOTIONS IN THE GROUP SPACE OF THREE-DIMENSIONAL ROTATIONS

V.P. YASHNIKOV [a,*] and H.J. BUNGE [b]

[a] *Institute of Solid State Physics, Russian Academy of Sciences, Chernogolovka 142 432, Moscow District, Russia;* [b] *Department of Physical Metallurgy, TU Clausthal, Germany*

(Received 10 January 1997)

Principal concepts and selected results relating to the inner geometry of the three-dimensional rotation group SO(3) are presented in a form which is appropriate for further applications to various problems of texture analysis. Starting from the basic concepts of regular and piecewise regular curves in the group space SO(3) we consider the functional of the angular length and introduce further geodesic curves. It is shown that the geodesics can be fully characterized, in the group-theoretical terms, as cosets of all possible one-parametric subgroups in the space SO(3). Two kinds of parallelism between geodesics in the group space are discussed as well as related congruences. Geodesic curves are characterized also in terms of their constitutive vectors. The related transformational rules under motions are obtained. The geometrical structure of general motions and non-euclidean rotations of the space SO(3) is described on the base.

Keywords: Crystallographic textures; Inner geometry; Angular length of a curve; Geodesic curves; Left and right parallelism; Constitutive vectors; Motions; Non-euclidean rotations; Geometric theory of textures; Symmetries; Group-theoretical approach

INTRODUCTION

The inner geometry of the group space SO(3) of three-dimensional rotations is the basis of texture analysis similarly to that as the conventional euclidean geometry is the basis of the classical geometrical

* Corresponding author.

crystallography. According to this "correspondence principle" all facts related to the basic geometrical constructions of three-dimensional texture analysis can be, and thus should be, expressed in terms of the non-euclidean inner geometry of the space SO(3) and its algebraic group structure. The realization of this program was begun in our previous paper (see Yashnikov and Bunge, 1995) in connection with the reduction problem for the orientation space of a crystallographic texture. It was introduced, in particular, in the mentioned paper some basic concepts such as the angular distance function between arbitrary rotations, left and right translations, inner automorphisms, motions of general form, and inversions of the group space SO(3). Also, a general procedure for the construction of Dirichlet–Voronoi partitions associated with an arbitrary proper point crystallographic group was given.

Further advances to an exhaustive geometrical characterization of Dirichlet–Voronoi domains (or primitive orientation cells in other terminology) as polyhedra in the non-euclidean space SO(3) require the introduction of geodesic curves together with a more detailed analysis, on this basis, of the structure of motions of general form in the space SO(3) and especially the so-called non-euclidean rotations of this space. The restrictions which have to be fulfilled for any point of a primitive orientation polyhedron of an arbitrary crystallographic texture should be formulated in terms of geodesic surfaces of the space SO(3). However, the latter surfaces can only be introduced and studied in detail on the basis of results concerning geodesic curves in this space. In other words, the "habitus" of any primitive orientation polyhedron can be described in terms of geodesic curves and geodesic surfaces of the group space SO(3).

Let us make, in addition to that, the following important remark. In the case that the polycrystalline material under consideration has a crystal lattice belonging to the triclinic crystal class the related proper point group contains no other elements except the identity rotation e, and thus any reduction procedure of the group space SO(3) for constructing the true orientation space is not necessary. The true orientation space for a texture of such kind is identical with the whole group space SO(3) if there are no statistical symmetries. Thus studying the inner geometry of the group SO(3) provides us with informations about the true orientation space for a crystallographic texture "without symmetries". If, on the contrary, the proper point group of the crystal lattice contains non-identical transformations, the related true orientation space being locally (i.e. in the neighborhood of an arbitrary point) isometric to the space SO(3), differs geometrically and topologically from it in the whole. The passage from the inner geometry of the space SO(3) to that for the related true orientation space and, in particular, interrelations between geodesic curves in these spaces will be considered separately in our future publications.

So, according to the above, the present paper provides an extensive explanation for the geometry of geodesic curves in the space SO(3) in a form which is appropriate for further applications to three-dimensional texture analysis. In addition to that, two concluding sections of the paper are focused on the analysis of interrelations between geodesics and motions of the space SO(3). Also, we discuss in detail the structure of the so-called non-euclidean rotations of general form, which play, for the group space SO(3), the same role that conventional rotations play for the euclidean space. The results obtained will be used for a geometrical classification of the motions which leave the orientation distribution function of a texture invariant in the presence of macroscopic (statistical) symmetries. Concerning symmetry properties of a texture in connection with the harmonic analysis of orientation distribution see Bunge (1982). Applications of geodesic curves to other problems in texture analysis such as texture goniometry and determination of grain orientations from channeling pattern data will be presented elsewhere.

Finally, it should be mentioned that the geometrical form of the principal primitive orientational cell (i.e. the Dirichlet–Voronoi domain having the identical element e as its center according to our terminology) was studied recently in a series of papers, in its dependence on the crystal class using different methods of parametrization of the rotation group SO(3). So, in particular, the problem was solved by Frank (1987, 1988, 1992) and by Heinz and Neumann (1991) using Rodrigues parameters introduced earlier in the context of texture analysis by Bonnet (1980). Matthies *et al.* (1990) analyzed the problem in terms of the Eulerian angles. The problem was considered also by Gertsman (1989) using computer methods and by Ibe (1993) on the basis of the quaternion formalism which was first proposed for the description of crystal orientations by Grimmer (1974). All these papers provide a rich information about various parametric representations of the Dirichlet–Voronoi domains for different proper crystallographic point groups, and we consider the totality of all the above-mentioned papers as a start position and an excellent motivation for a more systematic treatment of foundations of texture analysis from the point of view of the concepts of the inner geometry of the group space SO(3) in its invariant form using no particular parametrization but basing itself on the group-theoretical and topological argumentation.

LENGTH OF A CURVE IN THE GROUP SPACE SO(3)

The concept of the angular distance between rotations, which is invariant with respect to all possible left and right translations of the group space SO(3), is the basis of the approach, developed in this paper. The concept of the angular length of an arbitrary, sufficiently regular curve in the group space of rotations should be introduced independently of our intuitive understanding of this concept for a curve in the conventional three-dimensional euclidean space (i.e. specimen space in the context of texture analysis).

We will say that a continuous curve

$$g = g(t), \quad \alpha \leq t \leq \beta, \tag{1}$$

in the group space is given, if a rotation $g(t)$ is defined for each particular value of the variable parameter t running through the closed interval $\alpha \leq t \leq \beta$. The additional continuity assumption means that the increment of the angular distance

$$\mathrm{dist}(g(t); g(t + \Delta t)) \tag{2}$$

is infinitely small for any point t, if the related increment Δt of the parameter is infinitely small. It should be mentioned that any other group-invariant distance function in the space of rotations may be equivalently used instead of the angular distance in the previous definition. So, the totality of all possible continuous curves in the space SO(3) does not depend on the choice of the particular group-invariant distance. In particular, we may use in Eq. (2) the trace distance function, which was introduced in our previous paper (Yashnikov and Bunge, 1995). If the mapping (1) is biunivoque in the sense that different rotations correspond to any two different values of the parameter t varying within the open interval $\alpha < t < \beta$, this will be called a curve without self-intersections. As a rule, we will deal in all further considerations with curves of such kind. A curve of the form (1) will be called a closed one if, in addition, the following condition is fulfilled:

$$g(\alpha) = g(\beta). \tag{3}$$

In the opposite case we speak about an open (non-closed) curve having the rotation $g(\alpha)$ as its origin and, respectively, the rotation $g(\beta)$ as its end.

Let the interval $\alpha \leq t \leq \beta$ be subdivided with the help of an arbitrary increasing sequence of points which will be denoted as

$$\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta; \tag{4}$$

then the related rotations

$$g(t_0) = g(\alpha), g(t_1), \ldots, g(t_{n-1}), g(t_n) = g(\beta) \tag{5}$$

form a point sequence on the curve under consideration. We may make up the sum of angular distances between successive pairs of these rotations in the form

$$\sum_{i=1}^{n} \mathrm{dist}(g(t_i); g(t_{i-1})) \tag{6}$$

and denote the magnitude (6), which depends evidently on the subdivision (4), by the symbol

$$L(t_0, t_1, \ldots, t_{n-1}, t_n). \tag{7}$$

The latter magnitude has been interpreted, in an intuitively clear manner, as the "approximate" angular length of the curve under consideration. If there exists the limit L of the magnitude (7) by infinitely refining the subdivisions of the form (4), it will be called the angular length of the curve $g = g(t)$ under consideration.

It is seen that we do not use explicitly, by defining the length of a curve in the space SO(3), any approximation of the curve by the so-called "broken" lines. However, as it will be evident from further consideration, the length of a curve may be obtained as the limit of lengths of piecewise geodesic lines which approximate the given curve.

In addition to the above we will consequently distinguish the curve $g = g(t)$, $\alpha \leq t \leq \beta$, with its particular parametrization from the related one-dimensional continuum constituted by all possible rotations $g(t)$ when the variable parameter t runs through the closed interval $\alpha \leq t \leq \beta$. It should be mentioned that there are infinitely many different ways to parametrize continuously the point continuum of any given curve. By any reparametrization of such kind we obtain, according to our definition, a new curve having the same totality of its points as the original one. It should be emphasized that the length of a curve does not depend on the method of its parametrization, thus the length of an arbitrary curve is a metric characteristic of the one-dimensional continuum of its points in the space of rotations SO(3). In practice, certain preference that we may show for one method of parametrization in comparison with all others may be motivated by some simplifications, which could be reached by computations of the length of a curve.

The existence problem for the length L of an arbitrary curve is very complicated, since there are examples of the so-called "exotic" curves each point of which is a corner point. It is sufficient, however, for our particular purposes to restrict ourselves by consideration of the class including only all possible regular curves. Thereby a curve given in the parametric form (1) is called a regular one, if its image in the nine-dimensional euclidean space of coefficients by the orthogonal matrix representation given in Eq. (1) of our previous paper possesses a continuously varying tangent vector, when the parameter t runs through its interval of definition. A curve is called piecewise regular, if the interval $\alpha \leq t \leq \beta$ can be subdivided in a finite system of mutually disjoint subintervals, so that in the interior of each of them the curve is regular. So, the continuity condition for the tangent vector is not violated only for the ends of the intervals of the subdivision. As it may be rigorously shown with the help of some additional analytical considerations, the length exists for any regular or piecewise regular curve in the group space SO(3).

