Semi-classical Limit in a Semiconductor Superlattice

PHILIPPE BECHOUCHE*

Univ. Nice, Lab. J. A. Dieudonné URM 6621 du CNRS, Parc Valrose, F-06108 Nice, France

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In this paper, we perform a mathematical study of a semiconductor superlattice. Since the thickness of the layers is very small, quantization plays an important role. The modelling is therefore given by a Schrödinger equation with a periodic potential. The scaled lattice thickness is denoted by a small parameter $\varepsilon$ which is of the same order of magnitude as the Planck constant. When this parameter tends to zero, i.e., the semi-classical limit, we obtain classical transport of the charge carriers described by a Vlasov equation.

Keywords: Semiconductor, superlattice, quantum transport, Schrödinger equation, semi-classical limit

1. INTRODUCTION

Superlattices technology represent a modern technology used also in the design of semiconductors. In order to gain insight in the complex physical behaviour of these devices it is sometimes necessary to have very precise models which give a description on the atomic level. In this work we start with such a fundamental model that takes into account the quantum effects and investigate its semi-classical approximation in a mathematically rigorous way.

A superlattice can be described as material which consists of a stack of layers, those will be of two kinds: a layer $A$ and a layer $B$ as it is shown on Figure 1.

This is the model of Si/Ge, GaAs/AlAs or InAsSb/InSb superlattices for instance.

In the first part of this paper, we give a quantum description of such superlattices describing structures which are periodic in one direction $x_1$ but whose properties vary slowly in other directions $x_2, x_3$. This means that in this model we regard layers which have a slowly varying doping profile.

The motion of a charged particle in a stratified medium can be described by the Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial T} = -\frac{\hbar^2}{2m} \Delta x \Psi(X, T) + qV(X)\Psi(X, T) \quad (1.1)$$

$$X = (X_1, X_2, X_3) \in \mathbb{R}^3, \ T \in \mathbb{R}$$

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*e-mail: phbe@math.unice.fr
Let us introduce new variables $x, t, \varepsilon$:

$$x = \frac{X}{L}, \quad t = \frac{T}{T_0}, \quad \varepsilon = \frac{\lambda}{L}$$

We now scale the wave function.

We recall that $\Psi$ has to be normalized in such a way that:

$$\int_{\mathbb{R}^3} |\Psi(X, T)|^2 dX = 1.$$ 

So we put:

$$\psi(x, t) = L^{3/2} \Psi(Lx, T_0t)$$  \hfill (1.5)

We remark that $\psi$ is also normalized.

Let us choose $T_0$ and $L$ such that:

$$\frac{\hbar}{T_0} \cdot \frac{\tau^2}{m\lambda^2} = \varepsilon \quad \text{and} \quad \frac{\hbar^2}{2mL^2} \cdot \frac{\tau^2}{m\lambda^2} = \frac{\varepsilon^2}{2}$$  \hfill (1.6)

With this scaling Eq. (1.1) becomes:

$$i\varepsilon \partial_t \psi = -\frac{\varepsilon^2}{2} \Delta_x \psi(x, t) + v\left(\frac{X_1}{\varepsilon}, x_2, x_3\right) \psi(x, t)$$

$$x \in \mathbb{R}^3, \quad t \in \mathbb{R}$$  \hfill (1.7)

We want to study rigorously the semi-classical limit of 1.7 when $\varepsilon$ goes to zero. In particular we are concerned with the limit behavior of the density:

$$n_\varepsilon(x, t) = |\psi(x, t)|^2$$  \hfill (1.8)

Semi-classical limits of Schrödinger operators have been intensively studied. In the case of perturbed periodic potential we refer to [2]. In this works the limit behaviour of the Schrödinger operator is given. Our purpose is to study the limit behaviour of the particle dynamics. This problem has been already been investigated in the case of: $-v$ periodic [8, 6]$- v \in C^1[4, 7, 6] - v + u$ with $v$ periodic and $u$ a small perturbation [9, 1]. The result in the periodic case is that the limit concentration is given by the sum of the concentrations $n_p(x, t)$ $p = 1, \ldots, \infty$.
corresponding to the eigenvalue problems e.g. [8].

