ON A THIRD ORDER PARABOLIC EQUATION
WITH A NONLOCAL BOUNDARY CONDITION

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In this paper we demonstrate the existence, uniqueness and continuous dependence of a strong solution upon the data, for a mixed problem which combine classical boundary conditions and an integral condition, such as the total mass, flux or energy, for a third order parabolic equation. We present a functional analysis method based on an a priori estimate and on the density of the range of the operator generated by the studied problem.

Key words: Integral Condition, Third Order Parabolic Equation, A Priori Estimate, Strong Solution.

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1. Introduction

In the rectangle $Q = (0, l) \times (0, T)$, with $l < \infty$ and $T < \infty$, we consider the one-dimensional third order parabolic equation

$$
\mathcal{L}v = \frac{\partial v}{\partial t} - \frac{\partial^2}{\partial x^2} \left( a(x, t) \frac{\partial v}{\partial x} \right) = f(x, t). \tag{1.1}
$$

Assumption A: We shall assume that

$$
0 \leq a(x, t) \leq c_1, \quad \frac{\partial a(x, t)}{\partial t} \leq c_2,
$$

where $c_i > 0$, $(i = 0, 1, 2)$.

We pose the following problem for equation (1.1): to determine its solution $v$ in $Q$ satisfying

the initial condition

$$
\ell u = v(x, 0) = \Phi(x), \quad x \in (0, l), \tag{1.2}
$$

and the boundary conditions

$$
\frac{\partial v(0, t)}{\partial x} = \chi(t), \quad t \in (0, T), \tag{1.3}
$$
The data satisfies the following compatibility conditions:
\[
\frac{\partial \Phi(0)}{\partial x} = \chi(0), \quad \frac{\partial^2 \Phi(0)}{\partial x^2} = \vartheta(0), \quad \int_0^t \Phi(x) dx = m(0).
\]


Along a different line, mixed problems for second order parabolic equations, which combine classical and integral conditions, were considered by Ionkin [17], Cannon-van der Hoek [13, 14], Benouar-Yurchuk [3], Yurchuk [25], Cahn-Kulkarni-Shi [11], Cannon-Esteva-van der Hoek [15], and Shi [23]. A mixed problem with integral condition for second order pluriparabolic equation has been investigated in Bouziani [7]. Mixed problems with only integral conditions for a 2m-parabolic equation was studied in Bouziani [6], and for second order parabolic and hyperbolic equations in Bouziani-Benouar [8, 9].

In this paper, we demonstrate that problem (1.1)-(1.5) possesses a unique strong solution that depends continuously upon the data. We present a functional analysis method which is an elaboration of that in Bouziani [4, 5] and Bouziani-Benouar [10].

To achieve the purpose, we reduce the nonhomogeneous boundary conditions (1.3)-(1.5) to homogeneous conditions, by introducing a new, unknown function \( u \) defined as:
\[
\Phi(x, t) = \phi(x, t) - \Phi(x, t),
\]
where
\[
\Phi(x, t) = x \left( 1 - \frac{2x^2}{l^2} \right) \chi(t) + \frac{1}{2} \left( x^2 - \frac{l^2}{3} \right) \vartheta(t) + \frac{4x^3}{l^4} \phi(x, t).
\]
Then, the problem can be formulated as follows:
\[
\mathcal{L}u = f - \mathcal{L}u = f,
\]
\[
\mathcal{L}u = u(x, 0) = \Phi(x) - \mathcal{L}u = \varphi(x),
\]
\[
\frac{\partial u(0, t)}{\partial x} = 0,
\]
\[
\frac{\partial^2 u(0, t)}{\partial x^2} = 0.
\]
Here, we assume that the function $\varphi$, satisfies conditions of the form (1.8)-(1.10), i.e.,

$$
\frac{\partial \varphi(0)}{\partial x} = 0, \quad \frac{\partial^2 \varphi(0)}{\partial x^2} = 0 \quad \text{and} \quad \int_0^l \varphi(x) dx = 0.
$$

Instead of searching for the function $v$, we search for the function $u$. So, the strong solution of problem (1.1)-(1.5) will be given by: $v(x,t) = u(x,t) + \mathcal{U}(x,t)$.

