THE SOLUTION OF AN OPEN PROBLEM GIVEN BY H. HARUKI AND T.M. RASSIAS

BARA KIM

Korea Advanced Institute of Science and Technology (KAIST)
Department of Mathematics and Center for Applied Mathematics
373-1 Kusong-Dong, Yusong-Gu, Taejon 305-701, Korea
e-mail: bara@mathx.kaist.ac.kr

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Haruki and Rassias [1] generalized the Poisson kernel in two dimensions and discussed integral formulas for each case. They presented an open problem for an integral formula. In this paper, we give a solution to that problem.

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1. Introduction

Haruki and Rassias [1] introduced two types of generalizations of the Poisson kernel. One of them is defined by

\[ Q(\theta; a, b) \triangleq \frac{1 - ab}{(1 - ae^{i\theta})(1 - be^{-i\theta})}, \]

where \(a, b\) are complex parameters satisfying \(|a| < 1\) and \(|b| < 1\).

They proved the integral formulas:

\[ \frac{1}{2\pi} \int_0^{2\pi} Q(\theta; a, b)d\theta = 1, \]

\[ \frac{1}{2\pi} \int_0^{2\pi} Q(\theta; a, b)^2d\theta = \frac{1 + ab}{1 - ab}. \]

They set the open problem as follows:

"Let
where \( a, b \) are complex parameters satisfying \(|a| < 1\) and \(|b| < 1\). Compute \( I_n \) for \( n = 2, 3, 4, \ldots \). In the next section, we will give the solution to the problem.

2. Solution of the Problem

**Theorem 1:** \( I_n \), defined by (3), satisfies

\[
I_n = \sum_{j=0}^{n} \frac{(2n-j)!}{j!(n-j)!^2} \left( \frac{ab}{1-ab} \right)^{n-j},
\]

for \( n = 0, 1, 2, \ldots \), and complex values \( a, b \) are such that \(|a| < 1\) and \(|b| < 1\).

**Proof:** By the change of variables, with \( z = e^{i\theta} \), (3) becomes

\[
I_n = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})} \right)^{n+1} d\theta,
\]

where \( n = 0, 1, \ldots \), \( a, b \) are complex parameters satisfying \(|a| < 1\) and \(|b| < 1\).

In the next section, we will give the solution to the problem.
\[ I_n = \sum_{j=0}^{n} \binom{2n-j}{n-j} \binom{n}{j} \left( \frac{ab}{1-ab} \right)^{n-j} \]

\[ = \sum_{j=0}^{n} \frac{(2n-j)!}{j!(n-j)!^2} \left( \frac{ab}{1-ab} \right)^{n-j}. \]

Note that we obtain (1) and (2) by substituting \( n = 0 \) and \( n = 1 \), respectively.

References