\[- \frac{\varepsilon^2}{2} \frac{d^2}{dx_1^2} \psi_p(x_1, k_1) + v \left( \frac{x_1}{\varepsilon} \right) \psi_p(x_1, k_1) = E_p(k) \psi_p(x_1, k_1) \psi_p(x_1 + \varepsilon \gamma, k_1) = e^{i \varepsilon \gamma \gamma} \psi_p(x_1, k_1)\]

and the concentration are given by:

\[n_p(x, t) = \int f_p(x, k_1, t) \, dk_1\]

where the non negative distribution \(f_p\) solves the transport equations.

\[\partial_t f_p(x, k_1, t) + \partial_{k_1} E_p(k_1) \cdot \partial_{x_1} f_p(x, k_1, t) = 0.\]

In this work the main tools are the Wigner transform of wave functions and the Bloch decomposition of Schrödinger operators.

The Bloch decomposition e.g. [12, 4, 11] is the projection of wave functions on generalized eigenspaces, the Floquet spaces. In our framework they depend on \(x_2, x_3\). To each eigenspace \(V_p(x_2, x_3)\) corresponds an energy band \(E_p(k_1, x_2, x_3)\) and the projector \(P_p(x_2, x_3)\). The main difficulty is that the projectors \(P_p(x_2, x_3)\) do not commute with \((-\varepsilon^2/2) \Delta_{x_2, x_3}\).

We overcome this difficulty by using asymptotic expansions of Wigner transforms given in [6].

In the following of this paper, we will make the assumption that the energy Bands don’t cross. In fact it is possible to choose the eigenvectors of the Bloch decomposition such that the energy Bands do not cross. Such an choice can always been done for a Bloch decomposition in one dimension, which is the case here, but it has been proved that it is false in higher dimensions (e.g. [5, 3]).

The result is that the distribution function solves the transport equation:

\[\partial_t f_p(x, k_1, t) + \partial_{k_1} E_p(k_1, x_2, x_3) \cdot \nabla_{x_1} f_p(x, k_1, t)
+ \nabla_{x_2, x_3} E_p(k_1, x_2, x_3) \cdot \nabla_{k_1, k_2} f_p(x, k_1, t)\]

and the limit concentration \(n\) is given by:

\[n(x, t) = \sum_p \int_{[-\pi, \pi]^2 \mathbb{R}^2} f_p(x, k_1, t) \frac{dk_1}{2\pi} dk_2 dk_3\]

We remark in Eq. (1.9) that the band velocity in the first variable

\[\partial_{k_1} E_p(k_1, x_2, x_3)\]

depends on the two parameters \(x_2, x_3\) and that in the other variable the velocity corresponds to free transport.

This paper is organized as follows: In the next section we introduce the Bloch decomposition of the wave operators but only in the \(x_1\)-direction and examine the \(x_2, x_3\) dependence of the projectors and the energy bands. Then we derive a Wigner equation by using Wigner series in the direction \(x_1\) and Wigner transforms in the directions \(x_2, x_3\). In the last section we pass to the semi-classical limit and give the main results.

The main results concerning Bloch waves are used in [2, 4, 6] and those concerning Wigner transforms and series in [7, 4, 6].

In this paper we will assume that the initial data is \(\varepsilon\)-oscillating; more precisely we assume:

\[|\alpha| \leq 4 \int_{\mathbb{R}^3} |D^\alpha \psi_I(x)|^2 \, dx \leq \frac{C}{\varepsilon^{2|\alpha|}} \quad \alpha \in \mathbb{N}^3 \]

Remark 1.1 As in [6], this assumption can be weaken to the usual condition of \(\varepsilon\)-oscillation of [4].

