SELECTIONS OF SET-VALUED STOCHASTIC PROCESSES

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We show that $\mathcal{F}_t$-adapted set-valued stochastic processes satisfying mild continuity conditions admit, $\mathcal{F}_t$-adapted, stochastically continuous selections.

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1. Introduction

In this paper we prove several theorems on the existence of $\mathcal{F}_t$-adapted, continuous selections for $\mathcal{F}_t$-adapted, set-valued stochastic processes, as well as a continuous time version of Hess’ result on martingale selection [3]. Such results may be useful in the theory of the set-valued stochastic integral.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ (i.e., with a family of $\sigma$-fields $\mathcal{F}_t$), such that $0 \leq s \leq t$ implies that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$. We assume that all $P$-null sets are in $\mathcal{F}_0$. Let $\mathcal{F}_t^-=\sigma(\bigcup_{s \geq t} \mathcal{F}_s)$ and $\mathcal{F}_t^+=\bigcap_{s > t} \mathcal{F}_s$. Obviously, $\mathcal{F}_t^\pm \subset \mathcal{F}_t \subseteq \mathcal{F}_t^+$. For a random variable $\varphi: \Omega \rightarrow \mathbb{R}^n$ such that $E(|\varphi|) = \int_{\Omega} |\varphi| \, dP < +\infty$, by $E(\varphi \mid \mathcal{F}_t)$ we denote the conditional expectation of $\varphi$, (i.e., an $\mathcal{F}_t$-measurable mapping) such that

$$\int_A E(\varphi \mid \mathcal{F}_t) \, dP = \int_A \varphi \, dP$$

for each $A \in \mathcal{F}_t$.

We say that a set-valued mapping $\Phi: \Omega \rightarrow \mathbb{R}^n$ is a set-valued random variable iff $\Phi$ is $\mathcal{F}$-measurable (weakly measurable in the terminology of Himmelberg [5]), i.e.,

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The real-valued function \( d(z, \Phi) : \Omega \rightarrow \mathbb{R}^n \) defined by

\[
d(z, \Phi) = d(z, \Phi(\omega)) = \inf_{v \in \Phi(\omega)} \| z - v \|
\]

where \( \| w \| \) is the Euclidean norm of \( w \in \mathbb{R}^n \), is a random variable. Clearly, for a mapping \( \varphi : \Omega \rightarrow \mathbb{R}^n \) identified with the set-valued mapping \( \Phi = \{ \varphi \} \), this is equivalent to saying that \( \varphi \) is a random variable. Let \( (F_t) = (F_t)_{t \geq 0} \) be a set-valued stochastic process with closed values in \( \mathbb{R}^n \) (i.e., a family of \( \mathcal{F} \)-measurable set-valued mappings \( F_t : \Omega \rightarrow \mathbb{R}^n, t \geq 0 \), with closed values). We say that \( (F_t) \) is \( \mathcal{F}_t \)-adapted iff \( F_t \) is \( \mathcal{F}_t \)-measurable for each \( t \geq 0 \), and we denote an \( \mathcal{F}_t \)-adapted process \( (F_t) \) such that

\[
E(\| d(0, F_t) \|) < +\infty \text{ for each } t \geq 0, \text{ by } (F_t, \mathcal{F}_t).
\]

A selection of the process \( (F_t) \) is a single-valued stochastic process \( (f_t) \) such that for every \( t \geq 0 \), there holds \( f_t(\omega) \in F_t(\omega) \) for \( P \)-almost all \( \omega \). Additionally, if \( (f_t) \) is \( \mathcal{F}_t \)-adapted and satisfies

\[
E(\| f_t \|) < +\infty \text{ for each } t \geq 0,
\]

we will denote the process by \( (f_t, \mathcal{F}_t) \).