GEODESIC CURVES IN THE SPACE OF ROTATIONS AND THEIR CHARACTERIZATION IN TERMS OF THE CONSTITUTIVE VECTORS

Let $g=g(t)$, $\alpha \leq t \leq \beta$, be a regular curve in the group space SO(3). The curve is called a geodesic one, if for any two sufficiently near points $A_1=g(t_1)$ and $A_2=g(t_2)$ of this curve the related arc A_1A_2 (i.e. the curve $g=g(t)$, $t_1 \leq t \leq t_2$) possesses the minimal angular length in the class of all possible regular curves having A_1 and A_2 as their ends. Let $g=g(t)$, $\alpha \leq t \leq \beta$, and $h=h(t)$, $\mu \leq t \leq \nu$, be two geodesic curves in the space SO(3). Also it is assumed that the inequalities $\mu \leq \alpha$ and $\beta \leq \nu$ are fulfilled. The second curve is called the continuation (or the extension) of the first one if the following condition is satisfied:

$$g(t) = h(t) \tag{8}$$

when the variable parameter t runs through the interval $\alpha \leq t \leq \beta$. A geodesic curve is called a complete geodesic one if it admits no geodesic extension. The problem of an explicit characterization of all possible complete geodesics in a non-euclidean space of general nature is very complicated. The degree of its complexity depends, as it is known, both on the complexity of the interior topological structure of the space under consideration and on the analytical structure of the particular distance function, which we have used. This problem has been treated, as a rule, by variational methods, which permit to obtain ordinary differential equations for geodesic curves. These latter equations should be explicitly integrated in conclusion of the procedure. An other and essentially more simplified approach is possible in our particular case of the space SO(3), provided by the angular distance function. Thereby the following two important reasons should be taken in account. This space is topologically homogeneous, i.e. its local topological structure in the neighborhood of an arbitrary point of it is the same as that in the neighborhood of any other point. Besides that, the invariance property of the angular distance function with respect to all possible left and right translations of the group space SO(3) implies its geometrical homogeneity. This latter property means that the angular geometry in any spherical neighborhood of an arbitrary point g_1 is equivalent to that in the spherical neighborhood of the same radius for any other point g_2. The equivalence may be established, for example, with the help of the related left or right translations $L(g_2 g_1^{-1})$ or $R(g_2^{-1} g_1)$.

After these preliminary remarks we will characterize all possible complete geodesic curves passing through the point $g = e$, where e denotes, according to the notations of our previous paper, the identity rotation of the specimen space, i.e. the identity element of the group SO(3). First of all we will show that any one-parametric subgroup of this group is a complete geodesic curve. Let

$$G(\bar{c}; \bar{c}) = G(\bar{c}) \cup G(-\bar{c}) \tag{9}$$

be the one-parametric subgroup of all possible rotations about the axis passing through an arbitrarily chosen vector $\bar{c}$ of unit euclidean length in the specimen space. Thereby we have denoted by $G(\bar{c})$ the totality of all possible rotations $g(\bar{c}; \varphi)$ having the rotation angle in the interval $0 \le \varphi \le \pi$. Let further two rotations

$$g_1 = g(-\bar{c}; \varphi_1), \qquad g_2 = g(\bar{c}; \varphi_2) \tag{10}$$

be chosen so that the following restriction is fulfilled:

$$\varphi_1 + \varphi_2 \le \pi. \tag{11}$$

We will denote for instance by $\overline{g_1 g_2}$ the piece of the one-parametric subgroup $G(\bar{c}; \bar{c})$ containing both the rotations

$$g = g(-\bar{c}; \varphi), \quad 0 \le \varphi \le \varphi_1, \tag{12}$$

and those of the form

$$g = g(\bar{c}; \varphi), \quad 0 \le \varphi \le \varphi_2. \tag{13}$$

Besides that we obtain a convenient method for parametrization of the one-dimensional continuum $\overline{g_1 g_2}$ if we put, for an arbitrary one of its points g,

$$g = g(t), \quad 0 \le t \le \varphi_1 + \varphi_2, \tag{14}$$

where the related value t of the parameter is defined as the rotation angle of the product $g_1^{-1} g$. So we have, in particular,

$$g(0) = g_1, \qquad g(\varphi_1) = e, \qquad g(\varphi_1 + \varphi_2) = g_2. \tag{15}$$

Let us consider now the subdivision of the interval $0 \leq t \leq \varphi_1 + \varphi_2$ formed by an arbitrarily chosen sequence of the parameter values

$$t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = \varphi_1 + \varphi_2. \tag{16}$$

The related approximate length

$$L(t_0, t_1, \ldots, t_{n-1}, t_n) = \sum_{i=1}^{n} \mathrm{dist}(g(t_i);\, g(t_{i-1})) \tag{17}$$

of the curve given by Eq. (14) may be explicitly computed if we take into account the following equalities:

$$\mathrm{dist}(g(t_i);\, g(t_{i-1})) = t_i - t_{i-1}, \quad i = 1, \ldots, n. \tag{18}$$

After substitution of the latter expressions in Eq. (18) we obtain

$$L(t_0, t_1, \ldots, t_{n-1}, t_n) = t_n - t_0 = \varphi_1 + \varphi_2. \tag{19}$$

By virtue of the restriction (11) we may rewrite the latter equality in the following equivalent form:

$$L(t_0, t_1, \ldots, t_{n-1}, t_n) = \mathrm{dist}(g_1; g_2). \tag{20}$$

So we may see that the approximate length of the curve (14) is equal to the angular distance between its ends g_1, g_2 and independent of the particular method, which was used for the subdivision of the interval $0 \leq t \leq \varphi_1 + \varphi_2$. Therefore the same equality is valid for the true angular length of this curve

$$L(\overline{g_1 g_2}) = \mathrm{dist}(g_1; g_2). \tag{21}$$

On the other hand, let

$$h = h(s), \quad \alpha \leq s \leq \beta, \tag{22}$$

be an arbitrarily chosen regular curve having the rotations g_1 and g_2 as its ends, so that the conditions

$$h(\alpha) = g_1, \qquad h(\beta) = g_2 \tag{23}$$

are satisfied. If the variation range of the parameter s is subdivided with the help of any finite sequence

$$s_0 = \alpha < s_1 < \cdots < s_{n-1} < s_n = \beta, \tag{24}$$

the related point system on the curve (22)

$$g_1 = h(s_0), h(s_1), \ldots, h(s_{n-1}), h(s_n) = g_2 \tag{25}$$

must obey the following generalized triangle inequality:

$$\operatorname{dist}(g_1; g_2) \leq \sum_{i=1}^{n} \operatorname{dist}(h(s_{i-1}); h(s_i)), \tag{26}$$

which may be easily derived by the method of mathematical induction from the basic triangle inequality that was considered in our previous paper. The inequality (26) demonstrates, in particular, the fact that the approximate angular length of any regular curve cannot be smaller than the angular distance between its ends. Therefore, if we pass in the inequality (26) to the limit by infinitely refining the subdivision of the interval $\alpha \leq t \leq \beta$, we obtain the following natural inequality:

$$\operatorname{dist}(g_1; g_2) \leq L(H), \tag{27}$$

where H denotes the curve (22). So we may conclude that the angular length for any regular curve connecting the points g_1 and g_2 cannot be smaller than the angular distance between these points. In the regular case, when the condition

$$\varphi_1 + \varphi_2 < \pi \tag{28}$$

is satisfied, some additional considerations, which we however omit here, give the possibility to demonstrate that the true equality in Eq. (27) is realized if and only if the curve (22) is obtained from the arc (14) by any reparametrization. Thus the arc (14) of the one-parametric subgroup $G(\bar{c}; \bar{c})$ is a shortest curve and hence is a unique one in the class of all possible regular curves with the ends g_1 and g_2. The exceptional case when

$$\varphi_1 + \varphi_2 = \pi \tag{29}$$

should be considered separately.

Let us mention that we have supposed in the previous considerations a special mutual disposition of the points g_1 and g_2 on the one-parametric subgroup $G(\bar{c}; .\bar{c})$. The related restrictions are given by Eqs. (10) and (11). The general case may be treated in the following manner. If g_1 and g_2 are chosen arbitrarily, we may observe that the following inequality is fulfilled:

$$\operatorname{dist}(g_1; g_2) \leq \pi. \tag{30}$$

Since any one-parametric subgroup is topologically equivalent to a circumference, the pair g_1 and g_2 under consideration divides it into two mutually complementary arcs. If, for example, $g_1 = e$ and $g_2 = g(\bar{c}; \pi)$ we obtain the partition given by Eq. (9). If we apply the above-stated relation, Eq. (21), to the arc $G(-\bar{c})$ and $G(\bar{c})$ we obtain the equality

$$L(G(-\bar{c})) = L(G(\bar{c})) = \pi. \tag{31}$$

Thus we may conclude that the total angular length for any one-parametric subgroup is equal to 2π:

$$L(G(\bar{c}; \bar{c})) = 2\pi. \tag{32}$$

Coming back to the general case we will suppose at first that the rotations g_1 and g_2 obey the strict inequality

$$\operatorname{dist}(g_1; g_2) < \pi. \tag{33}$$

The latter inequality shows, in particular, that the angular lengths for related mutually complementary arcs of the subgroup $G(\bar{c}; \bar{c})$ having g_1 and g_2 as their ends cannot be equal. Let us denote by $\overline{g_1 g_2}$ one of the two arcs, the length of which is smaller, then the same argumentation as above leads to the validity of Eq. (21) in any case when Eq. (33) is satisfied. Thus, we may conclude that any arc $\overline{g_1 g_2}$ of an arbitrary one-parametric subgroup $G(\bar{c}; \bar{c})$ is a shortest curve and a unique one in the class of all possible regular curves with the ends g_1 and g_2, if the angular length of this arc is strictly smaller than π.

In the case if the alternative condition

$$\operatorname{dist}(g_1; g_2) = \pi \tag{34}$$

is fulfilled, the related rotation

$$g_1^{-1}g_2 = g(\bar{c};\pi) \tag{35}$$

is evidently of the second order in the sense that the equality $g^2 = e$ is valid for this rotation. Thus, the left translation generated by the element, Eq. (35), being applied to any arc of the partition, transforms it onto its complementary arc. Therefore, the related angular lengths both are equal to π. In contrast with the case of Eq. (33) we obtain, by the restriction (34), two geometrically different shortest curves in the class of all possible regular curves having g_1 and g_2 as their ends. So, we may summarize all previous considerations in the following form. If $G(\bar{c};\bar{c})$ is an arbitrary one-parametric subgroup of the group SO(3) any of its arcs $\overline{g_1g_2}$ having an angular length to be not greater then π furnishes us with a shortest curve in the totality of all possible regular ones with the ends g_1 and g_2. Thus we may conclude that any one-parametric subgroup of SO(3) is a geodesic curve in this space. Since any $G(\bar{c};\bar{c})$ is a closed curve it cannot be extended. Therefore, any one-parametric subgroup is a complete geodesic curve in the group space SO(3).