Since the energy of the system reads

\[E(t) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla \psi|^2 \, dx + \int \left( \frac{x_1}{\varepsilon}, x_2, x_3 \right) |\psi|^2\]

this implies that

\[\int_{\mathbb{R}^3} |\nabla \psi(x, t)|^2 \, dx \leq \frac{C}{\varepsilon^2} \]

(1.12)
2. BLOCH DECOMPOSITION

In this section, we will recall classical results about Bloch decomposition. But here we only make one-dimensional \((x_1)\) Bloch decomposition and then analyze the regularity of the eigenvalues and of the projectors w.r.t. \(x_2\) and \(x_3\). We want to "diagonalize" the Hamiltonian

\[
H\psi = -\frac{\varepsilon^2}{2} \frac{\partial^2 \psi}{\partial x_1^2} + V\left(\frac{x_1}{\varepsilon}, x_2, x_3\right) \psi
\]  

(2.1)

If we consider the microscopic scale (e.g., \([10, 1]\)) in the first variable,

\[
\psi_1(y_1, x_2, x_3) = \varepsilon^{1/2} \psi_2(\varepsilon y_1, x_2, x_3)
\]

As in [10, 4] or [6, 1], we expand the function \(\psi_1 \in L^2(\mathbb{R}^3)\) into the Bloch waves

\[
\phi(y_1, x_2, x_3, k) = \sum_{\gamma \in \mathbb{Z}} \psi_1(y_1 - \gamma, x_2, x_3)e^{ik_1\gamma},
\]

\(k \in \mathbb{R}\)

(2.2)

**Properties of the Bloch Waves**

**Property 2.1**

\[
\Psi_1(y_1, x_2, x_3) = \int_{-\pi}^{\pi} \phi(y_1, x_2, x_3, k_1) \frac{dk_1}{2\pi},
\]

\[\|\phi\|_{L^2([0, 1] \times \mathbb{R}^2 \times [-\pi, \pi])} = \|\psi_1\|_{L^2(\mathbb{R}^3)}
\]

(2.3) (2.4)

**Property 2.2** \(\psi\) is \(k\)-periodic i.e.,

\[
\phi(y_1 + \gamma, x_2, x_3, k_1) = e^{ik_1\gamma} \phi(y_1, x_2, x_3, k_1)
\]

(2.5)

Let \(H\) be the Hamiltonian defined on

\[
D(H) = \left\{ \psi \in L^2(\mathbb{R}^3), \frac{\partial^2}{\partial y_1^2} \psi \in L^2(\mathbb{R}^3) \right\}
\]

**Property 2.3** If \(H_{k_1}\) is the Hamiltonian which acts on \(k\)-periodic functions i.e.,

\[
D(H_{k_1}) = \{ \phi(y_1, x_2, x_3, k_1) \in L^2([0, 1] \times \mathbb{R}^2 \times [-\pi, \pi]), \\
\frac{\partial^2}{\partial y_1^2} \phi(y_1, x_2, x_3, k_1) \in L^2([0, 1] \times \mathbb{R}^2 \times [-\pi, \pi]) \}
\]

and

\[
\phi(y_1 - \gamma, x_2, x_3, k_1) = e^{ik_1\gamma} \phi(y_1, x_2, x_3, k_1)
\]

then e.g. [12, 11] \(H_{k_1}\) is a compact operator and diagonalizable. Moreover

\[
H\psi = \int_{-\pi}^{\pi} H_{k_1} \phi(\cdot, k_1) \frac{dk_1}{2\pi}
\]

(2.6)

where \(\phi\) is given by 2.2. Let us denote by \(E_1(k_1, x_2, x_3), \ldots, E_p(k_1, x_2, x_3), \ldots\) the eigenvalues on \(H_{k_1}\),

\[
E_1(k_1, x_2, x_3) \leq E_2(k_1, x_2, x_3) \leq \cdots \leq E_p(k_1, x_2, x_3) \leq \cdots
\]

and by \(\phi_p(\cdot, k_1) = \phi_p(y_1, x_2, x_3, k_1)\) the associated eigenvectors. Then we have

**Theorem 2.1** cf. [11, 12] There is a choice of the eigenvector family \(\{\phi_p(\cdot, k_1)\}\) such that it is an Hilbert Basis of \(L^2(\mathbb{R}^3)\) and such that the functions

\[
k_1 \mapsto \phi_p(y_1, \cdot, k_1)\bar{\phi_p}(z_1, \cdot, k_1)
\]

are all analytical. The \(\phi_p(\cdot, k_1)\) are \(k_1\)-periodic and \(\phi_p \in L^2([0, 1] \times \mathbb{R}^2 \times [-\pi, \pi])\).

