EXISTENCE AND UNIQUENESS OF A CLASSICAL SOLUTION TO A FUNCTIONAL-DIFFERENTIAL ABSTRACT NONLOCAL CAUCHY PROBLEM

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The aim of this paper is to investigate the existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem in a general Banach space. For this purpose, a special kind of a mild solution is introduced and the Banach contraction theorem and a modified Picard method are applied.

Key words: Abstract Cauchy Problem, Ordinary Functional-Differential Equation, Nonlocal Condition, Existence and Uniqueness of a Classical Solution, Mild Solution, Banach Contraction Theorem, Picard Method.

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1. Introduction

We present four theorems (Theorems 2.1-2.4) on the existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem in an arbitrary Banach space and give an approximation of the solution to the nonlocal problem. In the proofs of the theorems, we introduce a special kind of a mild solution and apply the Banach contraction theorem and a modified Picard method of successive approximations.

The functional-differential nonlocal problem, studied in this paper is of the form:

\[ u'(t) = f(t,u(t),u(a(t))), \quad t \in I, \]
\[ u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \]

where \( I = [t_0, t_0 + T], \ t_0 < t_1 < \ldots < t_p \leq t_0 + T, \ T > 0; \ f: I \times E^2 \rightarrow E \) and \( a: I \rightarrow I \) are given functions satisfying some assumptions; \( E \) is a Banach space with norm \( \| \cdot \|, \ x_0 \in E, \ c_k \neq 0 \ (k = 1, \ldots, p) \) and \( p \in \mathbb{N} \).
The results obtained are generalizations and continuations of those, reported previously in [1-4], with the nonlocal condition of type (1.2). Moreover, the results of the paper include, among other things, a special kind of a mild solution to nonlocal problem (1.1)-(1.2). Therefore, throughout the proofs of the theorems, we apply properties of function $f$ in a greater measure than in [1-3]. Consequently, in contrast with [1-3], now, even if $T$ is an arbitrary positive constant, then $c_k (k = 1, \ldots, p)$ from the nonlocal condition (1.2) can satisfy the inequalities $|c_k| > 1 \ (k = 1, \ldots, p)$. The special kind of a mild solution in this paper is a modification of a mild solution introduced by the author (in [5]), for nonlocal evolution problems. In the case when $c_k = 0 \ (k = 1, \ldots, p)$ and the right-hand side of the functional-differential equation does not depend on the functional argument, some results of Theorem 2.4 are reduced to those (given in [6]) on the existence and uniqueness of a classical solution to the abstract Cauchy problem with the standard initial condition.

If $c_k \neq 0 \ (k = 1, \ldots, p)$ then the results of the paper can be applied in kinematics to determine the evolution $t \mapsto u(t)$ of the location of a physical object for which we do not know the positions $u(t_0), u(t_1), \ldots, u(t_p)$, but we know that the nonlocal condition (1.2) holds. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $u(t_0) = x_0$.

2. Theorems About the Existence and Uniqueness of a Classical Solution

By $X$, we denote the Banach space $C(I, E)$ with the standard norm $\| \cdot \|_X$. So,

$$\| w \|_X = \sup_{t \in I} \| w(t) \|, \ w \in X.$$ 

Assume that $\sum_{k=1}^{p} c_k \neq -1$. A function $u \in X$, satisfying the integral equation

$$u(t) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t} f(\tau, u(\tau), u(a(\tau))) d\tau \right) / \left( 1 + \sum_{k=1}^{p} c_k \right)$$

$$+ \int_{t_0}^{t} f(\tau, u(\tau), u(a(\tau))) d\tau, \ t \in I,$$

is said to be a mild solution of the nonlocal problem (1.1)-(1.2).

A function $u: I \rightarrow E$ is said to be a classical solution of the nonlocal problem (1.1)-(1.2) if

(i) $u$ is continuous on $I$ and continuously differentiable on $I$,

(ii) $u'(t) = f(t, u(t), u(a(t)))$ for $t \in I$

and

(iii) $u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0$.