One-parametric subgroups in the group SO(3) do not exhaust the variety of all possible geodesic curves of this space. In fact, we may make the following observation based on the homogenity property of the group space SO(3). Let

$$g = g(t), \quad \alpha \leq t \leq \beta, \tag{36}$$

be an arbitrary complete geodesic curve in this space. If we have taken two rotations h_1 and h_2, the curve

$$h(t) = h_1 g(t) h_2^{-1} = M(h_1;h_2)\, g(t), \tag{37}$$

any point of which is obtained from the corresponding point of the curve (36) by the motion $M(h_1;h_2)$, is also a complete geodesic one, since any motion of the group space SO(3) preserves the angular distance between rotations and, as a consequence, the angular length of any regular curve in this space as it follows immediately from the definition of the angular length given in the previous section. If, in particular, we apply an arbitrary motion $M(h_1;h_2)$ to an arbitrarily chosen one-parametric subgroup $G(\bar{c};\bar{c})$ we obtain evidently another complete geodesic curve passing through the point

$$g = M(h_1; h_2)e = h_1 h_2^{-1}. \tag{38}$$

We may introduce the so-called constitutive vectors

$$\bar{x} = h_2\bar{c}, \tag{39}$$

$$\bar{y} = h_1\bar{c}, \tag{40}$$

then the geodesic curve of the form

$$G(\bar{x};\bar{y}) = M(h_1; h_2)G(\bar{c};\bar{c}) \tag{41}$$

obtained from the one-parametric subgroup $G(\bar{c};\bar{c})$ with the help of the motion $M(h_1; h_2)$ may be characterized as the totality of all possible rotations each of which transforms $\bar{x}$ to $\bar{y}$. So we may write the latter observation in the following form:

$$G(\bar{x};\bar{y}) = \{g \in \mathrm{SO}(3)\colon g\bar{x} = \bar{y}\}. \tag{42}$$

Expression (42) represents the most general form of geodesic curves in the group space SO(3). In fact, if two vectors $\bar{x}$ and $\bar{y}$ of unit length in the specimen space are chosen arbitrarily, then the related curve $G(\bar{x};\bar{y})$, defined by Eq. (42), is a complete geodesic one. This statement may be proved in the following way. Let $\bar{c}$ be a unit vector of the specimen space, then we may choose (and even in infinitely many different ways) appropriate rotations h_1 and h_2 so that both Eqs. (39) and (40) are fulfilled. It means that the related motion $M(h_1; h_2)$ transforms the one-parametric subgroup $G(\bar{c};\bar{c})$ onto the curve $G(\bar{x};\bar{y})$. Thus we may conclude that any curve of the form (42) is a complete geodesic one. Let further

$$g = g(t), \quad \alpha \leq t \leq \beta, \tag{43}$$

be an arbitrary complete geodesic curve in the group space SO(3). We will denote by H the one-dimensional continuum of its points, then any left translation $L(h^{-1})$, where h belongs to the curve under consideration, being applied to H transforms it onto the geodesic curve $L(h^{-1})H$ of the parametric form

$$h^{-1}g = h^{-1}g(t), \quad \alpha \leq t \leq \beta, \tag{44}$$

which passes through the identity rotation e. Thus $L(h^{-1})H$ is a one-parametric subgroup of SO(3) and we may write

$$L(h^{-1})H = G(\bar{c}; \bar{c}) \tag{45}$$

for an appropriate unit vector $\bar{c}$ of the specimen space. The latter equality shows, in particular, that the curve H may be obtained if we apply the left translation $L(h)$ to the one-parametric subgroup $G(\bar{c}; \bar{c})$:

$$H = L(h)G(\bar{c}; \bar{c}) \tag{46}$$

and, as a consequence, H may be represented in the form

$$H = G(\bar{c}; h\bar{c}). \tag{47}$$

So we conclude that there exist, in the group space SO(3), no other complete geodesic curves except the ones of the form (42) when the vectors $\bar{x}$ and $\bar{y}$ run independently over the unit sphere of the specimen space. It should be mentioned, however, that any two inversion-symmetric pairs of the constitutive vectors, such as $\bar{x}, \bar{y}$ and $-\bar{x}, -\bar{y}$, represent evidently one and the same geodesic curve and we may write

$$G(\bar{x}; \bar{y}) = G(-\bar{x}; -\bar{y}). \tag{48}$$

As it will be seen from further analysis, two pairs of constitutive vectors $\bar{x}_1; \bar{y}_1$ and $\bar{x}_2; \bar{y}_2$ generate one and the same geodesic curve if and only if either the conditions

$$\bar{x}_1 = \bar{x}_2, \qquad \bar{y}_1 = \bar{y}_2 \tag{49}$$

are satisfied or the equalities

$$\bar{x}_1 = -\bar{x}_2, \qquad \bar{y}_1 = -\bar{y}_2 \tag{50}$$

are valid. In addition to that, an arbitrarily chosen geodesic curve $G(\bar{x}; \bar{y})$ is a one-parametric subgroup if and only if $\bar{x} = \bar{y}$.

Since any one-parametric subgroup in SO(3) has been represented in the form $G(\bar{c};\bar{c})$, where $\bar{c}$ runs over the spherical surface of unit radius of the specimen space, and any two antipodal unit vectors constitute one and the same subgroup, we can conclude that the variety of all possible complete geodesic curves passing through the point e of the group space SO(3) is topologically equivalent to the space of all possible straight lines having the origin O_s of the specimen space as their common point of incidence. It is sufficient to consider, in correspondence to any one-parametric subgroup $G(\bar{c};\bar{c})$, the non-oriented straight line $l(\bar{c})$ passing through the point O_s of the specimen space and containing both $\bar{c}$ and $-\bar{c}$. The term "non-oriented" means that a positive direction on $l(\bar{c})$ is not fixed. So we may see that the variety GEOD(e) of all possible complete geodesics containing the identity rotation e is topologically equivalent to the real two-dimensional projective space RP^2.

The same result is valid for the variety GEOD(g) of all possible geodesics passing through an arbitrary point g of the space SO(3) since GEOD(g) is exhausted by the curves of form (47), when, similarly to the above, the related constitutive vector $\bar{c}$ runs over the unit sphere of the specimen space. Thus, in general, we may write the following relation of topological equivalence:

$$\mathrm{GEOD}(g) \cong \mathrm{RP}^2. \tag{51}$$

The variety GEOD(SO(3)) of all possible complete geodesic curves in the group space SO(3) may be described in the following manner. Let us consider the cartesian product $\Omega \times \Omega$ of two copies of the unit sphere Ω in the specimen space. By virtue of relation (48) the latter space is "surplus" and thus it should be reduced with respect to the simplest of its symmetry groups which only includes the identity transformation $E_6 = E_3 \times E_3$ acting by the formula

$$E_6(\bar{x};\bar{y}) = (E_3\bar{x};E_3\bar{y}) = (\bar{x};\bar{y}) \tag{52}$$

and the six-dimensional euclidean inversion $I_6 = I_3 \times I_3$ that is defined by the relation

$$I_6(\bar{x};\bar{y}) = (I_3\bar{x};I_3\bar{y}) = (-\bar{x};-\bar{y}). \tag{53}$$

We consider the transformations E_6 and I_6 as the ones in the cartesian product of two copies of the specimen space. Analogously, the space $\Omega \times \Omega$ has been considered as a four-dimensional surface imbedded in this six-dimensional space given by the following system of equations:

$$x_1^2 + x_2^2 + x_3^2 = 1, \tag{54}$$

$$y_1^2 + y_2^2 + y_3^2 = 1, \tag{55}$$

where $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ are referred to the specimen coordinate system. After identifying antipodal pairs, such as $(\bar{x}; \bar{y})$ and $(-\bar{x}; -\bar{y})$, we obtain instead of $\Omega \times \Omega$, the reduced space $p(\Omega \times \Omega)$ which is a geometrical model of the variety GEOD(SO(3)). Thereby an arbitrary geodesic $G(\bar{x}; \bar{y})$ has been imaged by the antipodal pair $\{(\bar{x}; \bar{y}), I_6(\bar{x}; \bar{y})\}$ of its constitutive vectors.

Let us emphasize that in this section we have considered all possible geodesics as one-dimensional continua without using any particular parametrization. Some natural methods for parametrizing an arbitrary geodesic in the group space SO(3) will be discussed in one of the following sections.

LEFT AND RIGHT PARALLELISM OF GEODESIC CURVES. RELATED CONGRUENCES OF GEODESICS

As it is well known the conventional three-dimensional euclidean space (for example the specimen space in the context of texture analysis) admits only one method for the parallel transfer of various geometrical objects (such as straight lines, curves, figures, planes, surfaces, three-dimensional bodies and others). Thereby any object parallel to an initial one can be obtained by an appropriate spatial translation. So the translation group is the basis of the parallelism in this space. In contrast to the euclidean space the group space SO(3) possesses two different types of translational transformations – left and the right ones. According to that, there exist, in the space SO(3), two different types of parallelism – the left and the right ones. Both these types of parallelism are important in various aspects of three-dimensional texture analysis, especially for a clearer understanding of interrelations between the orientation space and its reciprocal space (i.e. the weighted average of reciprocal spaces of crystals of all possible orientations, when the related orientation distribution function is used as a statistical weight). Therefore, both types of parallelism will be briefly accounted in this section. Besides that, we will observe that an arbitrary complete geodesic curve in the space SO(3) may be represented as a left or right coset of an appropriate one-parametric subgroup.

Let H_1 and H_2 be two complete geodesic curves in the group space SO(3). The curve H_2 is called left-parallel to H_1 if there exists a left translation $L(h)$ that transforms H_1 onto H_2:

$$L(h)H_1 = H_2. \tag{56}$$

Since

$$L(e)H = H \tag{57}$$

for an arbitrary complete geodesic H we may conclude that any geodesic is left-parallel to itself. Let us observe, in addition, that relation (56) is evidently equivalent to the condition

$$L(h^{-1})H_2 = H_1. \tag{58}$$

Thus the property of H_2 to be left-parallel to H_1 implies the property of H_1 to be left-parallel to H_2. If three geodesic curves H_1, H_2, H_3 are chosen so that H_1 is left-parallel to H_2 and H_2 is left-parallel to H_3 then we may easily deduce, using the basic relation (56), the statement that H_1 is left-parallel to H_3. In fact, our assumptions mean that there exist two rotations h_1, h_2 of such kind that the following basic relations are fulfilled:

$$L(h_1)H_1 = H_2, \tag{59}$$

$$L(h_2)H_2 = H_3. \tag{60}$$

The multiplication rules for left translations given us

$$L(h_2 \cdot h_1) = L(h_2)L(h_1). \tag{61}$$

Thus, if we apply the translation (61) to the geodesic curve H_1 and take into account the conditions (59) and (60), we obtain the equality

$$L(h_2 \cdot h_1)H_1 = L(h_2) \cdot [L(h_1)H_1] = L(h_2)H_2 = H_3. \tag{62}$$

In other words, the property of geodesic curves to be left-parallel to one another establishes, from the set-theoretical point of view, a kind of the equivalence relation in the variety GEOD(SO(3)) of all possible complete geodesics. Being defined only for selected pairs of curves connected by transformations of the form (56), this equivalence relation is reflexive, symmetric, and transitive.