2. Preliminaries

We employ certain function spaces to investigate our problem. Let $L^2(0,l)$, $L^2(0,T;L^2(0,l)) = L^2(Q)$ be the standard functional spaces, $\| \cdot \|_{0,Q}$ and $(\cdot, \cdot)_{0,Q}$ denote the norm and the scalar product in $L^2(Q)$, $L^2_{2}(0,l)$ be the weighted space of square integrable functions on $(0,l)$ with the finite norm

$$
\| u \|_{L^2_{2}(0,l)}^2 := \int_0^l (l-x) u^2 dx,
$$

$B^1_2(0,l)$ be the Hilbert space defined, for the first time in [6], by

$$
B^1_2(0,l) = \{ u/\mathcal{T}_x u \in L^2(0,l) \},
$$

where $\mathcal{T}_x u = \int_0^l u(\xi, t) d\xi$, and let $L^2(0,T;B^1_2(0,l))$ be the space of all functions which are square integrable on $(0,T)$ in the Bochner sense, i.e., Bochner integrable and satisfying

$$
\int_0^T \| u \|_{B^1_2(0,l)}^2 dt < \infty.
$$

Problem (1.6)-(1.10) is equivalent to the operator equation

$$
Lu = \mathcal{F},
$$

where $\mathcal{F} = (f, \varphi), L = (\mathcal{L}, \ell)$ with the domain $D(L)$ consisting of all functions $u \in L^2(Q)$ with $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial t \partial x^3}, \frac{\partial^3 u}{\partial t \partial x^2} \in L^2(Q)$ and $u$ satisfying conditions (1.8)-(1.10); the operator $L$ is on $B$ into $F$; $B$ is the Banach space obtained by the completion of $D(L)$ in the form

$$
\| u \|_B^2 = \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;B^1_2(0,l))}^2 + \sup_{0 \leq \tau \leq T} \left\| \frac{\partial u(x, \tau)}{\partial x} \right\|_{L^2_2(0,l)}^2
$$

and $F$ is the Hilbert space of the vector-valued functions $\mathcal{F} = (f, \varphi)$ with the norm.
\[ \| \mathcal{F} \|_F^2 = \| f \|_{0,Q}^2 + \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^2(0,1)}^2. \]

Let \( \bar{L} \) be the closure of the operator \( L \) with the domain \( D(\bar{L}) \).

**Definition:** A solution of the operator equation

\[ \bar{L} u = \mathcal{F} \]

is called a **strong solution** of the problem (1.6)-(1.10).

We now introduce the family of operators \( \rho^{-1}_\varepsilon \theta \) and \( (\rho^{-1}_\varepsilon)^* \theta \) defined by the formulas

\[ \rho^{-1}_\varepsilon \theta = \frac{1}{\varepsilon} \int_0^t e^{\frac{1}{\varepsilon}(\tau - t)} \theta(x, \tau) d\tau, \quad \varepsilon > 0, \]

\[ (\rho^{-1}_\varepsilon)^* \theta = -\frac{1}{\varepsilon} \int_t^T e^{\frac{1}{\varepsilon}(t - \tau)} \theta(x, \tau) d\tau, \quad \varepsilon > 0, \]

which we use as smoothing operators with respect to \( t \). These operators provide the solutions of the problems

\[
\begin{align*}
\rho^{-1}_\varepsilon \theta(x,0) &= 0, \quad (2.1) \\
\rho^{-1}_\varepsilon \theta(x,T) &= 0, \quad (2.2)
\end{align*}
\]

and

\[
\begin{align*}
-\varepsilon \frac{\partial (\rho^{-1}_\varepsilon)^* \theta}{\partial t} + (\rho^{-1}_\varepsilon)^* \theta &= \theta, \quad (2.3) \\
\left(\rho^{-1}_\varepsilon\right)^* \theta(x,T) &= 0 \quad (2.4)
\end{align*}
\]

respectively. They have the following properties.

**Lemma 1:** For all \( \theta \in L^2(0,T) \), we have

(i) \( \rho^{-1}_\varepsilon \theta(x,t) \in H^1(0,T) \) and \( \rho^{-1}_\varepsilon \theta(x,0) = 0 \);

(ii) \( \left(\rho^{-1}_\varepsilon\right)^* \theta(x,t) \in H^1(0,T) \) and \( \left(\rho^{-1}_\varepsilon\right)^* \theta(x,T) = 0 \).