**Theorem 2.1:** Suppose that $f: I \times E^2 \rightarrow E$, $a: I \rightarrow I$ and $\sum_{k=1}^{p} c_k \neq -1$. If $u$ is a classical solution of the nonlocal problem (1.1)-(1.2), then $u$ is a mild solution of this problem.
Proof: Let $u$ be a classical solution of the nonlocal problem (1.1)-(1.2). Then $u$ satisfies equation (1.1) and, consequently,

$$u(t) = u(t_0) + \int_{t_0}^{t} f(\tau, u(\tau), u(a(\tau)))d\tau, \quad t \in I. \quad (2.2)$$

From (2.2),

$$u(t_k) = u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \quad (k = 1, \ldots, p). \quad (2.3)$$

By (1.2) and (2.3),

$$u(t_0) + \sum_{k=1}^{p} c_k \left(u(t_0) + \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau\right) = x_0. \quad (2.4)$$

Since $\sum_{k=1}^{p} c_k \neq -1$, then (2.4) implies

$$u(t_0) = \left(x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau\right) \left(1 + \sum_{k=1}^{p} c_k\right). \quad (2.5)$$

From (2.2) and (2.5), we obtain that $u$ is a mild solution of the nonlocal problem (1.1)-(1.2). The proof of Theorem 2.1 is complete.

Theorem 2.2: Suppose that $f \in C(I \times E^2, E)$, $a : I \rightarrow I$ and $\sum_{k=1}^{p} c_k \neq -1$. If $u$ is a mild solution of the nonlocal problem (1.1)-(1.2) then $u$ is a classical solution of this problem.

Proof: Let $u$ be a mild solution of the nonlocal problem (1.1)-(1.2). Then $u$ satisfies equation (1.1) and, from the continuity of $f, u \in C^1(I, E)$. Now, we will show that $u$ satisfies the nonlocal condition (1.2). For this purpose, observe that, by (2.1),

$$u(t_0) = \left(x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau\right) \left(1 + \sum_{k=1}^{p} c_k\right) \quad (2.6)$$

and

$$u(t_i) = \left(x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau\right) \left(1 + \sum_{k=1}^{p} c_k\right) + \int_{t_0}^{t_i} f(\tau, u(\tau), u(a(\tau)))d\tau \quad (i = 1, \ldots, p). \quad (2.7)$$

From (2.6) and (2.7), and from some computations,
\[ u(t_0) + \sum_{i=1}^{p} c_i u(t_i) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right) + \sum_{i=1}^{p} c_i \int_{t_0}^{t_i} f(\tau, u(\tau), u(a(\tau)))d\tau = x_0. \]

Therefore, the proof of Theorem 2.2 is complete.

As a consequence of Theorems 2.1 and 2.2, we obtain:

**Theorem 2.3:** Suppose that \( f \in C(I \times E^2, E) \), \( a: I \rightarrow I \) and \( \sum_{k=1}^{p} c_k \neq -1 \). Then \( u \) is the unique classical solution of the nonlocal problem (1.1)-(1.2) if and only if \( u \) is the unique mild solution of this problem.

Now, we will prove the main theorem of the paper.

**Theorem 2.4:** Assume that:

(i) \( a \in C(I, I) \), \( f: I \times E^2 \rightarrow E \) is continuous with respect to the first variable on \( I \) and there is \( L > 0 \) such that

\[
\| f(s, z_1, z_2) - f(s, \tilde{z}_1, \tilde{z}_2) \| \leq L \sum_{i=1}^{2} \| z_i - \tilde{z}_i \| \tag{2.8}
\]

for \( s \in I \), \( z_i, \tilde{z}_i \in E \) \((i = 1, 2)\),

(ii) \( \sum_{k=1}^{p} c_k \neq -1 \)

and

(iii) \[
2LT \left( 1 + \left| \left( \sum_{k=1}^{p} c_k \right) \right| \left( 1 + \sum_{k=1}^{p} c_k \right) \right) < 1. \]

Then the nonlocal Cauchy problem (1.1)-(1.2) has a unique classical solution \( u \). Moreover, the successive approximations \( u_n (n = 0, 1, 2, \ldots) \), defined by the formulas

\[
u(0): = x_0 \text{ for } t \in I \tag{2.9}\]

and

\[
u_n(t) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, u(\tau), u(a(\tau)))d\tau \right) \left/ \left( 1 + \sum_{k=1}^{p} c_k \right) \right. + \int_{t_0}^{t} f(\tau, u_n(\tau), u_n(a(\tau)))d\tau \] \text{ for } t \in I \quad (n = 0, 1, 2, \ldots), \tag{2.10}\]

converge uniformly on \( I \) to the unique classical solution \( u \).