The above-mentioned general properties of the left parallelism give us the possibility to obtain more detailed information about left-parallel geodesics in the space of three-dimensional rotations. In fact, as it follows immediately from Eqs. (40) and (41), an arbitrary left translation $L(h)$ transforms a one-parametric subgroup $G(\bar{c};\bar{c})$ according to the formula

$$L(h)\,G(\bar{c};\bar{c}) = G(\bar{c};h\bar{c}). \tag{63}$$

Thus, any complete geodesic of the form given by Eq. (63) is left-parallel to the subgroup $G(\bar{c};\bar{c})$. If h in the latter equality runs through the group SO(3) and $\bar{c}$ is assumed to be fixed then the related vector $\bar{y}$ of the form

$$y = h\bar{c} \tag{64}$$

runs over the whole unit sphere Ω in the specimen space. Therefore we may conclude that any geodesic curve of the form $G(\bar{c};\bar{y})$ is left-parallel to the original one-parametric subgroup $G(\bar{c};\bar{c})$. By virtue of the symmetry and reflexivity of the left parallelism any two geodesic curves $G(\bar{c};\bar{y}_1)$ and $G(\bar{c};\bar{y}_2)$, both being left-parallel to the subgroup $G(\bar{c};\bar{c})$, are also left-parallel to one another.

A general criterion for the left parallelism in terms of constitutive vectors may be given in the following form. Two geodesic curves $G(\bar{x}_1;\bar{y}_1)$ and $G(\bar{x}_2;\bar{y}_2)$ in the group space SO(3) are left-parallel if and only if either the equality

$$x_1 = x_2 \tag{65}$$

is satisfied or the vectors $\bar{x}_1$ and $\bar{x}_2$ are antipodal, i.e. the following relation is valid:

$$\bar{x}_1 = -\bar{x}_2. \tag{66}$$

These two different conditions arise, as it follows from the considerations of the previous section, due to the possibility to represent any complete geodesic curve by two antipodal pairs of its consitutive vectors (see, for instance, Eq. (48)). In fact, if we deal with the geodesic $G(\bar{x};\bar{y}_1)$ and $G(-\bar{x};\bar{y}_2)$ we may write

$$G(-\bar{x};\bar{y}_2) = G(\bar{x};-\bar{y}_2), \tag{67}$$

then the left parallelism between the curves is evident.

Now we will consider in more detail the family of geodesic curves in the space SO(3) which are left-parallel to a given geodesic. Families of such kind in the space of rotations play the role which is analogous to that of families (or in other terms "congruences") of parallel straight lines in the conventional euclidean space.

Let $\bar{c}$ be a vector of unit length in the specimen space. If $\bar{c}$ is chosen arbitrarily and fixed, all possible geodesic curves $G(\bar{c};\bar{y})$ form, when the movable constitutive vector $\bar{y}$ runs over the whole spherical surface of the radius $R=1$, the family of those geodesics which are left-parallel to the related one-parametric subgroup $G(\bar{c};\bar{c})$. Any family of such kind will be denoted by the symbol $\mathrm{CONG_{left}}(\bar{c})$. The related one-parametric subgroup will be called the generating one. Let us observe some simple properties of the curves belonging to $\mathrm{CONG_{left}}(\bar{c})$. Any rotation belongs at least to one of the curves of $\mathrm{CONG_{left}}(c)$. In fact, if g is chosen arbitrarily, it is evidently a point of the geodesic $G(\bar{c};g\bar{c})$. Whenever two vectors $\bar{y}_1, \bar{y}_2$ are distinct the related curves $G(\bar{c};\bar{y}_1)$ and $G(\bar{c};\bar{y}_2)$ of the family $\mathrm{CONG_{left}}(\bar{c})$ cannot have any common point. So, all possible curves of an arbitrary family $\mathrm{CONG_{left}}(\bar{c})$ fill the whole group space of the three-dimensional rotations. Thereby any two curves of the family under consideration are mutually disjoint. In addition to that, the position of an arbitrary curve $G(\bar{c};\bar{y})$ in the space SO(3) depends regularly (i.e. continuously and, even more, differentiably) on the vector parameter $\bar{y}$ running over the unit sphere Ω of the specimen space, when the first constitutive vector $\bar{c}$ is fixed. We will call any family of the form

$\mathrm{CONG}_{\mathrm{left}}(\bar{c})$ the congruence of the geodesics, which are left-parallel to the related one-parametric subgroup $G(\bar{c};\bar{c})$.

It should be mentioned, in addition to the above, that an arbitrary curve $G(\bar{c};\bar{y})$ of the congruence $\mathrm{CONG}_{\mathrm{left}}(\bar{c})$ may be algebraically interpreted as a left coset of the one-parametric subgroup generating this congruence. In fact, if we choose an arbitrary rotation h belonging to $G(\bar{c};\bar{y})$ we obtain evidently the following representation of this curve in the form of a left coset:

$$G(\bar{c};\bar{y}) = hG(\bar{c};\bar{c}). \tag{68}$$

It is immediately seen that the one-parametric subgroup with respect to which the geodesic $G(\bar{c};\bar{y})$ is a left coset is determined uniquely. The latter subgroup coincides evidently with $G(\bar{c};\bar{c})$. Besides that, we may observe that two arbitrary rotations h_1, h_2, both belonging to the curve $G(\bar{c};\bar{y})$ obey the identity

$$h_2 = h_1(h_1^{-1}h_2) = h_1 g, \tag{69}$$

where the rotation

$$g = h_1^{-1}h_2 \tag{70}$$

leaves the constitutive vector $\bar{c}$ stable and, as a consequence, belongs to the subgroup $G(\bar{c};\bar{c})$. Thus, independently of the choice of the representative h on the curve $G(\bar{c};.\bar{y})$, the expression on the right-hand side of Eq. (68) defines one and the same left coset of $G(\bar{c};\bar{c})$. So we may conclude that any complete geodesic curve in the space SO(3) is represented uniquely as a left coset of an appropriate one-parametric subgroup.

It is not difficult to see, on the other hand, that any left coset of an arbitrary one-parametric subgroup in SO(3) is, at the same time, a complete geodesic curve in the space SO(3). In fact, let h_1 be a rotation and let $G(\bar{c};\bar{c})$ be an arbitrarily chosen one-parametric subgroup of the group SO(3). By virtue of Eqs. (40) and (41) of the previous section the left coset $h_1G(\bar{c};\bar{c})$ may also be represented in the following form:

$$h_1G(\bar{c};\bar{c}) = L(h_1)G(\bar{c};\bar{c}) = G(\bar{c};\bar{y}), \tag{71}$$

where we have put

$$\bar{y} = h_1\bar{c}. \tag{72}$$

Although the latter expression determines the constitutive vector $\bar{y}$ with the help of a particular representative of the left coset under consideration, one can verify that $\bar{y}$ will be the same if we take any other representative of the left coset $h_1G(\bar{c};\bar{c})$. In fact, if h_2 belongs to $h_1G(\bar{c};\bar{c})$ it may be written in the form (69), where g is an element of $G(\bar{c};\bar{c})$, thus we obtain

$$h_1\bar{c} = h_2\bar{c}. \tag{73}$$

So, we may conclude that any left coset of an arbitrary one-parametric subgroup of the space SO(3) is a complete geodesic curve and that there exist no other geodesics in this space except left cosets of all possible one-parametric subgroups. This result shows, in particular, that the invariant inner geometry of the space of rotations is closely connected with its algebraical (group) structure.

The right-sided parallelism in the space SO(3) may be introduced similarly to the above. Let us consider two complete geodesic curves H_1, H_2. Then the curve H_2 is called right-parallel to H_1 if there exists an appropriate right translation $R(h)$ which transforms H_1 onto H_2, i.e. we may write the following equality:

$$R(h)H_1 = H_1h^{-1} = H_2. \tag{74}$$

Analogous to the case of the left parallelism, the equality

$$R(e)H = H \tag{75}$$

is evidently valid for any complete geodesic curve. Besides that, Eq. (74) implies the equality

$$R(h^{-1})H_2 = H_1. \tag{76}$$

Thus, the relation of right parallelism is reflexive and symmetric. If, in addition, three geodesics H_1, H_2, H_3 are chosen so that H_1 is right-parallel to H_2 and H_2 is right-parallel to H_3 then we may conclude H_1

is right parallel to H_3. In fact, we have evidently as in the case of the left parallelism

$$R(h_1)H_1 = H_2, \tag{77}$$

$$R(h_2)H_2 = H_3 \tag{78}$$

for two appropriate rotations h_1, h_2. According to the multiplication rules for right translations we have the equality

$$R(h_2h_1) = R(h_2)R(h_1). \tag{79}$$

Thus, if we apply the right translation, Eq. (79), to the curve H_1, we obtain the following relation:

$$R(h_2h_1)H_1 = R(h_2)(R(h_1)H_1) = R(h_2)H_2 = H_3. \tag{80}$$

So, the right parallelism is transitive similarly to the left one.

By virtue of Eqs. (39) and (41) an arbitrary right translation $R(h)$ transforms a one-parametric subgroup $G(\bar{c};\bar{c})$ according to the formula

$$R(h)G(\bar{c};\bar{c}) = G(h\bar{c};\bar{c}). \tag{81}$$

Thus any geodesic curve of the form $G(\bar{x};\bar{c})$ is right-parallel to the related $G(\bar{c};\bar{c})$. As a consequence, two geodesics $G(\bar{x}_1;.\bar{c})$, $G(\bar{x}_2;\bar{c})$ are right-parallel independently of the choice of the constitutive vectors $\bar{x}_1, \bar{x}_2$ and $\bar{c}$. Since the correspondence between geodesic curves and related constitutive vectors is two-valued, we may state that two geodesics $H_1 = G(\bar{x}_1;\bar{y}_1), H_2 = G(\bar{x}_2;\bar{y}_2)$ are right-parallel if and only if either

$$\bar{y}_1 = \bar{y}_2 \tag{82}$$

or

$$\bar{y}_1 = -\bar{y}_2. \tag{83}$$

Analogous to the case of the left parallelism the family of all possible geodesic curves of the form $G(\bar{x};\bar{c})$, where $\bar{c}$ is chosen arbitrarily and fixed and $\bar{x}$ runs over the unit sphere Ω of the specimen space, forms the so-called congruence of the geodesics which are right-parallel to the one-parametric subgroup $G(\bar{c};\bar{c})$. An arbitrary congruence of such kind will be denoted $\mathrm{CONG}_{\mathrm{right}}(\bar{c})$. Similar to the case of the left parallelism all possible curves belonging to $\mathrm{CONG}_{\mathrm{right}}(\bar{c})$ fill the group space SO(3). Besides that two curves $G(\bar{x}_1;\bar{c}), G(\bar{x}_2;\bar{c})$ are disjoint if $\bar{x}_1$ and $\bar{x}_2$ are different. Thus, an arbitrarily chosen rotation g belongs to one and only one curve of the congruence.