**Definition 2.2** Let us define now the Wannier functions:

\[
\psi_p(y_1, x_2, x_3) = \int_{-\pi}^{\pi} \phi_p(y_1, x_2, x_3, k_1) \frac{dk_1}{2\pi}
\]

(2.7)

These functions are associated to the Bloch waves \(\phi_p(\cdot, k_1)\). They also verify the Property 2.1 and the following properties.
Property 2.4

\[ \phi_p(\gamma, x_2, x_3, k_1) = \sum_{\gamma \in \mathbb{Z}} \psi_p(-\gamma, x_2, x_3) e^{ik_1 \gamma i} \]

and \( \psi_p(-\gamma, x_2, x_3) \) are the Fourier coefficients of \( \phi_p(\gamma, x_2, x_3, k_1) \).

Property 2.5

\[ \int_{\mathbb{R}^3} |\psi_p(y_1, x_2, x_3)|^2 dy_1 dx_2 dx_3 = 1 \]

and

\[ \int_{\mathbb{R}} \psi_p(y_1, x_2, x_3) \overline{\psi_p}(y_1 - \gamma, x_2, x_3) dy_1 = 0 \quad \gamma \neq 0 \]

Definition 2.3 Let us define the Floquet subspaces by

\[ V_p(x_2, x_3) = \left\{ \psi \in L^2(\mathbb{R}^3), \psi(y_1, x_2, x_3) = \int_{-\pi}^{\pi} \alpha_p(k_1, x_2, x_3) \right. \]

\[ \psi_p(y_1, x_2, x_3) (2.8) \]

where the \( \phi_p \) are the Bloch waves associated to \( H_{k_1} \).

Theorem 2.4 cf. [11, 12]

\[ L^2(\mathbb{R}^3) = \bigoplus_p V_p(x_2, x_3), \quad x_2, x_3 \in \mathbb{R} \]

Let \( \hat{E}_p(\gamma, x_2, x_3) \), \( \gamma \in \Gamma \), be the Fourier coefficient of \( E_p(k_1, x_2, x_3) \). We also have \( V_p(x_2, x_3) \in D(H) \) and

\[ H(\psi) = \sum_{\gamma \in \mathbb{Z}} \hat{E}_p(\gamma, x_2, x_3) \psi(y_1 + \gamma, x_2, x_3) \]

\[ \psi \in V_p(x_2, x_3); \]

moreover we have \( H(\psi) \in V_p(x_2, x_3) \).

For the proof of Theorem 2.4 we refer to [10, 11, 8, 6].

Remark 2.5 The function \( E_p(\gamma, x_2, x_3) \) is even. Therefore its Fourier coefficients satisfy

\[ \hat{E}_p(\gamma, x_2, x_3) = \hat{E}_p(-\gamma, x_2, x_3) \in \mathbb{R} \]

Remark 2.6 The equation

\[ H_{k_1} \psi_p(y_1, x_2, x_3, k_1) = \hat{E}_p(k_1, x_2, x_3) \phi_p(y_1, x_2, x_3, k_1) \]

implies (by derivating in direction \( x_2 \), multiplying by \( \phi_p \) and integrating in \( y_1 \)) that

\[ \frac{\partial}{\partial x_2} E_p(k_1, x_2, x_3) \]

\[ = \int_{\mathbb{R}} \frac{\partial}{\partial x_2} \psi(y_1, x_2, x_3, k_1) \phi_p(y_1, x_2, x_3, k_1) \]

\[ \quad dy_1 \]

(2.10)

Let us denote by \( \Pi_p(x_2, x_3) \) the projectors from \( L^2(\mathbb{R}^3) \) on \( V_p(x_2, x_3) \). Let us define the operator

\[ G_k = \frac{1}{2} \left( \frac{\partial}{\partial y_1} + ik_1 \right)^2 + V(y_1, x_2, x_3). \]

\[ = e^{ik_1 \gamma} H_k(e^{-ik_1 \cdot}) \]

\[ DG_k = \left\{ \psi \in H^2(\mathbb{R}^3), \psi(y_1 + \gamma, x_2, x_3) = \psi(y), \quad \gamma \in \Gamma \right\} \]