Let us mention that for the unique \( \sigma \)-field \( \mathcal{F} \), the result on convergence of measurable selections being extracted from the sequence of measurable set-valued mappings, that converge in the distribution, has been investigated by Salinetti and Wets [9, Theorem 5.1, Corollary 5.2]. On the other hand, Hess has proven the existence of martingale selections for discrete time, set-valued martingales and discussed the convergence of set-valued martingales.

### 3. Selection Theorem Results

Our first simple result concerns the case when almost all paths \( t \mapsto F_t(\omega) \) are continuous, and similar to the results of Salinetti and Wets, are based on the regularity of metric projections. For \( z \in \mathbb{R}^n \) and the closed, convex set \( A \subset \mathbb{R}^n \), we denote by \( \text{Pr}(z, A) \) the metric projection of \( z \) onto \( A \) with respect to Euclidean norm (i.e., a unique element \( \text{Pr}(z, A) \in A \) such that \( \| \text{Pr}(z, A) - z \| = d(z, A) \)). The Wijsman topology for the family \( \text{Cl}(\mathbb{R}^n) \) of all nonempty, closed convex subsets of \( \mathbb{R}^n \), is the weakest topology such that for every \( y \in \mathbb{R}^n \), the function \( A \mapsto d(y, A) \) is continuous [10]. We will need the following lemma.

**Lemma 1:** The mapping \( A \mapsto \text{Pr}(z, A) \) of \( \text{Cl}(\mathbb{R}^n) \) into \( \mathbb{R}^n \) is continuous with respect to the Wijsman topology.

**Proof:** For \( A, A_0 \in \text{Cl}(\mathbb{R}^n) \) and \( z \in \mathbb{R}^n \), let us denote \( y_0 = \text{Pr}(y_0, A), y = \text{Pr}(z, A) \). Clearly,

\[
\| y - y_0 \| \leq \| y - \text{Pr}(y_0, A) \| + \| \text{Pr}(y_0, A) - y_0 \| = \| y - \text{Pr}(y_0, A) \| + d(y_0, A).
\]

By the parallelogram equality, we have

\[
\| y - \text{Pr}(y_0, A) \|^2 = 2 \| y - z \|^2 + 2 \| \text{Pr}(y_0, A) - z \|^2 - 4 \| \frac{y + \text{Pr}(y_0, A)}{2} - z \|^2 \leq 2 \| \text{Pr}(y_0, A) - z \|^2 - 2d(z, A)^2.
\]

But
\[ \| \Pr(y_0, A) - z \| \leq \| \Pr(y_0, A) - y_0 \| + \| y_0 - z \| = d(y_0, A) + d(z, A_0). \]

Thus,

\[ \| y - \Pr(y_0, A) \|^2 \leq 2(d(y_0, A) + d(z, A_0) - d(z, A))(d(y_0, A) + d(z, A_0) + d(z, A)). \]

Consequently,

\[ \| y - y_0 \| \leq d(y_0 < A) + \sqrt{2(d(y_0, A) + d(z, A_0) - d(z, A))(d(y_0, A) + d(z, A_0) + d(z, A))}. \]

From this it follows immediately that \( A \mapsto \Pr(z, A) \) is continuous with respect to the Wijsman topology.

**Theorem 1:** If the stochastic process \((F_t, \mathcal{F}_t)\) has closed convex values and for every \( z \in \mathbb{R}^n \), the functions \( t \mapsto d(z, F_t)(\omega) \) is continuous for a.e. \( \omega \in \Omega \), then for any \( y \in \mathbb{R}^n \), the process \((f_t)\) defined by \( f_t(\omega) = \Pr(y, F_t(\omega)) \) is an \( \mathcal{F}_t \)-adapted selection of \( F \) such that \( t \mapsto f_t(\omega) \) is continuous for \( \mathcal{P} \)-a.e. \( \omega \in \Omega \).