**Lemma 2:** For all \( \theta \) and all \( h \) in \( L^2(Q) \), we have

\[ \int_Q \rho^{-1}_\varepsilon \theta h dx dt = \int_Q \theta \left(\rho^{-1}_\varepsilon\right)^* h dx dt. \]

This lemma states that the operators \( \left(\rho^{-1}_\varepsilon\right)^* \) are conjugate to \( \rho^{-1}_\varepsilon \).

**Lemma 3:** For all \( \theta \in L^2(0,T) \), we have

\[ \rho^{-1}_\varepsilon \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial t}\rho^{-1}_\varepsilon \theta + \frac{1}{\varepsilon} e^{-t/\varepsilon} \theta(x,0). \]
For the proof of the above lemma, it suffices to integrate by parts the expression $\rho_\epsilon^{-1} \frac{\partial \theta}{\partial \tau}.$

**Lemma 4:** For all $\theta \in L^2(0, T)$, we have

(i) \[
\int_0^T \| \rho_\epsilon^{-1} \theta \|_{0,(0,t)} dt \leq \int_0^T \| \theta \|_{0,(0,t)} dt
\]

and

(ii) \[
\int_0^T \| \rho_\epsilon^{-1} \theta - \theta \|_{0,(0,t)} dt \to 0 \text{ for } \epsilon \to 0;
\]

Proof of Lemma 4 is similar to the proof of the lemma of Section 2.18 in [1].

We easily get the following lemma.

**Lemma 5:** If

\[
A(t)u = \frac{\partial^2}{\partial x^2} \left( a(x,t) \frac{\partial u}{\partial x} \right)
\]

then

\[
A(t)\rho_\epsilon^{-1} = \rho_\epsilon^{-1} A(\tau) + \epsilon \rho_\epsilon^{-1} A'(\tau) \rho_\epsilon^{-1},
\]

where $A'(t)$ is the operator of form (2.5) whose coefficient is the first derivative with respect to $t$ of the corresponding coefficient of $A(t)$.

3. A Priori Estimate and Its Consequences

**Theorem 1:** Under Assumption $A$, there exists a positive constant $c$, independent of $u$, such that

\[
\| u \|_B \leq c \| Lu \|_F.
\]

**Proof:** We multiply equation (1.6) by an integro-differential operator

\[
M u = (l - x)\frac{\partial u}{\partial t} - 2 \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t}
\]

and integrate over $Q^\tau$, where $Q^\tau = (0, l) \times (0, \tau)$. Consequently,

\[
\int_{Q^\tau} \int_{Q^\tau} \mathcal{L} u \cdot M u dx dt = \int_{Q^\tau} \int_{Q^\tau} \frac{\partial u}{\partial t} - 2 \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} dx dt - \int_{Q^\tau} \int_{Q^\tau} \frac{\partial u}{\partial t} - 2 \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} dx dt
\]

(3.2)
\[
- \int \int_{Q^r} \left( \frac{\partial^2}{\partial x^2} \left( a(x,t) \frac{\partial u}{\partial x} \right)(l-x) \frac{\partial u}{\partial t} \right) dx dt + 2 \int \int_{Q^r} \left( \frac{\partial^2}{\partial x^2} \left( a(x,t) \frac{\partial u}{\partial x} \right) \right) \frac{\sigma^2 \partial u}{\partial t} dx dt.
\]

We know from the integration by parts that

\[
\int \int_{Q^r} \frac{\partial u}{\partial t} (l-x) \frac{\partial u}{\partial t} dx dt = -\frac{1}{2} \int \int_{Q^r} \left( \frac{\partial}{\partial x} \frac{\partial u}{\partial t} \right)^2 dx dt, \quad (3.3)
\]

\[
-2 \int \int_{Q^r} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x \partial t} dx dt = 2 \int \int_{Q^r} \left( \frac{\partial}{\partial x} \frac{\partial u}{\partial t} \right)^2 dx dt, \quad (3.4)
\]

\[
- \int \int_{Q^r} \frac{\partial^2}{\partial x^2} \left( a(x,t) \frac{\partial u}{\partial x} \right)(l-x) \frac{\partial u}{\partial t} dx dt = \frac{1}{2} \int l \int_{0}^{l} (l-x)a(x,\tau) \left( \frac{\partial u(x,\tau)}{\partial x} \right)^2 dx \]