Therefore the projectors \( \Pi_p(k_1, x_2, x_3) \) on the eigenspace corresponding to \( E_p(k_1, x_2, x_3) \) for the operator \( H_k, G_k \) can be defined by

\[ P_p(k_1, x_2, x_3) \]

\[ = \int_{|\zeta| = \rho} (G_k - E_p, (k_1, x_2, x_3) - \zeta)^{-1} d\zeta \]

\[ \Pi_p(k_1, x_2, x_3) = e^{ik_1 \gamma} P_p(k_1, x_2, x_3) \]

(2.11)

Remark 2.7 The functions \( E_p(\cdot, x_2, x_3), P_p(\cdot, x_2, x_3) \)

and \( \Pi_p(\cdot, x_2, x_3) \) are analytical since we suppose
that the energy bands $E_p(\varepsilon_1, x_2, x_3)$ never cross for every $p \geq 0$.

**Remark 2.8** Of course we have for $\psi \in L^2(\mathbb{R}^3)$

$$\Pi_p(x_2, x_3)\psi = \int_{-\pi}^{\pi} \Pi_p(k_1, x_2, x_3)\phi_p(k_1, x_2, x_3) \frac{dk_1}{2\pi}$$

where $\phi_p$ is the Bloch wave associated to $\psi$ and Eq. (2.9) implies that $\Pi_p$ commutes with $H$. More exactly we have

$$H \Pi_p\psi(y_1, x_2, x_3) = \Pi_p H \psi(y_1, x_2, x_3)$$

$$= \sum_{\gamma \in \mathbb{Z}} E_p(\gamma, x_2, x_3) \Pi_p\psi(y_1 + \gamma, x_2, x_3)$$

(2.12)

Returning to the microscopic variable $y_1 = (x_1/\varepsilon)$ Eq. (2.12) becomes

$$\left[ -\frac{\varepsilon^2}{2} \frac{\partial^2}{\partial x_1^2} + V\left(\frac{x_1}{\varepsilon}, x_2, x_3\right) \right] \Pi_p\psi(x_1, x_2, x_3)$$

$$= \sum_{\gamma \in \mathbb{Z}} E_p(\gamma, x_2, x_3) \Pi_p\psi(x_1 + \varepsilon\gamma, x_2, x_3)$$

(2.13)

which is a “diagonalisation” of $H$ defined in (2.1).

We now turn back to Eq. (1.7), by taking the projection of (1.7) on $V_p$ we get

$$i\varepsilon \partial_t \Pi_p \psi = H \Pi_p \psi - \frac{\varepsilon^2}{2} \Pi_p (\Delta_{x_2, x_3} \psi)$$

(2.14)

where $H$ is the Hamiltonian defined in (2.1) and $\Delta_{x_2, x_3} = (\partial^2/\partial x_2^2) + (\partial^2/\partial x_3^2)$ is the Laplacian in both variable $x_2$ and $x_3$.

### 3. WIGNER TRANSFORMS

In this paper since we have periodicity in the direction $x_1$, we will associate Wigner series in the direction $x_1$, and classical Wigner transforms in the direction $x_2, x_3$.

For a couple of functions $\psi, \varphi \in L^2(\mathbb{R}^3)$, let us define

$$W^\varepsilon(\psi, \varphi)(x, k)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^3} \sum_{\gamma \in \mathbb{Z}} \psi\left(x_1 - \frac{\varepsilon\gamma}{2}, x_2, x_3 - \frac{\varepsilon x_3}{2}\right) \varphi\left(x_1 + \frac{\varepsilon\gamma}{2}, x_2 + \frac{\varepsilon x_2}{2}, x_3 + \frac{\varepsilon x_3}{2}\right)$$

$$e^{ik_1\gamma} e^{i(k_2x_2 + k_3x_3)} \, dx_2 \, dx_3.$$  

(3.1)