**Proof:** By virtue of Lemma 1, from the assumption that the functions \( t \mapsto d(z, F_t)(\omega) \), \( z \in \mathbb{R}^n \), and a.e. \( \omega \in \Omega \) are continuous, it follows that for every \( y \in \mathbb{R}^n \), a.e. \( \omega \in \Omega \), the mapping \( t \mapsto \Pr(y, F_t(\omega)) \) is continuous. To see that \( f_t \) is \( \mathcal{F}_t \)-measurable note that

\[ \text{Graph } f_t \subset \{(\omega, z) : \| y - z \| - d(y, F_t(\omega)) \leq 0 \} \cap \text{Graph } F_t. \]

Hence, by virtue of [5, Theorem 3.5 and Corollary 6.3], \( f_t \) is \( \mathcal{F}_t \)-measurable.

In the following theorems we dispense completely with the upper semicontinuity assumption for the process \((F_t, \mathcal{F}_t)\). We do not adopt any lower semicontinuity assumption for the functions \( t \mapsto d(y, F_t)(\omega) \); we assume only the stochastic upper semicontinuity of these functions, which means the stochastic lower semicontinuity of the process \((F_t, \mathcal{F}_t)\). We utilize a well-known theorem on measurable selections due to Kuratowski and Ryll-Nardzewski, as well as theorems on continuous selections of lower semicontinuous, set-valued mappings due to Michael [7] and to Antosiewicz, Cellina (see e.g., [1, Theorem 3]), respectively. We will need the following lemma.

**Lemma 2:** Assume that for the stochastic process \((F_t, \mathcal{F}_t)\), \( s \geq 0 \) and every \( z \in \mathbb{R}^n \), \( A \in \mathcal{F}_s \), the real-valued function \( t \mapsto E(\chi_A d(z, F_t)) \) is right-hand (respectively: left-hand) usc at \( s \). Then for any \( \mathcal{F}_s \)-measurable random variable \( \varphi \) with \( E(| \varphi |) < +\infty \), the function \( t \mapsto E(d(\varphi, F_t)) \) is right-hand (respectively: left-hand) usc at \( s \).

**Proof:** Let \( \epsilon > 0 \). By assuming that for any constant function, \( \varphi \equiv z \), we have \( E(d(\varphi, F_t)) < E(d(\varphi, F_s)) + \frac{\epsilon}{2} \) whenever \( t \in [s, s + \delta) \) (respectively, \( t \in (s - \delta, s] \)) for sufficiently small \( \delta \). For a step random variable \( \varphi = \sum_{i=1}^m z_i \chi_{A_i} \), \( A_i \in \mathcal{F}_s \), we have

\[ E(d(\varphi, F_t)) = \sum_{i=1}^m E(\chi_{A_i} d(z_i, F_t)) \leq \sum_{i=1}^m (E(\chi_{A_i} d(z_i, F_s)) + \frac{\epsilon}{2^i}) \leq E(d(\varphi, F_s)) + \epsilon, \]

whenever \( t \in [s, s + \delta) \) (\( t \in (s - \delta, s] \)) for sufficiently small \( \delta \). For an arbitrary \( \mathcal{F}_s \)-measurable \( \varphi \), first choose a sequence of \( \mathcal{F}_s \)-measurable step functions \( \varphi_n \) such that
Then choose \( n \) such that \( E(\| \varphi - \varphi_n \|) < \frac{\varepsilon}{3} \) and let \( \delta > 0 \) be such that \( E(d(\varphi_n, F_t)) < E(d(\varphi_n, F_s)) + \frac{2\varepsilon}{3} \) for \( t \in [s, s + \delta) \) \((t \in (s - \delta, s])\). Then,

\[
E(d(\varphi, F_t)) \leq E(\| \varphi - \varphi_n \|) + E(d(\varphi_n, F_t)) < E(d(\varphi_n, F_s)) + \frac{2\varepsilon}{3} \\
\leq E(\| \varphi_n - \varphi \|) + E(d(\varphi, F_s)) + \frac{2\varepsilon}{3} < E(d(\varphi, F_s)) + \varepsilon,
\]

whenever \( t \in [s, s + \delta) \) \((t \in (s - \delta, s])\).