\[
- \frac{1}{2} \int_{0}^{l} (l-x)a(x,0) \left( \frac{\partial \varphi}{\partial x} \right)^2 dx - \frac{1}{2} \int (l-x) \frac{\partial a(x,t)}{\partial t} \left( \frac{\partial u}{\partial x} \right)^2 dx dt \quad (3.5)
\]

\[
- 2 \int \int_{Q^r} a(x,t) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt,
\]

\[
2 \int \int_{Q^r} \frac{\partial^2}{\partial x^2} \left( a(x,t) \frac{\partial u}{\partial x} \right) \frac{\sigma^2 \partial u}{\partial t} dx dt = 2 \int \int_{Q^r} a(x,t) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt. \quad (3.6)
\]

Substituting (3.3)-(3.6) into (3.2), we obtain

\[
\frac{3}{2} \int \int_{Q^r} \left( \frac{\partial}{\partial x} \frac{\partial u}{\partial t} \right)^2 dx dt + \frac{1}{2} \int l \int_{0}^{l} (l-x)a(x,\tau) \left( \frac{\partial u(x,\tau)}{\partial x} \right)^2 dx
\]

\[
= \int \int_{Q^r} \left[ f \left( l-x \right) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} - 2\varphi^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right] dx dt + \frac{1}{2} \int l \int_{0}^{l} (l-x)a(x,0) \left( \frac{\partial \varphi}{\partial x} \right)^2 dx \quad (3.7)
\]

\[
+ \frac{1}{2} \int \int_{Q^r} (l-x) \frac{\partial a(x,t)}{\partial t} \left( \frac{\partial u}{\partial x} \right)^2 dx dt.
\]

Further, by virtue of inequality (2.2) in [6] and the Cauchy inequality, the first integral on the right-hand side of (3.7) is estimated as follows

\[
\int \int_{Q^r} f \left( l-x \right) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} dx dt
\]

\[
\leq \frac{3l^2}{2} \int \int_{Q^r} f^2 dx dt + \int \int_{Q^r} \left( \frac{\partial u}{\partial x} \right)^2 dx dt. \quad (3.8)
\]
Substituting (3.8) in (3.7) and using Assumption A, we get
\[ L^2(0, r; B_{12}(0, l)) - \| \frac{\partial u(x, \tau)}{\partial x} \|_{L^2(0, l)}^2 \leq c_3 \left( \| f \|_{0, Q^r}^2 + \| \frac{\partial \varphi}{\partial x} \|_{L^2(0, l)}^2 \right) + c_4 \| \frac{\partial u}{\partial x} \|_{L^2(0, \tau; L^2(0, l))}^2, \]
where
\[ c_3 = \frac{\max(3l^2, c_1)}{\min(1, c_0)} \]
and
\[ c_4 = \frac{c_2}{\min(1, c_0)}. \]

We eliminate the last term on the right-hand side of (3.9). To do this we use the following lemma.

**Lemma 6:** If \( f_i(\tau) \) (\( i = 1, 2, 3 \)) are nonnegative functions on \( (0, T) \), \( f_1(\tau) \) and \( f_2(\tau) \) are integrable on \( (0, T) \), and \( f_3(\tau) \) is nondecreasing on \( (0, T) \) then it follows, from
\[ \mathcal{T}_\tau f_1 + f_2 \leq f_3 + c\mathcal{T}_\tau f_2, \]
that
\[ \mathcal{T}_\tau f_1 + f_2 \leq \exp(c\tau).f_3, \]
where
\[ \mathcal{T}_\tau f_i = \int_0^\tau f_i(t)dt, \quad (i = 1, 2). \]

The proof of the above lemma is similar to that of Lemma 7.1 in [16]. \( \square \)