We define as in [7, 8, 10] an algebra of test functions by

$$A = \left\{ w = w(x, k) = \frac{1}{(2\pi)^3} \sum_{\gamma \in \mathbb{Z}} e^{ik_1\gamma} \int_{\mathbb{R}^3} \hat{w}(k_1, \gamma, \eta_2, \eta_3) e^{i(k_2 \eta_2 + k_3 \eta_3)} \, d\eta_2 \, d\eta_3 ; \quad \hat{w} \in L^1(\mathbb{Z} ; L^1(d\eta_2 \times \eta_3 ; C^0(\mathbb{R}^3))) \right\}$$

(3.2)

$A$ is equipped with its natural norm:

$$\|w\|_A = \sum_{\gamma \in \mathbb{Z}} \|\hat{w}(k_1, \gamma, \eta_2, \eta_3)\|_{L^1(d\eta_2 \times d\eta_3)}$$

Let us recall a theorem from [7, 10, 6]:

**Theorem 3.1** Let $(\psi^\varepsilon)$ be a bounded family of $L^2(\mathbb{R}^3)$; then the family $(W^\varepsilon)$ is bounded in $A'$. The accumulation points in $A'$ weak $- \ast$ when $\varepsilon \to 0$ are bounded non negative measures.

$$W^\varepsilon \to f \quad \text{in } A' \text{ weak } - \ast.$$

Moreover we have

$$|\psi^\varepsilon|^2 \to \int_{[-\pi, \pi] \times \mathbb{R}^2} f(\varepsilon, k) \frac{dk}{2\pi} \quad \text{in } C^0(\mathbb{R}^3) \text{ weak } - \ast$$

if $(\psi^\varepsilon)$ is $\varepsilon$-oscillatory for the variable $x_2, x_3$.
expansion of Wigner transform with a pseudo differential operator \( p(x, \varepsilon D) \).

**Definition 3.2** Let us consider a symbol
\[
p(x, \xi) \in \mathcal{S}(\mathbb{R}^m_x \times \mathbb{R}^m_\xi)
\]
then for \( f \in \mathcal{S}'(\mathbb{R}^m) \)
\[
P(x, \varepsilon D)f(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m_x \times \mathbb{R}^m_\xi} p(x, \xi)f(y) e^{i(x-y)\cdot \xi} d\xi dy
\]
(3.3)

**Definition 3.3** The Weyl operator associated to the symbol \( p(x, \xi) \) is defined by
\[
P^w(x, \varepsilon D)f(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m_x \times \mathbb{R}^m_\xi} p\left(\frac{x+y}{2}, \varepsilon \xi\right)f(y) e^{i(x-y)\cdot \xi} d\xi dy
\]
(3.4)

**Definition 3.4** If the notation \( \{p, q\} \) denotes the Poisson bracket of \( p = p(x, \varepsilon) \) and \( q = q(x, \xi) \) defined by
\[
\{p, q\} (x, \xi) = \nabla_x p(x, \xi) \cdot \nabla_x q(x, \xi) - \nabla_\xi p(x, \xi) \cdot \nabla_\xi q(x, \xi)
\]
(3.5)

**Remark 3.5** We can weaken the regularity of \( p \) if we take a more regular function \( f \).

**Definition 3.6** Classical Wigner transform expansion with a pseudo differential operator. The classical Wigner transform of \( (f, g) \in L^2(\mathbb{R}^m) \) is defined by
\[
w_{\varepsilon}(f, g) = \frac{1}{2\pi} \int_{\mathbb{R}^m} f\left(x - \frac{\varepsilon z}{2}\right) g^*\left(x + \frac{\varepsilon z}{2}\right) e^{i\varepsilon z \cdot \xi} d\xi dz
\]
(3.6)

**Property 3.6** Let \( p \in C^\infty(\mathbb{R}^m_x \times \mathbb{R}^m_\xi) \) satisfy for some \( M \geq 0 \):
\[
\forall \alpha \in (N_+^m \times N_+^m) : |\partial_x^\alpha p(x, \xi)| \leq C_\alpha (1 + |\xi|)^M.
\]
If \( f \) and \( g \) lie in a bounded set of \( L^2(\mathbb{R})^m \), we have the expansion
\[
w_{\varepsilon}(P^w(x, \varepsilon D)f, g) = pw_{\varepsilon}(f, g) + \frac{\varepsilon}{2i} \{p(x, \xi), w_{\varepsilon}(f, g)\} + \varepsilon^2 r_{\varepsilon}
\]
(3.7)
where \( r_{\varepsilon} \) is bounded in \( \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^m) \) uniformly w.r.t. \( \varepsilon \).