It should be emphasized that any complete geodesic in the space SO(3) may be interpreted, in addition to the above, as a right coset of an appropriate one-parametric subgroup. In fact, if a rotation h obeys the relation

$$h\bar{c} = \bar{x}, \tag{84}$$

we obtain evidently for $G(\bar{x};\bar{c})$ the following representation in the form of a right coset:

$$G(\bar{x};\bar{c}) = G(\bar{c};\bar{c})h^{-1}. \tag{85}$$

So we see that any congruence of the form $\mathrm{CONG}_{\mathrm{right}}(\bar{c})$ may be equivalently interpreted as the congruence of all possible right cosets with respect to the same one-parametric subgroup $G(\bar{c};\bar{c})$. Also it should be mentioned that, if we compare two possible congruences such as $\mathrm{CONG}_{\mathrm{left}}(\bar{c})$ and $\mathrm{CONG}_{\mathrm{right}}(\bar{c})$ generated by one and the same arbitrarily chosen subgroup $G(\bar{c};\bar{c})$, we observe that there exist two and only two geodesics which belong to both these congruences. Those are the generating subgroup $G(\bar{c};.\bar{c})$ and the curve $G(\bar{c};-\bar{c})$. If we consider a geodesic of the form $G(\bar{c};\bar{y})$ with $\bar{y} \neq \pm\bar{c}$ belonging to $\mathrm{CONG}_{\mathrm{left}}(\bar{c})$ and a curve of the form $G(\bar{x};.\bar{c})$ with $\bar{x} \neq \pm\bar{c}$ taken arbitrarily from the congruence $\mathrm{CONG}_{\mathrm{right}}(\bar{c})$, one of the two following possibilities may be realized: either these curves are not coincident or they admit one and only one point of coincidence.

If two unit vectors $\bar{c}_1, \bar{c}_2$ are chosen to be different, and $\bar{c}_1 \neq -\bar{c}_2$, the congruence $\mathrm{CONG}_{\mathrm{left}}(\bar{c}_1)$ possesses, as above, two geodesic curves which, at the same time, belong to the congruence $\mathrm{CONG}_{\mathrm{right}}(\bar{c}_2)$. As it is easily seen these two common geodesics are identical to the curves $G(\bar{c}_1;\bar{c}_2)$ and $G(\bar{c}_1;-\bar{c}_2)$ respectively.

In conclusion of this section we will mention a method to characterize an arbitrary geodesic curve, which is especially appropriate to texture analysis. We will suppose that a cartesian reference system in the specimen space is chosen and fixed. The system will be denoted, following our previous paper, by O_s, X_s, Y_s, Z_s where O_s is the origin of the specimen space. The term "rotation" means, at the given moment, a rotation about the point O_s. Let us consider, in the group space SO(3), an arbitrary complete geodesic curve which will be denoted by $G(\bar{c}_1; \bar{c}_2)$. Let further g be an arbitrarily chosen point of that curve. It is convenient to associate with the rotation g the reference system $O_s, X_s(g), Y_s(g), Z_s(g)$ the axes of which are obtained from those of the specimen reference system by the rotation g under consideration. Since the condition

$$g\bar{c}_1 = \bar{c}_2 \tag{86}$$

is satisfied the coordinates of the vector $\bar{c}_2$ with respect to the reference system $O_s, X_s(g), Y_s(g), Z_s(g)$ are equal to the coordinates of $\bar{c}_1$ with respect to the specimen reference system. If, on the other hand, a cartesian reference system O_s, X', Y', Z' is chosen so that the coordinates x_2', y_2', z_2' of the vector $\bar{y}$ with respect to O_s, X', Y', Z' obey to the equalities

$$x_1 = x_2', \qquad y_1 = y_2', \qquad z_1 = z_2', \tag{87}$$

where x_1, y_1, z_1 denote the coordinates of $\bar{c}_1$ with respect to the specimen reference system, the rotation g which transforms O_s, X_s, Y_s, Z_s to the system O_s, X', Y', Z', transforms evidently $\bar{c}_1$ to $\bar{c}_2$. Thus g is a point of the geodesic curve $G(\bar{c}_1; \bar{c}_2)$. It should be emphasized in this connection that the requirement for the rotation g to be transforming the system O_s, X_s, Y_s, Z_s to O_s, X', Y', Z' includes both the conditions that the straight line O_sX' is obtained from O_sX_s, the line O_sY' is obtained from O_sY_s, and respectively the line O_sZ' is obtained from O_sZ_s as well as the important restriction that the rotation g preserves the correspondence between the positive directions which are a priori given for these coordinate lines. The rotation g will also be called the orientation of the reference system O_s, X', Y', Z with respect to the specimen reference system O_s, X_s, Y_s, Z_s. If we denote by x_2, y_2, z_2 the coordinates of the second constitutive vector $\bar{c}_2$

referred to the specimen system we may evidently write the following representation:

$$\begin{aligned} x_2' &= g_{11}x_2 + g_{12}y_2 + g_{13}z_2, \\ y_2' &= g_{21}x_2 + g_{22}y_2 + g_{23}z_2, \\ z_2' &= g_{31}x_2 + g_{32}y_2 + g_{33}z_2, \end{aligned} \tag{88}$$

where the proper orthogonal matrix $\|g_{ij}\|$, $ij = 1, 2, 3$, denotes the rotation g referred to the specimen coordinate system. After substituting the latter expressions into relations (87) we obtain, for the geodesic curve $G(\bar{c}_1; \bar{c}_2)$, the following system of equations:

$$\begin{aligned} x_2 &= g_{11}x_1 + g_{12}y_1 + g_{13}z_1, \\ y_2 &= g_{21}x_1 + g_{22}y_1 + g_{23}z_1, \\ z_2 &= g_{31}x_1 + g_{32}y_1 + g_{33}z_1, \end{aligned} \tag{89}$$

where the numbers x_1, y_1, z_1 as well as x_2, y_2, z_2 are given and they obey the normalization restrictions

$$x_1^2 + y_1^2 + z_1^2 = 1, \qquad x_2^2 + y_2^2 + z_2^2 = 1. \tag{90}$$

Besides that the orthogonality relations

$$g_{i1}g_{j1} + g_{i2}g_{j2} + g_{i3}g_{j3} = g_{ij}, \qquad i, j = 1, 2, 3, \tag{91}$$

as well as the determinant condition

$$\det\|g_{ij}\| = 1 \tag{92}$$

for the unknown matrix elements are to be accounted. So we may conclude that Eq. (89) being considered together with Eqs. (91) and (92) furnish us with a complete system of equations for a geodesic curve in the so-called matrix representation. This system of equations determines $G(\bar{c}_1; \bar{c}_2)$ as a closed curve in the nine-dimensional space of the matrix coefficients referred to the specimen coordinate system O_s, X_s, Y_s, Z_s.

GEODESIC CURVES AND MOTIONS OF GENERAL FORM IN THE SPACE SO(3)

The transformational properties of one-parametric subgroups under left and right translations in the space of rotations SO(3) were discussed in detail in the two previous sections of the paper. Now we will make some additional remarks concerning the transformational properties of general geodesic curves in the group SO(3) under left and right translations. In addition to that we will consider the transformational properties of geodesics under inner automorphisms of the group space and motions of general form. The related invariant geodesic will be found.

Let us recall that the inner automorphism $A(h)$ generated by an arbitrary element h of the group SO(3) was defined in our previous paper (Yashnikov and Bunge, 1995) as the symmetry which transforms the points of the group space according to the formula

$$A(h)g = hgh^{-1}, \tag{93}$$

where g runs through the group SO(3). As it follows immediately from the latter expression any inner automorphism may be represented in the form of the product of two commuting group translations $L(h)$, $R(h)$:

$$A(h) = L(h)R(h) = R(h)L(h). \tag{94}$$

Therefore we may write, for an arbitrary one-parametric subgroup $G(\bar{c};\bar{c})$, the following sequence of relations:

$$\begin{aligned} A(h)G(\bar{c};\bar{c}) &= L(h)(R(h)G(\bar{c};\bar{c})) \\ &= L(h)G(h\bar{c};c) = G(h\bar{c};h\bar{c}). \end{aligned} \tag{95}$$

Thus any inner automorphism maps a one-parametric subgroup onto an other one-parametric subgroup. Besides that any inner automorphism leaves the identity element e of the group SO(3) unchanged, i.e. we may write

$$A(h)e = e. \tag{96}$$

Since, in addition, inner automorphisms preserve the angular distance between rotations we may conclude that any inner automorphism "rotates" an arbitrary one-parametric subgroup about the fixed point e

as "a rigid body" in the non-euclidean space SO(3). The latter observation becomes especially clear if we use the descriptive-geometrical representation of elements of the group SO(3) with the help of their rotation axes and rotation angles. If, indeed, $g = g(\bar{a}; \varphi)$ and $h = g(\bar{b}; \psi)$ are two arbitrarily chosen rotations, both represented in these variables (for details see our previous paper), the automorphism (93) may be written in the following form:

$$A(h)g = A(h)g(\bar{a}; \varphi) = g(h\bar{a}; \varphi). \tag{97}$$

The rotation $g = g(\bar{a}; \varphi)$ may be represented in the closed ball $\bar{B}(O_s; \pi)$ of the specimen space by the vector $\varphi\bar{a}$ which is proportional to $\bar{a}$ with the coefficient which is equal to the related rotation angle φ. Let us recall that, according to the definition given in our previous paper, the ball $\bar{B}(O_s; \pi)$ contains those vectors of the specimen space the euclidean lengths of which are either strictly smaller than π or equal to π. Relation (97) demonstrates that the transformed rotation $A(h)g$ is represented in this ball by the vector $\varphi(h\bar{a}) = h(\varphi\bar{a})$ which is obtained from the vector $\varphi\bar{a}$ by the euclidean rotation h, i.e. by means of the operation of clockwise rotation of $\varphi\bar{a}$ about the axis $\bar{b}$ at the angle ψ. So we see that the parametrization of the group SO(3) with the help of the variables "axis-angle of rotation" possesses the following important property. If we represent any rotation $g = g(\bar{a}; \varphi)$ by the related vector $\varphi\bar{a}$ in the ball $\bar{B}(O_s; \pi)$ any inner automorphism $A(h)$ of the group space SO(3) is represented (modelled) by the euclidean rotation h which generates the automorphism under consideration.