Returning to the proof of Theorem 1, we denote the first term on the left-hand side of (3.9) by \( f_1(\tau) \), the remaining term on the same side on (3.9) by \( f_2(\tau) \), and the sum of two first terms on the right-hand side of (3.9) by \( f_3(\tau) \). Consequently, Lemma 6 implies the inequality
\[ \| \frac{\partial u}{\partial t} \|_{L^2(0, \tau; B_{12}(0, l))}^2 + \| \frac{\partial u(x, \tau)}{\partial x} \|_{L^2(0, l)}^2 \leq c_3 e^{c_4 \tau} \left( \| f \|_{0, Q^r}^2 + \| \frac{\partial \varphi}{\partial x} \|_{L^2(0, l)}^2 \right) \]
\[ \leq c_5 \left( \| f \|_{0, Q^r}^2 + \| \frac{\partial \varphi}{\partial x} \|_{L^2(0, l)}^2 \right), \]
where
\[ c_5 = c_3 \exp(c_4 T). \]

Since the right-hand side of the above inequality does not depend on \( \tau \), in the left-hand side we take the upper bound with respect to \( \tau \) from 0 to \( T \). Therefore, we obtain inequality (3.1), where \( c = c_5^{1/2} \).

**Proposition 1:** The operator \( L \) from \( B \) into \( F \) is closable.

The proof of this proposition is analogous to the proof of the proposition in [7]. \( \square \)
Since the points of the graph of \( \overline{L} \) are limits of the sequences of points of the graph of \( L \), we can extend (3.1) to apply to strong solutions by taking the limits.

**Corollary 1:** Under Assumption A, there is a constant \( c > 0 \), independent of \( u \), such that
\[
||u||_B \leq c ||\overline{L}u||_F, \quad \forall u \in D(\overline{L}). \tag{3.11}
\]

Let \( R(L) \) and \( R(\overline{L}) \) denote the set of values taken by \( L \) and \( \overline{L} \), respectively. Inequality (3.11) implies the following corollary.

**Corollary 2:** The range \( R(\overline{L}) \) is closed in \( F \), \( R(L) = R(\overline{L}) \) and \( \overline{L}^{-1} = \overline{L}^{-1} \), where \( \overline{L}^{-1} \) is the extension of \( L^{-1} \) by continuity from \( R(L) \) to \( R(\overline{L}) \).

4. Solvability of the Problem

**Theorem 2:** Let Assumption A be satisfied and let \( \frac{\partial a}{\partial x} \) and \( \frac{\partial^2 a}{\partial x \partial t} \) be bounded. Then for arbitrary \( f \in L^2(Q) \) and \( \frac{\partial \varphi}{\partial x} \in L^2_a(0,1) \), problem (1.6)-(1.10) admits a unique strong solution \( u = \overline{L}^{-1} \varphi = \overline{L}^{-1} \varphi \).

**Proof:** Corollary 1 asserts that, if a strong solution exists, it is unique and depends continuously on \( \varphi \). (If \( u \) is considered in the topology of \( B \) and \( \varphi \) is considered in the topology of \( F \).) Corollary 2 states that, to prove that (1.6)-(1.10) has a strong solution for an arbitrary \( \varphi = (f, \varphi) \in F \) it is sufficient to show the equality \( R(L) = F \). To this end, we need the following proposition.

**Proposition 2:** Let the assumptions of Theorem 2 hold and let \( D_0(L) \) be the set of all \( u \in D(L) \) vanishing in a neighborhood of \( t = 0 \). If, for \( \varphi \in L^2(Q) \) and for all \( u \in D_0(L) \), we have
\[
(Lu, h)_{L^2(Q)} = 0, \tag{4.1}
\]
then \( h \) vanishes almost everywhere in \( Q \).

**Proof of the proposition:** We can write (4.1) as follows
\[
\int \int_Q \frac{\partial u}{\partial t} \cdot h \, dx \, dt = \int \int_Q A(t)u \cdot h \, dx \, dt. \tag{4.2}
\]

Replacing \( u \) by the smooth function \( \rho^{-1}_\varepsilon u \) in (4.2), this yields, from Lemma 5, that
\[
\int \int_Q \frac{\partial \rho^{-1}_\varepsilon}{\partial t} \cdot h \, dx \, dt = \int \int_Q \rho^{-1}_\varepsilon A u \cdot h \, dx \, dt + \int A'(\rho^{-1}_\varepsilon) \rho^{-1}_\varepsilon u \cdot h \, dx \, dt. \tag{4.3}
\]