**Theorem 3.7** Let \( p \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) satisfy for some \( M \geq 0 \),
\[
\forall \alpha \in (N_+^2 \times N_+^2) : |\partial_{x_1,x_2,k_1,k_2} p(x_2, x_3, k_2, k_3)|
\]
\[
< C_\alpha (1 + |\xi|)^M
\]
(3.8)

Then as \( \phi \) and \( \varphi \) lie in a bounded set of \( L^2(\mathbb{R}^3) \) we have the expansion
\[
W^\varepsilon(p^w(x_2, x_3, \varepsilon D_{k_1, k_2}))(\psi, \varphi) = p(x_2, x_3, k_2, k_3) W^\varepsilon(\psi, \varphi)
\]
\[
+ \frac{\varepsilon}{2i} \{p(x_2, x_3, k_2, k_3), W^\varepsilon(\psi, \varphi)\} + \varepsilon^2 R_{\varepsilon}
\]
(3.9)
where \( R_{\varepsilon} \) is bounded in \( \mathcal{S}'(\mathbb{R})^6 \) uniformly w.r.t. \( \varepsilon \).

**Theorem 3.8** Let \( a \in C^0(\mathbb{R}^2 \times \mathbb{R}, C^1(\mathbb{R}^k)) \) 1-periodic with respect to \( k_1 \). Then as \( \psi \) and \( \varphi \) lie in a bounded set of \( L^2(\mathbb{R}^3) \) we have the expansion
\[
W^\varepsilon(a(\varepsilon D_1, x_2, x_3))(\psi, \varphi) = (a(D_1, x_2, x_3)) W^\varepsilon(\psi, \varphi)
\]
\[
+ \frac{\varepsilon}{2i} \nabla_{x_2,x_3} a(k_1, x_2, x_3) \cdot \nabla_{k_1} W^\varepsilon(\psi, \varphi)
\]
\[
+ \frac{\varepsilon}{2i} \nabla_{k_1} a(k_1, x_2, x_3) \cdot \nabla_{x_1} W^\varepsilon(\psi, \varphi) + \varepsilon^2 T_{\varepsilon}
\]
(3.10)
in \( \mathcal{S}' \) with \( T_{\varepsilon} \) bounded in \( \mathcal{S}'(\mathbb{R}^6) \) uniformly w.r.t. \( \varepsilon \).
The proof of Theorem (3.7) and Theorem (3.8) is almost the same as in [6]: we make a Taylor expansion of the Fourier transform of the Wigner function and we get the desired result.

4. COMMUTATOR ESTIMATES

Let $\psi_\varepsilon$ be a bounded family of functions in $L^2(\mathbb{R}^3)$ satisfying Eq. (1.7). $\Pi_\varepsilon \psi_\varepsilon$ satisfies Eq. (2.14) with $\Pi_\varepsilon \psi_\varepsilon = \psi_\varepsilon \in V_\varepsilon$. In order to pass to the limit $\varepsilon \to 0$, let us rewrite Eq. (2.14) in another way:

\[ i\partial_t \Pi_\varepsilon \psi = \imath \varepsilon E_\varepsilon (\varepsilon D_1, x_2, x_3) \Pi_\varepsilon \psi - \frac{\varepsilon^2}{2} \Delta_{x_2,x_3} \Pi_\varepsilon \psi + \frac{\varepsilon^2}{2} [\Delta_{x_2,x_3}; \Pi_\varepsilon] \psi \]

(4.1)

where $[\Delta_{x_2,x_3}; \Pi_\varepsilon]$ is a commutator defined by

\[ [\Delta_{x_2,x_3}; \Pi_\varepsilon] \psi = \Delta_{x_2,x_3} \Pi_\varepsilon \psi - \Pi_\varepsilon \Delta_{x_2,x_3} \psi \]

(4.2)

**Proposition 4.1**

\[ \int_\mathbb{R}^3 n_p \, dx \leq \frac{C}{\varepsilon \rho^d} \]

(4.5)

Therefore the sequence $\Sigma_\varepsilon n_p$ is uniformly convergent w.r.t. $\varepsilon$.