As it follows from Eq. (95) the transformed one-parametric subgroup $A(h)G(\bar{c}; \bar{c})$ differs geometrically from the original one $G(\bar{c}; \bar{c})$ and, as a rule, the identity rotation e is the unique common point for these subgroups. It should be mentioned, however, that any one-parametric subgroup admits two and only two families of inner automorphisms which transform the subgroup onto itself. In fact, if h belongs to $G(\bar{c}; \bar{c})$ we have evidently the equality

$$h\bar{c} = \bar{c}. \tag{98}$$

Thus we may write

$$A(h)G(\bar{c}; \bar{c}) = G(\bar{c}; \bar{c}) \tag{99}$$

and, what is more, we may observe that any point of $G(\bar{c};\bar{c})$ is stable under $A(h)$ in the case of Eq. (98) because of the commutativity of an arbitrary one-parametric subgroup.

If, alternatively, h belongs to the geodesic curve $G(\bar{c};-\bar{c})$ we obtain according to Eq. (95)

$$\begin{aligned} A(h)G(\bar{c};\bar{c}) &= G(h\bar{c};h\bar{c}) \\ &= G(-\bar{c};-\bar{c}) = G(\bar{c};\bar{c}). \end{aligned} \tag{100}$$

So the totality of all possible inner automorphisms $A(h)$, when h runs through $G(\bar{c};-\bar{c})$, furnishes us with the second one of the two families with the invariance property (99). As it may be immediately verified

$$A(h)g = g^{-1} \tag{101}$$

if h belongs to $G(\bar{c};-\bar{c})$ and g is an arbitrary point of $G(\bar{c};\bar{c})$. Thus, in contrast to the preceeding case, elements of the one-parametric $G(\bar{c};\bar{c})$ are not stable under automorphisms $A(h)$ if h belongs to $G(\bar{c};-\bar{c})$. We see that any inner automorphism of such kind transforms points of the subgroup $G(\bar{c};\bar{c})$ according to the group inversion. Also it should be mentioned that any rotation h belonging to the geodesic $G(\bar{c};-\bar{c})$ is an element of second order, thus the algebraical condition

$$h^2 = e \tag{102}$$

is fulfilled for every point of the curve. The latter condition means that $G(\bar{c};-\bar{c})$ is the totality of all possible rotations of the form

$$h = g(\bar{b};\pi), \tag{103}$$

where $\bar{b}$ is perpendicular to the vector $\bar{c}$. As a consequence, in the variables "axis-angle of rotation" any inner automorphism of the form $A(h)$, where h belongs to $G(\bar{c};-\bar{c})$, can be interpreted as a rotation of the ball $\bar{B}(O_S;\pi)$ by an angle of 180° about an appropriate axis perpendicular to the diameter of the ball representing the subgroup $G(\bar{c};\bar{c})$. The invariance property of $G(\bar{c};\bar{c})$ under transformations of such kind is geometrically evident by this interpretation. If a rotation $h = g(\bar{b};\psi)$ and a one-parametric subgroup $G(\bar{c};\bar{c})$ are in "general position", i.e. h does not belong to this subgroup as well as h is not a point of the curve $G(\bar{c}; -\bar{c})$, the angle between $\bar{c}$ and $\bar{b}$ cannot be equal to 0°, 90° or 180°. Thus the transformed subgroup $A(h)G(c; c)$ differs geometrically from $G(\bar{c}; \bar{c})$.

Expressions (39)–(41) show in which way the constitutive vectors of a one-parametric subgroup change under an arbitrary motion of the group space SO(3). The relations can be extended to the general case in the following manner:

$$M(h_1; h_2)G(\bar{x}; \bar{y}) = G(h_2\bar{x}; h_1\bar{y}), \tag{104}$$

where $\bar{x}$ and $\bar{y}$ are arbitrary unit vectors of the specimen space. In fact, if g belongs to the curve $G(\bar{x}; \bar{y})$, the equality

$$g\bar{x} = \bar{y} \tag{105}$$

is valid, thus the rotation

$$M(h_1; h_2)g = h_1 g h_2^{-1} \tag{106}$$

being applied to the vector $h_2\bar{x}$ gives us

$$h_1 g h_2^{-1}(h_2\bar{x}) = h_1(g\bar{x}) = h_1\bar{y}. \tag{107}$$

If we specify Eq. (104) for left translations of the group space we obtain evidently the following useful relation:

$$L(h)G(\bar{x}; \bar{y}) = G(\bar{x}; h\bar{y}). \tag{108}$$

Analogously, specifying Eq. (104) for right translations furnishes us with an other important transformational rule such as the following:

$$R(h)G(\bar{x}; \bar{y}) = G(h\bar{x}; \bar{y}). \tag{109}$$

If h is not identical to e the related left translation $L(h)$ admits no stable point in the space SO(3). However, any left translation admits infinitely many complete geodesic curves each of which is invariant (as a point set) with respect to it. If, indeed,

$$h = g(\bar{c}; \varphi) \tag{110}$$

the related unit vector $\bar{c}$ is an eigenvector for h, thus we may write

$$h\bar{c} = \bar{c}. \tag{111}$$

As it follows immediately from Eq. (108), the equality

$$L(h)G(\bar{x};\bar{c}) = G(\bar{x};\bar{c}) \tag{112}$$

holds independently of the choice of the unit vector $\bar{x}$. Let us mention now that the curves $G(\bar{x};\bar{c})$, when $\bar{x}$ runs over the spherical surface of unit radius in the specimen space, form the congruence of complete geodesics which are right-parallel to the related one-parametric subgroup $G(\bar{c};\bar{c})$. Thus we see that any geodesic belonging to the congruence $\mathrm{CONG}_{\mathrm{right}}(c)$ is invariant under $L(h)$ if condition (111) is satisfied.

An analogous result is valid for any right translation. In fact, if condition (111) is fulfilled we may write

$$R(h)G(\bar{c};\bar{y}) = G(\bar{c};\bar{y}), \tag{113}$$

where $\bar{y}$ is an arbitrary unit vector. In other words, if $\bar{c}$ is an eigenvector of unit length for a rotation h, any curve of the congruence $\mathrm{CONG}_{\mathrm{left}}(\bar{c})$ is invariant under $R(h)$. Thereby it should be emphasized once more that any curve of the form $G(\bar{c};.\bar{y})$ is invariant under $R(h)$ as a point set. Each particular point g belonging to $G(\bar{c};\bar{y})$ is not stable under $R(h)$ and thus $R(h)g$ is not identical to g.

If we combine formulae (108) and (109) we obtain the following most general rule for transforming geodesic curves under motions of the group space SO(3):

$$M(h_1;h_2)G(\bar{x};\bar{y}) = G(h_2\bar{x};h_1\bar{y}), \tag{114}$$

where h_1, h_2 are two abitrarily taken rotations and $\bar{x},\bar{y}$ are two unit vectors of the specimen space. Let us observe that, if we have for h_1 and h_2 the representations

$$h_1 = g(\bar{a};\varphi), \qquad h_2 = g(\bar{b};\psi), \tag{115}$$

the related geodesic curve $G(\bar{b};\bar{a})$ is invariant under $M(h_1;h_2)$ i.e.

we may write

$$M(h_1; h_2)G(\bar{b}; \bar{a}) = G(\bar{b}; \bar{a}) \tag{116}$$

since

$$h_1\bar{a} = \bar{a}, \qquad h_2\bar{b} = \bar{b}. \tag{117}$$

Also the antipodal vector $-\bar{b}$ is an eigenvector for h_2, thus the geodesic $G(\bar{b}; -\bar{a})$ is invariant with respect to the motion $M(h_1; h_2)$ under consideration

$$M(h_1; h_2)G(\bar{b}; -\bar{a}) = G(\bar{b}; -\bar{a}). \tag{118}$$

So, any motion of the group space SO(3) admits at least two different invariant geodesic curves. As it may be easily proved, if both conditions

$$h_1 \neq e, \qquad h_2 \neq e, \tag{119}$$

$$h_1^2 \neq e, \qquad h_2^2 \neq e \tag{120}$$

are satisfied the related motion $M(h_1; h_2)$ admits no other invariant geodesic curves except $G(\bar{a}; \bar{b})$ and $G(\bar{a}; -\bar{b})$, where $\bar{a}$ and $\bar{b}$ are determined by Eq. (117). Indeed, the invariance condition

$$M(h_1; h_2)G(\bar{y}; \bar{x}) = G(\bar{y}; \bar{x}), \tag{121}$$

being fulfilled for a geodesic $G(\bar{y}; \bar{x})$, implies for the related constitutive vectors either the equalities

$$h_1\bar{x} = \bar{x}, \qquad h_2\bar{y} = \bar{y} \tag{122}$$

or, if we take into account the fact that the antipodal pair $(-\bar{x}; .-\bar{y})$ determines the same geodesic curve, the following alternative ones

$$h_1\bar{x} = -\bar{x}, \qquad h_2\bar{y} = -3ry. \tag{123}$$

However, the case of Eq. (123) may be realized if and only if h_1, h_2 both are rotations of second order. Since the latter possibility is excluded due to assumption (120) our statement is established.

Let us consider separately the exceptional family of the motions $M(h_1; h_2)$ which are defined by the algebraical restrictions

$$h_1^2 = e, \qquad h_2^2 = e. \tag{124}$$

Any motion belonging to this family may also be characterized by the property

$$(M(h_1; h_2))^2 = E, \tag{125}$$

where E is the identical transformation of the group space SO(3). If two unit vectors $\bar{a}, \bar{b}$ are determined by Eq. (117) the related h_1, h_2, by virtue of Eq. (124), may be written in the form

$$h_1 = g(\bar{a}; \pi), \qquad h_2 = g(\bar{b}; \pi). \tag{126}$$

Similar to the above-considered general case both the geodesic curves $G(\bar{b}; \bar{a})$ and $G(\bar{b}; -\bar{a})$ are invariant under $M(h_1; h_2)$. However, in the case under consideration, there is an additional continuous family of invariant geodesics, which includes all possible curves of the form $G(\bar{v}; \bar{u})$, where $\bar{u}$ is orthogonal to $\bar{a}$ and $\bar{v}$ is orthogonal to $\bar{b}$. In fact, if the orthogonality conditions

$$(\bar{u}; \bar{a}) = 0, \qquad (\bar{v}; \bar{b}) = 0 \tag{127}$$

are satisfied, we have evidently

$$h_1 \bar{u} = -\bar{u}, \qquad h_2 \bar{v} = -\bar{v} \tag{128}$$

by virtue of Eq. (126). Thus, if we apply the motion $M(h_1; h_2)$ to related geodesic curve $G(\bar{v}; \bar{u})$, we obtain

$$\begin{aligned} M(h_1; h_2) G(\bar{v}; \bar{u}) &= G(h_2 \bar{v}; h_1 \bar{u}) \\ &= G(-\bar{v}; -\bar{u}) = G(\bar{v}; \bar{u}) \end{aligned} \tag{129}$$

since antipodal pairs of constitutive vectors correspond to one and the same geodesic curve. If, in particular, the condition

$$h_1 = h_2 = h \tag{130}$$

is fulfilled we deal with an inner automorphism $A(h)$ of second order. The equality

$$\bar{a} = \pm\bar{b} \tag{131}$$

for related eigenvectors is valid, thus the constitutive vectors $\bar{u}, \bar{v}$, for every additional invariant geodesic curve $G(\bar{v};\bar{u})$ of $A(h)$ both belong to the plane, which is perpendicular to $\bar{a}$.