Applying Lemma 3 to the left-hand side of (4.3), and Lemma 2 to the obtained equality, we obtain
\[
\int \int_Q \frac{\partial u}{\partial t} \cdot \left( \rho^{-1}_\varepsilon \right)^* h \, dx \, dt \tag{4.4}
\]
\[
= \int \int_Q A u \cdot \left( \rho^{-1}_\varepsilon \right)^* h \, dx \, dt + \int A' \rho^{-1}_\varepsilon u \cdot \left( \rho^{-1}_\varepsilon \right)^* h \, dx \, dt.
\]

The standard integration by parts with respect to \( t \) in the left-hand side of (4.4)
leads to

\[
\int \int_{Q} u \cdot \frac{\partial \left( \rho_{c}^{-1} \right)^{*}}{\partial t} \, dx \, dt = \int \int_{A} Au \cdot \left( \rho_{c}^{-1} \right)^{*} \, h \, dx \, dt \\
+ \epsilon \int \int_{Q} A'\rho_{c}^{-1}u \cdot \left( \rho_{c}^{-1} \right)^{*} \, h \, dx \, dt.
\]

(4.5)

The operator \( A(t) \) with boundary conditions (1.8)-(1.10) has, on \( L^{2}(0,l) \), the continuous inverse. Hence,

\[
A'\rho_{c}^{-1}u = A'\rho_{c}^{-1}A^{-1}Au = \Lambda_{c}Au.
\]

(4.6)

Thus, from (4.5) and (4.6), we obtain

\[
\int \int_{Q} u \cdot \frac{\partial \left( \rho_{c}^{-1} \right)^{*}}{\partial t} \, dx \, dt = \int \int_{Q} Au \cdot \left( \rho_{c}^{-1} \right)^{*} \, h \, dx \, dt + \epsilon \int \int_{Q} \Lambda_{c}Au \cdot \left( \rho_{c}^{-1} \right)^{*} \, h \, dx \, dt
\]

\[
= \int \int_{Q} Au \cdot \left( I + \epsilon \Lambda_{c}^{*} \left( \rho_{c}^{-1} \right)^{*} \right) \, h \, dx \, dt.
\]

(4.7)

Defining \( A^{-1}(t) \), we apply operator \( \mathcal{T}^{2}_{x} \) to both sides of \( A(t)u = g \). After this operation, we get

\[
\frac{\partial u}{\partial x} = \frac{1}{a(x,t)} \int_{0}^{x} (x - \xi)g(\xi,t) \, d\xi.
\]

(4.8)

We now integrate each term of (4.8) over \([0,x]\) with respect to \( \xi \). Consequently,

\[
A^{-1}(t)g = \int_{0}^{x} \frac{d\xi}{a(\xi,t)} \int_{0}^{\xi} (\xi - \eta)g(\eta,t) \, d\eta + c_{6}.
\]

(4.9)

To compute the constant \( c_{6} \) in (4.9), we multiply (4.8) by \((l-x)\) and integrate the obtained equation over \([0,l]\). Therefore,

\[
\int_{0}^{l} (l-x) \frac{\partial u}{\partial x} \, dx = \int_{0}^{l} \frac{(l-x) \, dx}{a(x,t)} \int_{0}^{x} (x - \xi)g(\xi,t) \, d\xi.
\]

(4.10)

Integration by parts of the left-hand side of (4.10), gives

\[
c_{6} = -\frac{1}{l} \int_{0}^{l} \frac{(l-x) \, dx}{a(x,t)} \int_{0}^{x} (x - \xi)g(\xi,t) \, d\xi.
\]

Note that for the determination of \( \Lambda_{c} \) and \( \Lambda_{c}^{*} \), the corresponding calculations are not difficult, but they are long. Therefore, we only give the final results of the computations:
\[ \Lambda_\epsilon Au = \left( \frac{\partial^3 a(x, t)}{\partial x^3 \partial t} \rho_\epsilon^{-1} - 2 \frac{\partial^2 a(x, t)}{\partial x \partial t} \rho_\epsilon^{-1} \frac{1}{a(x, \tau)} \frac{\partial a(x, \tau)}{\partial x} + \frac{\partial a(x, t)}{\partial t} \rho_\epsilon^{-1} \frac{1}{(a(x, \tau))^2} \right) \left( \frac{\partial (a(x, \tau))}{\partial x} \right)^2 - \frac{\partial a(x, t)}{\partial t} \rho_\epsilon^{-1} \frac{1}{a(x, \tau)} \frac{\partial^2 a(x, \tau)}{\partial x^2} \right) \frac{1}{a(x, \tau)} \left( \int_0^x (x - \xi) Au(\xi, \tau) d\xi \right) \] (4.11)