**Proof of Proposition (4.1)** In the following, we will note with a prime a derivation w.r.t. the variable $x_2$ or $x_3$ and define $\psi_\varepsilon = \Pi_\varepsilon \psi$. We have

\[ \Pi_\varepsilon \varepsilon^2 \psi'' = \varepsilon^2 \psi''_p + \sum_{q \neq p} (\psi''_q - \Pi_\varepsilon (\varepsilon^2 \psi'')) \]

Therefore we have

\[ W^\varepsilon (\Pi_\varepsilon \varepsilon^2 \psi'', \psi_p) = W^\varepsilon (\varepsilon^2 \psi''_p, \psi_p) + \sum_{q \neq p} (W^\varepsilon (\psi''_q, \psi_p) - W^\varepsilon (\Pi_\varepsilon (\varepsilon^2 \psi''), \psi_p)) \]

Since the series $\Sigma_{q \neq p} \Pi_\varepsilon \psi$ and $\Sigma_{q \neq p} \Pi_\varepsilon \varepsilon^2 \psi''$ converge uniformly by Lemma (4.2) see also [6, 8], we only have to verify that for every fixed $q$, the terms with different indices converge to zero in $S'(\mathbb{R}^6)$. By a little computation using Lemma (4.1) and the
asymptotic expansion of Theorem (3.7), we have

\[ W^\varepsilon(\psi_p', \psi_p) - W^\varepsilon(\Pi_q(\varepsilon^2 \psi_p'), \psi_p) \]
\[ = \varepsilon \text{div}_x \left[ W^\varepsilon \left( \frac{x_1}{\varepsilon}, x_2, x_3 \right) \right. \]
\[ \left. \times (2\varepsilon \Pi_q' \psi_p' + \varepsilon^2 \Pi_q'' \psi_p, \psi_p) \right] + O(\varepsilon^2) \]
\[ = O(\varepsilon^2) \]

We are done.

5. SEMI-CLASSICAL LIMIT

We now compute the semi-classical limit i.e., \( \varepsilon \to 0 \) of the Wigner equation.

**Theorem 5.1** As \( \varepsilon \to 0 \), if for some subsequence \( W^\varepsilon \to f_p \) in \( \mathcal{A}'\)-weak*, then we have \( W^\varepsilon \to f_p \) in \( C^0(0, T, S'_{\mathbb{R}^6}) \)

where \( f_p \geq 0 \) verifies the Vlasov equation:

\[ \frac{\partial_t f_p}{f_p} = \{ \mathcal{E}_p, f_p \} \] (5.1)
\[ f_p(0, x, k) = f_{p0}(x, k). \] (5.2)

where

\[ \mathcal{E}_p(k, x_2, x_3) = E_p(k_1, x_2, x_3) + \frac{|k_2|^2}{2} + \frac{|k_3|^2}{2} \]

and \( n'(t, x) \to n(t, x) \) in \( C(0, T, C^0(\mathbb{R}^3) - \text{weak}^*) \) where:

\[ n(x, t) = \sum_p \int_{[-\pi, \pi] \times \mathbb{R}^2} f_p(x, k, t) \frac{dk}{2\pi}. \]

**Remark 5.2** In extended form the Eq. (5.1) reads

\[ \frac{\partial_t f_p}{f_p} = \left( \frac{k_2}{k_3} \right) \cdot \nabla_x f_p (t, x, k) \]
\[ - \nabla_{x_2, x_3} E_p(t, x, k) \cdot \nabla_{k_2, k_3} f_p(t, x, k) \] (5.3)

The proof of Theorem (5.1) follows immediately from proposition (4.1) by using the asymptotic expansions from Theorems (3.7) and (3.8) applied to Eq. (4.1).

**References**


**Author Biography**

**Philippe Bechouche** is a Researcher in Mathematics at the University of Nice Sophia-Antipolis. His research interests are in modeling classical and quantum semiconductors—more specifically in how semiclassical limits describe the link between quantum equations and kinetic equations and permit us to derive classical models of semiconductors with quantum corrections.
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