GEODESIC CURVES AND NON-EUCLIDEAN ROTATIONS OF GENERAL FORM

It was mentioned in the previous section that any inner automorphism of the group SO(3) may be interpreted geometrically as a non-euclidean rotation of the group space about the fixed point $g=e$. Thereby the algebraical object such as the group of all possible inner automorphisms Aut(SO(3)) is, from the geometrical point of view, the group of all possible non-euclidean rotations Rot(SO(3)); e) of this latter space provided by the angular distance between its points. Let now $g=g_0$ be an arbitrarily chosen point of the group space and $A(h)$ be an inner automorphism. We can observe that any motion of the form

$$R(g_0^{-1})A(h)R(g_0) = M(h; g_0^{-1}hg_0) \tag{132}$$

leaves the point g_0 stable independently of the choice of the element h. In fact, we may write evidently

$$M(h; g_0^{-1}hg_0)g_0 = hg_0g_0^{-1}h^{-1}g_0 = g_0. \tag{133}$$

On the other hand, let $M(h; h_2)$ be an arbitrary motion possessing the additional property to leave the point g_0 stable:

$$M(h_1; h_2)g_0 = g_0. \tag{134}$$

We obtain immediately from the latter condition

$$h_1g_0h_2^{-1} = g_0 \tag{135}$$

and thus, as a consequence, we see that the element h_2 is algebraically conjugated to h_1 with the help of element g_0^{-1}

$$h_2 = g_0^{-1} h_1 g_0. \tag{136}$$

Therefore the motion $M(h_1; h_2)$ can be represented in the form given by Eq. (132), where h is equal to h_1:

$$M(h_1; h_2) = R(g_0^{-1}) A(h_1) R(g_0). \tag{137}$$

After these preliminary considerations we define a non-euclidean rotation of the space SO(3) about an arbitrary point g_0 as a motion which obeys the restriction (134). The totality of all possible non-euclidean rotations having g_0 as their common fixed point will be denoted Rot(SO(3); g_0). As it follows from the definition, Rot(SO(3); g_0) is a subgroup of the group M(SO(3)) of all possible motions of the space SO(3). If we put $g_0 = e$ we obtain the group of all possible inner automorphisms

$$\mathrm{Rot}(\mathrm{SO}(3); e) = \mathrm{Aut}(\mathrm{SO}(3)). \tag{138}$$

If $g_0 \neq e$ the related group Rot(SO(3)); g_0) is, as it follows from the representation (137), algebraically conjugated to the group of all possible inner automorphisms

$$\mathrm{Rot}(\mathrm{SO}(3); g_0) = R(g_0^{-1}) \mathrm{Aut}(\mathrm{SO}(3)) R(g_0). \tag{139}$$

Let us observe that the condition (134) is equivalent to the equality

$$h_1 = g_0 h_2 g_0^{-1}. \tag{140}$$

Thus an arbitrary non-euclidean rotation $M(h_1; h_2)$ admits, together with the representation (137), the following alternative one:

$$M(h_1; h_2) = L(g_0) A(h_2) L(g_0^{-1}) \tag{141}$$

using related left translation $L(g_0)$ and $L(g_0^{-1})$ instead of the analogous right ones. Also the related inner automorphisms $A(h_1)$ and $A(h_2)$ are evidently connected by the relation

$$A(h_2) = A(g_0^{-1}h_1g_0) \tag{142}$$

as it follows from Eq. (136). So we see that the group of non-euclidean rotations Rot(SO(3); g_0) may also be obtained from the group of inner automorphisms by conjugation with the help of the left translation $L(g_0)$:

$$\mathrm{Rot}(\mathrm{SO}(3); g_0) = L(g_0)\mathrm{Aut}(\mathrm{SO}(3))L(g_0^{-1}). \tag{143}$$

Let us make now an important remark concerning invariant geodesic curves of non-euclidean rotations. As it was established in the above, an arbitrary motion $M(h_1; h_2)$ admits, in non-exceptional cases (i.e. when the conditions (119) and (120) are satisfied), exactly two different invariant geodesics which have the form $G(\bar{b}; \bar{a})$ and $G(\bar{b}; -\bar{a})$. The related constitutive vectors $\bar{a}, \bar{b}$ are determined from Eq. (117). If, in addition, $M(h_1; h_2)$ leaves g_0 stable the rotation h_1 is algebraically conjugated to h_2 with the help of g_0, thus $g_0\bar{b}$ is an eigenvector of unit length for h_1. So the two invariant geodesic curves of $M(h_1; h_2)$ may be written in the form $G(\bar{b}; .g_0\bar{b})$ and $G(\bar{b}; .-g_0\bar{b})$. Therefore, in the case if a motion $M(h_1; h_2)$ belongs to the group Rot(SO(3); g_0), one of the two of its invariant geodesics passes through the point g_0. This is evidently $G(\bar{b}; g_0\bar{b})$, where $\bar{b}$ is determined from Eq. (117). Let us observe, in addition to the above, that if $M(h_1; h_2)$ leaves g_0 stable, each point of its invariant geodesic $G(\bar{b}; g_0\bar{b})$ is stable under $M(h_1; h_2)$. If, indeed, g is an arbitrary point of $G(\bar{b}; g_0\bar{b})$ we obtain, by virtue of the representation (141), the following expression:

$$\begin{aligned} M(h_1; h_2)g &= L(g_0)A(h_2)L(g_0^{-1})g \\ &= g_0h_2g_0^{-1}gh_2^{-1}. \end{aligned} \tag{144}$$

The rotations g_0, g both belong to the left coset

$$g_0G(\bar{b}; \bar{b}) = G(\bar{b}; g_0\bar{b}). \tag{145}$$

Thus the product $g_0^{-1}g$ is an element of related one-parametric subgroup $G(\bar{b}; \bar{b})$. As a consequence h_2 and $g_0^{-1}g$ are permutable, and we may rewrite Eq. (144) in the following equivalent form:

$$\begin{aligned} M(h_1;h_2)g &= g_0(g_0^{-1}g)(h_2^{-1}h_2) \\ &= (g_0g_0^{-1})g = g. \end{aligned} \tag{146}$$

The above-established property is evidently a characteristic feature of non-euclidean rotations of the space SO(3). So we may state that a motion $M(h_1; h_2)$ is a non-euclidean rotation if and only if at least one of the two of its invariant geodesics consists of stable points of this motion.

The preceding result may be formulated more precisely if, in addition to the assumption that a motion $M(h_1;h_2)$ belongs to Rot(SO(3); g_0), the conditions

$$M(h_1;h_2) \neq E, \tag{147}$$

$$(M(h_1;h_2))^2 \neq E \tag{148}$$

are fulfilled. In this latter case we can state that there exist no other stable points for $M(h_1;h_2)$ except all possible points of the invariant geodesic $G(\bar{b};g_0\bar{b})$. In fact, if we assume the existence of a stable point $\hat{g}_0$ which does not belong to $G(\bar{b};g_0\bar{b})$, we obtain, by virtue of the above, that each point of the curve $G(\bar{b};\hat{g}\bar{b})$ should be stable under $M(h_1;h_2)$. Thus, in particular, $G(\bar{b};\hat{g}\bar{b})$ is invariant under this motion. Under the restrictions (147), (148) which are evidently equivalent to those in Eq. (119), (120), the motion $M(h_1;h_2)$ admits only the two above-mentioned different invariant geodesics. Therefore $G(\bar{b};\hat{g}\bar{b})$ must be identical to the curve $G(\bar{b}; -g_0\bar{b})$. As a consequence we obtain the following equality:

$$\hat{g}\bar{b} = -g_0\bar{b}, \tag{149}$$

which shows that the stable point $\hat{g}$ must be a point of $G(\bar{b}; -g_0\bar{b})$. As a consequence of Eq. (149) we obtain also the relation

$$(g_0^{-1}\hat{g})\bar{b} = -\bar{b}, \tag{150}$$

which demonstrates, in addition, that the rotation $g_0^{-1}\hat{g}$ belongs to the geodesic curve $G(\bar{b}; -\bar{b})$. The latter geodesic is invariant under the inner automorphism $A(h_2)$ and, if we apply it to the point $g_0^{-1}\hat{g}$, we come to the equality

$$A(h_2)(g_0^{-1}\hat{g}) = g_0^{-1}\hat{g}, \tag{151}$$

which follows immediately from our main assumption that $\hat{g}$ is an additional stable point of the non-euclidean rotation

$$M(h_1;h_2) = M(g_0h_2g_0^{-1};h_2) \tag{152}$$

about g_0. Besides that relation (150) shows that the auxiliary rotation $g_0^{-1}\hat{g}$ being represented in the variables "axis-angle of rotation"

$$g_0^{-1}\hat{g} = g(\bar{c};\theta) \tag{153}$$

has its rotation vector $\bar{c}$ orthogonal to $\bar{b}$ and the rotation angle $\theta=\pi$. Thus equality (151) may be fulfilled if and only if the rotation h_2 has the form

$$h_2 = g(\bar{b};\pi). \tag{154}$$

This means, in particular, that $h_2^2 = e$ and thus we obtain an evident contradiction with the condition (148). So we conclude that a non-euclidean rotation obeying the conditions (147), (148) admits one and only one complete geodesic curve each point of which is a stable point.