\[ + 2 \left( \frac{\partial^2 a(x, t)}{\partial x \partial t} \rho_\epsilon^{-1} - \frac{\partial a(x, t)}{\partial t} \rho_\epsilon^{-1} \frac{\partial a(x, \tau)}{\partial x} \right) \frac{1}{a(x, \tau)} \left( \int_0^x Au(\xi, \tau) d\xi \right) \]

\[ + \frac{\partial a(x, t)}{\partial t} \rho_\epsilon^{-1} \frac{1}{a(x, \tau)} Au; \]

\[ \Lambda_\epsilon \left( \rho_\epsilon^{-1} \right)^* h = \frac{1}{a(x, t)} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial a(x, \tau)}{\partial \tau} \left( \rho_\epsilon^{-1} \right)^* h + \int_x^l \frac{(x - \xi)}{a(\xi, t)} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial^3 a(\xi, \tau)}{\partial \tau \partial \xi^2} \] (4.12)

\[ - 2 \frac{1}{a(x, t)} \frac{\partial a(\xi, t)}{\partial \xi} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial^2 a(\xi, \tau)}{\partial \tau \partial \xi} \frac{1}{a(\xi, t)} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial a(\xi, \tau)}{\partial \tau} + 2 \frac{1}{(a(\xi, t))^2} \left( \frac{\partial a(\xi, t)}{\partial x} \right)^2 \left( \rho_\epsilon^{-1} \right)^* \frac{\partial a(\xi, \tau)}{\partial \tau} \]

\[ + \frac{1}{a(\xi, t)} \frac{\partial a(\xi, t)}{\partial \xi} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial a(\xi, \tau)}{\partial \tau} \frac{1}{a(\xi, t)} \left( \rho_\epsilon^{-1} \right)^* h(\xi, \tau) d\xi \]

\[ + 2 \int_x^l \frac{1}{a(\xi, t)} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial^2 a(\xi, \tau)}{\partial \tau \partial \xi} - \frac{1}{a(\xi, t)} \frac{\partial a(\xi, t)}{\partial \xi} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial a(\xi, \tau)}{\partial \tau} \left( \rho_\epsilon^{-1} \right)^* h(\xi, \tau) d\xi. \]

The left-hand side of (4.7) shows that the mapping \( \int \int_Q Au \cdot K_\epsilon \left( \rho_\epsilon^{-1} \right)^* h dx dt \) is a continuous linear functional of \( u \), where

\[ K_\epsilon \left( \rho_\epsilon^{-1} \right)^* h = (I + \epsilon \Lambda_\epsilon^*) \left( \rho_\epsilon^{-1} \right)^* h. \] (4.13)

Consequently, this assertion holds true, if the function \( K_\epsilon \) has the following properties

\[ \frac{\partial K_\epsilon}{\partial x} \in L^2(Q), \frac{\partial^2 K_\epsilon}{\partial x^2} \in L^2(Q) \text{ and } \frac{\partial^3 K_\epsilon}{\partial x^3} \in L^2(Q), \]

and satisfies the following conditions:

\[ K_\epsilon \left|_{x=l} = 0, \frac{\partial K_\epsilon}{\partial x} \bigg|_{x=l} = 0, \frac{\partial^2 K_\epsilon}{\partial x^2} \bigg|_{x=0} = 0 \text{ and } \frac{\partial^2 K_\epsilon}{\partial x^2} \bigg|_{x=l} = 0. \] (4.14)

From (4.12), we deduce that the operator \( \Lambda_\epsilon^* \) is bounded on \( L^2(Q) \). Hence, the norm of \( \epsilon \Lambda_\epsilon^* \) on \( L^2(Q) \) is smaller than 1 for sufficiently small \( \epsilon \). So, the operator \( K_\epsilon \) has the continuous inverse operator in \( L^2(Q) \).