**Theorem 2:** Assume that a set-valued stochastic process \((F_t, \mathcal{F}_t)\) has closed convex values and for every \( z \in \mathbb{R}^n \), \( s \geq 0 \), and \( A \in \mathcal{F}_s \), the real-valued function \( t \mapsto E(X_A d(z, F_t)) \) is right-hand usc at \( s \). Then \((F_t, \mathcal{F}_t)\) has a \( L^1 \)-right-hand continuous selection \((f_t, \mathcal{F}_t)\).

**Proof:** Define a set-valued mapping \( G: [0, + \infty) \rightarrow L^1(\Omega, \mathcal{F}, \mathbb{R}^n) \) by

\[
G(t) = \{ \varphi \in L^1(\Omega, \mathcal{F}, \mathbb{R}^n) : \varphi \text{ is } \mathcal{F}_t \text{-measurable selection of } F_t \}.
\]

Based on the assumption \( E(d(z, F_t)) < + \infty \) for each \( t \geq 0 \), the mapping \( G \) has non-empty values by virtue of the Kuratowski and Ryll-Nardzewski measurable selection theorem (see e.g., [5, Theorem 5.1]). Moreover, the sets \( G(t) \) are closed and convex because the set-valued random variables \( F_t \) have closed, convex values. If we equip \([0, + \infty)\) with the arrow topology \( \tau_\rightarrow \) (i.e., the topology generated by the intervals \([s, t), 0 \leq s < t\)), then it follows from the assumptions that \( G: [0, + \infty) \rightarrow L^1(\Omega, \mathcal{F}, \mathbb{R}^n) \) is a lower semicontinuous, set-valued mapping. Indeed, it suffices to show that

\[
d(\varphi, G(t)) = \inf_{\psi \in G(t)} E(\| \varphi - \psi \|) \rightarrow 0 \text{ as } t \uparrow s \text{ for any } \varphi \in G(s), s \geq 0.
\]

Since \( \varphi \) is \( \mathcal{F}_t \)-measurable for \( t \geq s \), as a consequence of Kuratowski and Ryll-Nardzewski selection theorem, we have that

\[
d(\varphi, G(t)) = E(d(\varphi, F_t))
\]

for \( t \geq s \), (see Hiai and Umegaki [4, Theorem 2.2] and Rybiński [8, Lemma 6]). But by virtue of Lemma 2 we have that \( E(d(\varphi, F_t)) \rightarrow 0 \) as \( t \uparrow s \). This shows that \( G \) is lower semicontinuous on \([(0, + \infty), \tau_\rightarrow]\). Since \([(0, + \infty), \tau_\rightarrow]\) is a Lindelöf space, hence paracompact (see Engelking [2]), we can then apply the Michael continuous selection theorem to \( G \) ([7, Theorem 3.2']), and get a continuous mapping \( g: [0, + \infty) \rightarrow L^1(\Omega, \mathcal{F}, \mathbb{R}^n) \) such that \( g(t) \in G(t) \) for all \( t \geq 0 \). Obviously, continuity with respect to \( \tau_\rightarrow \) means the right-hand continuity of \( g \). We can then define the stochastic process \((f_t)_t \geq 0 \) by \( f_t(\omega) = g(t)(\omega) \). Clearly, a selection \((f_t)_t \) is \( \mathcal{F}_t \)-adapted. Since \( E(\| f_t - f_s \|) = E(\| g(t) - g(s) \|) \rightarrow 0 \) as \( t \uparrow s \), then by the Chebyshev inequality, \( P(\| f_t - f_s \| > \varepsilon) \rightarrow 0 \) as \( t \rightarrow s \). Thus, \((f_t, \mathcal{F}_t)\) is stochastically right-hand continuous.