**Proof:** Introduce an operator \( A \) by the formula

\[
(Aw)(t) = \left( x_0 - \sum_{k=1}^{p} c_k \int_{t_0}^{t_k} f(\tau, w(\tau), w(a(\tau)))d\tau \right) \left/ \left( 1 + \sum_{k=1}^{p} c_k \right) \right. \] \quad (2.11)
It is easy to see that
\[ A: X \to X. \quad (2.12) \]

Now, we will show that \( A \) is a contraction on \( X \). For this purpose observe that

\[
(Aw)(t) - (A\tilde{w})(t) = \left( - \sum_{k=1}^{p} c_k \int_{t_0}^{t} \left[ f(\tau, w(\tau), \tilde{w}(\tau), u(\tau)) - f(\tau, \tilde{w}(\tau), \tilde{w}(\tau), u(\tau)) \right] d\tau \right) \left( 1 + \sum_{k=1}^{p} c_k \right)
\]

\[
+ \int_{t_0}^{t} \left[ f(\tau, w(\tau), w(\tau)) - f(\tau, \tilde{w}(\tau), \tilde{w}(\tau)) \right] d\tau, \quad w, \tilde{w} \in X, \quad t \in I.
\]

From (2.13) and (2.8),

\[
\| (Aw)(t) - (A\tilde{w})(t) \| \leq 2LT \left( 1 + \left( \sum_{k=1}^{p} c_k \right) \left( 1 + \sum_{k=1}^{p} c_k \right) \right) \| w - \tilde{w} \|_X, \quad w, \tilde{w} \in X, \quad t \in I.
\]

Let

\[
q = 2LT \left( 1 + \left( \sum_{k=1}^{p} c_k \right) \left( 1 + \sum_{k=1}^{p} c_k \right) \right).
\]

Then, by (2.14), (2.15) and assumption (iii),

\[
\| Aw - A\tilde{w} \|_X \leq q \| w - \tilde{w} \|_X \text{ for } w, \tilde{w} \in X
\]

with \( 0 < q < 1 \).

Consequently, by (2.12) and (2.16), operator \( A \) satisfies all the assumptions of the Banach contraction theorem. Therefore, in space \( X \) there is only one fixed point \( u \) of \( A \) and this point is the mild solution of the nonlocal problem (1.1)-(1.2). Consequently, from Theorem 2.3, \( u \) is the unique classical solution of the nonlocal problem (1.1)-(1.2).

Now, we will prove the second part of the thesis of Theorem 2.4. To this end, observe that by (2.10) and (2.9),

\[
\| u_1 - u_0 \|_X = \sup_{t \in I} \| u_1(t) - u_0(t) \| \leq \left\| \left( - \sum_{k=1}^{p} c_k \int_{t_0}^{t} f(\tau, u_0(\tau), u_0(a(\tau)) d\tau \right) \left( 1 + \sum_{k=1}^{p} c_k \right) \right\|
\]
Next, assume that 
\[ \| u_n - u_{n-1} \|_X \leq MT \left( 1 + \left| \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right| \right) \] (2.18)

for some natural \( n \geq 2 \).

Then, by (2.10), (2.9), (2.8) and (2.18),
\[ \| u_{n+1} - u_n \|_X = \sup_{t \in I} \| u_{n+1}(t) - u_n(t) \| \] (2.19)

Therefore, from (2.17), (2.18), (2.19), and from mathematical induction,
\[ \| u_n - u_{n-1} \|_X \leq MT \left( 1 + \left| \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right| \right) \left[ 2LT \left( 1 + \left| \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right| \right) \right]^{n-1} \] (2.20)
for all \( n = 1, 2, \ldots \).

Inequalities (2.20) and assumption (iii) imply, by the Weierstrass theorem, the uniform convergence of the series

\[
\sum_{n=1}^{\infty} (u_{n+1} - u_n)
\]

on the interval \( I \) and, consequently, the uniform convergence of the sequence \( u_n \) on \( I \). Let

\[
\lim_{n \to \infty} u_n(t) = u^*_n(t) \quad \text{for } t \in I.
\]

Since \( u_n \) tends uniformly to \( u^*_n \) on \( I \) then, by (2.9), (2.10) and (2.8), \( u^*_n \) is a classical solution of the nonlocal problem (1.1)-(1.2) on \( I \). But, from the first part of the thesis of Theorem 2.4, we know that there exists only one classical solution \( u \) of the nonlocal problem (1.1)-(1.2) on \( I \). So, \( u^*_n = u \) on \( I \).

The proof of Theorem 2.4 is complete.

References