Let us consider separately the exceptional case when the motion under consideration obeys the restrictions

$$M(h_1;h_2) \neq E, \tag{155}$$

$$(M(h_1;h_2))^2 = E. \tag{156}$$

Condition (156) is evidently equivalent to the following one:

$$h_1^2 = h_2^2 = e. \tag{157}$$

Therefore h_1, h_2 have the form

$$h_1 = g(\bar{a};\pi), \qquad h_2 = g(\bar{b};\pi). \tag{158}$$

As it is well known the geodesic $G(\bar{b};\bar{a})$ is invariant under the motion $M(h_1;h_2)$. We will prove, in addition, that each point of the curve is stable under $M(h_1;h_2)$. If, indeed, g_0 is an arbitrarily chosen point of $G(\bar{b};\bar{a})$ we obtain by virtue of the rule (97)

$$\begin{aligned} A(g_0)h_2 &= g_0^{-1}h_2g_0 \\ &= g(g_0\bar{b};\pi) = g(\bar{a};\pi) = h_1. \end{aligned} \tag{159}$$

Thus h_1 and h_2 are algebraically conjugated in the group SO(3) and the conjugacy relation

$$h_1 = g_0h_2g_0^{-1} \tag{160}$$

is valid when g_0 runs through the geodesic $G(\bar{b};\bar{a})$. As a consequence we have

$$M(h_1;h_2)g_0 = g_0h_2g_0^{-1}g_0h_2^{-1} = g_0 \tag{161}$$

if g_0 belongs to $G(\bar{b};\bar{a})$. Let us observe that an analogous result may be obtained for the second invariant curve $G(\bar{b};-\bar{a})$. In fact, if g_0 is an arbitrary point of the curve, we have like Eq. (159)

$$\begin{aligned} A(g_0)h_2 &= g(g_0\bar{b};\pi) \\ &= g(-\bar{a};\pi) = g(\bar{a};\pi) = h_1. \end{aligned} \tag{162}$$

Thus the conjugacy relation (160) is also valid when g_0 runs through the curve $G(\bar{b};-\bar{a})$. Therefore we can conclude that each point of $G(\bar{b};-\bar{a})$ is stable under $M(h_1;h_2)$. So we see that, if $M(h_1;h_2)$ obeys the conditions (155), (156), the points of both geodesics $G(\bar{b};\bar{a})$ and of $G(\bar{b};-\bar{a})$ are stable under this motion. As it can be shown under the restrictions (155), (156), there are no other stable points for $M(h_1;h_2)$ except the points of $G(\bar{b};\bar{a})$ and $G(\bar{b};-\bar{a})$. In fact, the stability condition

$$M(h_1;h_2)\hat{g} = \hat{g} \tag{163}$$

being fulfilled for a point $\hat{g}$ of the group space SO(3) implies the conjugacy relation

$$h_1 = \hat{g}h_2(\hat{g})^{-1}. \tag{164}$$

Thus the vector $\hat{g}\bar{b}$ is an eigenvector of unit length for the rotation h_1. As a consequence one of the two following possibilities may be realized:

$$\hat{g}\bar{b} = \bar{a}, \tag{165}$$

$$\hat{g}\bar{b} = -\bar{a}. \tag{166}$$

In the case of Eq. (165) $\hat{g}$ belongs to the curve $G(\bar{b};\bar{a})$: in the case if Eq. (166) holds, $\hat{g}$ is a point of $G(\bar{b};-\bar{a})$.

We will give, at the end of this section, another characterization of non-euclidean rotations of the group space SO(3). Let $M(h_1;.h_2)$be a motion having $h_1 = g(\bar{a};\varphi)$ and $h_2 = g(\bar{b};\psi)$. The motion under consideration is a non-euclidean rotation if and only if the related rotation angles φ, ψ are equal:

$$\varphi = \psi. \tag{167}$$

If, indeed, $M(h_1;h_2)$ is a non-euclidean rotation then the euclidean rotations h_1, h_2 are algebraically conjugated:

$$h_1 = g_0 h_2 g_0^{-1}, \tag{168}$$

where g_0 is an arbitrarily chosen stable point of $M(h_1;h_2)$. As it is well known, the angle of any rotation is invariant under the operation of conjugation in the group SO(3), thus Eq. (167) is valid. Let us consider, on the other hand, an arbitrary motion $M(h_1;h_2)$ that obeys the additional restriction (167). Thus we have

$$h_1 = g(\bar{a};\varphi), \qquad h_2 = g(\bar{b};\varphi). \tag{169}$$

Let us consider now the auxiliary geodesic curve $G(\bar{b};\bar{a})$. If g_0 is an arbitrarily chosen point of the curve we may apply $A(g_0)$ to the point h_2 then we obtain, according to the rule (97),

$$A(g_0)h_2 = g(g_0\bar{b};\varphi) = g(\bar{a};\varphi) = h_1. \tag{170}$$

So the conjugacy relation of the form (168) is fulfilled for each point of the curve $G(\bar{b};\bar{a})$. Thus each point of the curve is stable under the motion $M(h_1;h_2)$. As a consequence $M(h_1;h_2)$ is a non-euclidean rotation.

All above-presented considerations about non-euclidean rotations can be summarized in the following manner. In the non-exceptional case (see for instance Eqs. (147), (148)) a non-euclidean rotation $M(h_1; h_2)$ admits one and only one complete geodesic curve each point of which is a stable point for it. This curve has the form $G(\bar{b}; \bar{a})$, if $h_1 = g(\bar{a}; \varphi)$ and $h_2 = g(\bar{b}; \varphi)$. We may evidently interpret the geodesic $G(\bar{b}; \bar{a})$ as the non-euclidean analog of the rotation axis for $M(h_1; h_2)$. The related non-euclidean analog of the rotation angle for $M(h_1; h_2)$ may be naturally defined as the common value φ of the rotation angles of h_1 and h_2. So, in the non-exceptional case, the assumption that a motion has at least one stable point implies the consequence that the motion possesses the non-euclidean axis of rotation passing through the stable point. So we have obtained a non-euclidean analog of the classical Euler's theorem which states that a motion of the three-dimensional euclidean space, having at least one stable point, is a rotation about an appropriate axis passing through this point. As it is well known, an arbitrary rotation of the three-dimensional euclidean space admits one and only one rotation axis. In contrast to that, in the non-euclidean space SO(3) every motion of the second order, being automatically a rotation of the group space, admits exactly two geometrically different stable geodesics each of which can be called a rotation axis. We will agree to speak, in this exceptional case, about non-euclidean rotations having two different axes. The related angle of the non-euclidean rotation is equal to π in the exceptional case. Let us mention, in conclusion, that similar to the identical rotation e in the specimen space the identical rotation $M(e; e)$ leaves each point of the group space SO(3) stable. Therefore every complete geodesic curve in the space SO(3) can be considered as its rotation axis. We put the value of the rotation angle for $M(e; e)$ equal to 0. Let us observe that, by these agreements, the totality of all possible non-euclidean rotations about an arbitrarily chosen geodesic curve $G(\bar{x}; \bar{y})$ is a group. The latter group will be denoted $\mathrm{Rot}(G(\bar{x}; .\bar{y}))$. It is evident that the group $\mathrm{Rot}(G(\bar{x}; \bar{y}))$ is a subgroup of the point group $\mathrm{Rot}(\mathrm{SO}(3); g_0)$ if g_0 belongs to the curve $G(\bar{x}; \bar{y})$.

CONCLUDING REMARKS

We have presented in this paper a collection of results concerning geodesic curves and motions in the space of three-dimensional rotations SO(3). It was demonstrated, in particular in the second section of the paper, that any one-parametric subgroup of the group SO(3) as well as any left and right coset of an arbitrary one-parametric subgroup is a geodesic curve. So the geometry of the space SO(3) provided by the angular distance between rotations is completely determined by the natural group structure of the space. Although the concept of geodesic curves and the concept of motions play, in principle, the same role in the space SO(3) as the concept of straight lines and motions in the conventional euclidean space (i.e. the specimen space or the space of a grain in texture analysis). We had many cases to convince ourselves that the properties of geodesics and motions in SO(3) are essentially distinct from those of the straight lines and motions in the euclidean space. The group space SO(3) admits, for example, in contrast to the euclidean one, two different types of natural parallelism. Any motion of the group space possesses at least two different invariant geodesic curves. There are rotations of the space SO(3) which have more than one non-euclidean rotation axes, etc. All above-mentioned geometrical phenomena and many others, which will be analyzed in detail in future papers, are caused both by the specific topological structure of the space of rotations SO(3), which is evidently essentially distinct from that of the euclidean space, any by the presence, in the space SO(3), of the natural algebraical structure of a non-commutative group.

On the other hand, the angular geometry of the group space SO(3) is the basis of three-dimensional texture analysis similarly to that as the conventional euclidean geometry is a base for the classical geometrical crystallography. According to this "correspondence principle" all facts related to the basic geometrical constructions of texture analysis can be and should be expressed in terms of the non-euclidean inner geometry of the space SO(3) and its group structure. This program was partially realized in our previous paper (Yashnikov and Bunge, 1995). We may mention, in addition, that the concept of right parallelism, which is considered in detail in present paper, is closely connected with the crystallographic symmetry of a texture. The left parallelism of geodesics is important in connection with the macroscopic symmetry for a crystallographic texture. Also, the latter type of parallelism gives us the possibility to interpret the texture diffractometry (i.e. pole densities measurements) as a complicated tomography of the orientation space of a texture. These geometrical aspects of the three-dimensional texture analysis was briefly discussed by Yashnikov (1987, 1993).

The collection of results concerning various properties of geodesic curves in the space SO(3) presented in this paper does not exhaust the minimal set of those which are necessary for the needs of the texture analysis. A series of additional important properties of geodesics, non-euclidean rotations, and motions together with applications to the characterization of axial and spiral textures, symmetry classification of textures, reduction problem for their orientation spaces, etc. will be given in future publications.

References

Bonnet, R. (1980). *Acta Cryst. A*, **36**, 116.
Bunge, H.J. (1982). *Texture Analysis in Materials Science.* Butterworth, London.
Frank, F.C. (1987). *Proc. of 8th Int. Conf. on Textures of Materials.* Santa Fe, USA, 3.
Frank, F.C. (1988). *Metal. Trans. A*, **19**, 403.
Frank, F.C. (1992). *Phil. Mag. A*, **65**, 1141.
Gertsman, V.Y. (1989). *Proc. Int. Conf. on Intergranular and Interface Boundaries.* Paris, France, C1–145.
Grimmer, H. (1974). *Acta Cryst. A*, **30**, 685.
Heinz, A. and Neumann, P. (1991). *Acta Cryst. A*, **47**, 780.
Ibe, G. (1993). *Textures of Materials.* ICOTOM-10 Ed. H.J. Bunge. Materials Science Forum 157–162, 369 Trans. Tech. Publ.
Mattheis, S., Helming, K. and Kunze, K. (1990). *Phys. Stat. Sol. (b)*, **157**, 489.
Yashnikov, V.P. (1987). *Preprint of ISSP.* Russian Academy of Sciences. Chernogolovka (In Russian).
Yashnikov, V.P. (1993). *Collected Abstracts of 10th Int. Conf. on Textures of Materials.* Clausthal, Germany, 182.
Yashnikov, V.P. and Bunge, H.J. (1995). *Textures and Microstructures*, **23**, 201.