For the proof of the next selection theorem, we will need also the following consequence of Levy's martingale convergence theorem.

**Proposition 1:** \( \mathcal{F}_t^{} = \mathcal{F}_t^{} \) if and only if the function \( s \mapsto E(\varphi \mid \mathcal{F}_s^{}) \) is \( P \)-almost everywhere left-hand continuous at \( t \) for each \( \mathcal{F}_t \)-measurable \( \varphi \) such that \( E(\| \varphi \|) < + \infty \). Analogously, \( \mathcal{F}_t^{} = \mathcal{F}_t^{} \) if and only if the function \( s \mapsto E(\varphi \mid \mathcal{F}_s^{}) \) is \( P \)-almost everywhere right-hand continuous at \( t \) for each \( \mathcal{F} \)-measurable \( \varphi \) such that \( E(\| \varphi \|) < + \infty \).

**Proof:** If \( \mathcal{F}_t^{} = \mathcal{F}_t^{} \), then by Levy's theorem (see Liptser and Shiraev [6, p. 24])
we have that \( E(\varphi \mid \mathcal{F}_s) \rightarrow E(\varphi \mid \mathcal{F}_t) \) whenever \( s_n \uparrow t \). Conversely, observe that for \( A \in \mathcal{F}_t \), \( E(\chi_A \mid \mathcal{F}_s) \rightarrow E(\chi_A \mid \mathcal{F}_t) \) by Levy's theorem whenever \( s_n \uparrow t \). On the other hand, by assumption \( E(\chi_A \mid \mathcal{F}_s) \rightarrow E(\chi_A \mid \mathcal{F}_t) = \chi_A \), thus \( \chi_A = E(\chi_A \mid \mathcal{F}_t) \) \( P \)-almost everywhere. Therefore, for \( B = (E(\chi_A \mid \mathcal{F}_t))^{-1}(1) \in \mathcal{F}_t \), \( P((A \setminus B) \cup (B \setminus A)) = 0 \). Since all \( P \)-null sets are in \( \mathcal{F}_t \), we conclude that \( A \in \mathcal{F}_t \). The analogous statement regarding \( \mathcal{F}_t^+ \) can be verified in the same way.

**Theorem 3:** Let \( \mathcal{F}_t = \mathcal{F}_t^+ \) for each \( t \geq 0 \). Assume that a set-valued stochastic process \((F_t, \mathcal{F}_t)\) has closed values and for every \( z \in \mathbb{R}^n \), \( s \geq 0 \), \( A \in \mathcal{F}_s \), the real-valued function \( t \mapsto E(X_A d(z, F_t)) \) is used at \( s \). Assume also that \( P \) is nonatomic or \((F_t, \mathcal{F}_t)\) has convex values. Then \((F_t, \mathcal{F}_t)\) has an \( L^1 \)-continuous selection \((f_t, \mathcal{F}_t)\).

**Proof:** We consider \([0, +\infty)\) with the usual topology and will show that \( G \) (defined in the proof of Theorem 2) is lower semicontinuous. The right-hand lower semicontinuity can be proved exactly in the same way as in Theorem 2, so it suffices to show that for fixed \( s \geq 0 \), \( \varphi \in G(s) \), we have \( d(\varphi, G(t)) \rightarrow 0 \) as \( t \uparrow s \). But for \( t < s \), we have

\[
d(\varphi, G(t)) \leq E(\varphi - E(\varphi \mid \mathcal{F}_t)) + d(E(\varphi \mid \mathcal{F}_t), G(t))
\]

\[
= E(\varphi - E(\varphi \mid \mathcal{F}_t)) + d(E(\varphi \mid \mathcal{F}_t), F_t))
\]

\[
\leq E(\varphi - E(\varphi \mid \mathcal{F}_t)) + E(E(\varphi \mid \mathcal{F}_t) - \varphi) + d(\varphi, F_t)).
\]

By Proposition 1 we have \( E(\varphi - E(\varphi \mid \mathcal{F}_t)) \rightarrow 0 \) as \( t \uparrow s \), and by Lemma 2 we have \( E(d(\varphi, F_t)) \rightarrow 0 \) as \( t \uparrow s \). Therefore \( G \) is a lower semicontinuous set-valued mapping with closed values. Suppose now that \( P \) is nonatomic. Clearly, the sets \( G(t) \) are decomposable (i.e., \( \varphi \chi_A + \psi \chi_{\Omega \setminus A} \in G(t) \) whenever \( \varphi, \psi \in G(t) \) and \( A \in \mathcal{F}_t \)). We can apply the Antosiewicz-Cellina continuous selection theorem (see Bressan and Colombo [1, Theorem 3]) to \( G \), and get a continuous mapping \( g: [0, +\infty) \rightarrow L^1(\Omega, \mathcal{F}, R^n) \) such that \( g(t) \in G(t) \) for all \( t \geq 0 \). If \((F_t, \mathcal{F}_t)\) has convex values, as in the proof of Theorem 2, we get a continuous selection \( g \) applying Michael's theorem. Thus, the stochastic process \((f_t)\) defined by \( f_t(\omega) = g(t)(\omega) \) has desired properties.

If we assure the continuity of the conditional expectation operator \( t \mapsto E(\varphi \mid \mathcal{F}_t) \), then we can extend Hess's result [3, Theorem 3.2] on the martingale selection of discrete time set-valued martingale and obtain a continuous martingale selection result. A set-valued process \((F_t, \mathcal{F}_t)\) is a set-valued martingale if

\[
\{ \varphi \in L^1(\Omega, \mathcal{F}, \mathcal{P}) : \varphi \text{ is } \mathcal{F}_s\text{-measurable selection of } F_s \}
\]

\[
= \text{cl}\{E(\varphi \mid \mathcal{F}_s) : \varphi \text{ is } \mathcal{F}_t\text{-measurable selection of } \mathcal{F}_t \}
\]

for any \( 0 \leq s \leq t \), (see Hiai and Umegaki [4], Hess [3]). We propose the following continuous time version of Hess's theorem.

**Proposition 2:** Let \((F_t, \mathcal{F}_t)\) be a set-valued martingale. If for every \( t \geq 0 \) we have \( \mathcal{F}_t = \mathcal{F}_t^- \), then \((F_t, \mathcal{F}_t)\) admits a martingale selection \((f_t, \mathcal{F}_t)\) with \( P \)-almost all paths left-hand continuous. If for every \( t \geq 0 \) we have \( \mathcal{F}_t = \mathcal{F}_t^+ \), then \((F_t, \mathcal{F}_t)\) admits a martingale selection \((f_t, \mathcal{F}_t)\) with \( P \)-almost all paths right-hand continuous.

**Proof:** Consider the discrete time set-valued martingale \((F_n)_{n=0,\ldots} \) obtained
from \((F_t, \mathcal{F}_t)\) by taking \(t = 0, 1, \ldots\). By the Hess result, \((F_n)\) has a martingale selection \((f_n)\) (i.e., there exists a sequence of \(\mathcal{F}_n\)-measurable mappings \(f_n: \Omega \to \mathbb{R}^n\) such that \(f_n\) is a selection of \(F_n\) and \(f_n = E(f_{n+1} | \mathcal{F}_n)\) for \(n = 0, 1, \ldots\)). For \(t \in [0, +\infty) \setminus \{0, 1, 2, \ldots\}\) we define \(f_t: \Omega \to \mathbb{R}^n\) by \(f_t = E(f_n | \mathcal{F}_t)\) where \(n - 1 < t < n\). Clearly, \((f_t)\) is a martingale selection of \(F\). By Proposition 1, \((f_t)\) has \(P\)-almost all paths left-hand (respectively, right-hand) continuous.

References