From (4.12) and (4.14), we deduce that

\[ \left( I + \epsilon \frac{1}{a(x, t)} \left( \rho_\epsilon^{-1} \right)^* \frac{\partial a(x, \tau)}{\partial \tau} \left( \rho_\epsilon^{-1} \right)^* h \right) \bigg|_{x=l} = 0, \] (4.15)
For each fixed $x \in [0, l]$ and sufficiently small $\epsilon$, the operator
\[
\left( I + \epsilon \frac{1}{a(x,t)} \left( \rho^{-1} \right)^* \frac{\partial a(x, \tau)}{\partial \tau} \right) \left( \rho^{-1} \right)^* h \bigg|_{x = l} = 0,
\]
has the continuous inverse operator on $L^2(0, T)$. Hence, (4.15)-(4.18) imply that
\[
\left( \rho^{-1} \right)^* h \bigg|_{x = l} = 0, \quad \frac{\partial \left( \rho^{-1} \right)^* h}{\partial x} \bigg|_{x = l} = 0, \quad \frac{\partial^2 \left( \rho^{-1} \right)^* h}{\partial x^2} \bigg|_{x = 0} = 0,
\]
\[
\frac{\partial^2 \left( \rho^{-1} \right)^* h}{\partial x^2} \bigg|_{x = l} = 0.
\]
In other words, (4.15)-(4.18) imply that
\[
\hat{h} \bigg|_{x = l} = 0, \quad \frac{\partial \hat{h}}{\partial x} \bigg|_{x = l} = 0, \quad \frac{\partial^2 \hat{h}}{\partial x^2} \bigg|_{x = 0} = 0, \quad \frac{\partial^2 \hat{h}}{\partial x^2} \bigg|_{x = l} = 0.
\]
Set
\[
\hat{h} = \left( (l - x) \hat{\tau}_x z - 2 \rho_x \hat{\tau}_x z \right).
\]
Differentiating (4.20) with respect to $x$, we obtain
\[
\begin{cases}
\frac{\partial \hat{h}}{\partial x} = (\rho_x \hat{\tau}_x z - (l - x)z) \in L^2(Q), \\
\frac{\partial^2 \hat{h}}{\partial x^2} = -(l - x) \frac{\partial \hat{z}}{\partial x} \in L^2(Q), \\
\frac{\partial^3 \hat{h}}{\partial x^3} = - \frac{\partial}{\partial x} \left( (l - x) \frac{\partial \hat{z}}{\partial x} \right) \in L^2(Q).
\end{cases}
\]
From (4.20), (4.21), and (4.19), we deduce that the conditions
\[
\hat{\mathcal{T}}_{l} z = 0, \hat{\mathcal{T}}^{2}_{l} z = 0, (l - x) \frac{\partial \hat{z}}{\partial x} \bigg|_{x = 0} = 0, (l - x) \frac{\partial \hat{z}}{\partial x} \bigg|_{x = l} = 0
\]
are met.
In (4.2), we replace $h$ by its representation (4.20). Consequently,

\[
\int \int_{Q} \frac{\partial u}{\partial t}((l-x)\mathcal{F}_x \mathcal{F}_z - 2\mathcal{F}_x^2 \mathcal{F}_z) dx dt = \int \int_{Q} A(t)u((l-x)\mathcal{F}_x \mathcal{F}_z - 2\mathcal{F}_x^2 \mathcal{F}_z) dx dt
\]

\[
= - \int \int_{Q} a(x,t)\frac{\partial u}{\partial x}(l-x) \frac{\partial z}{\partial x} dx dt. \tag{4.23}
\]

Substituting (2.3) in (4.23) (with $\theta = z$) and integrating by parts (with respect to $x$), by taking into account (4.22), we obtain

\[
\int \int_{Q} \frac{\partial u}{\partial t}((l-x)\mathcal{F}_x \mathcal{F}_z - 2\mathcal{F}_x^2 \mathcal{F}_z) dx dt = \int \int_{Q} a(x,t)\frac{\partial u}{\partial x}(l-x) \frac{\partial^2 (\rho_{\epsilon}^{-1})^z}{\partial x \partial t} dx dt
\]

\[
- \int \int_{Q} a(x,t)\frac{\partial u}{\partial x}(l-x) \frac{\partial (\rho_{\epsilon}^{-1})^z}{\partial t} dx dt. \tag{4.24}
\]

Putting

\[
u = \int \int_{Q} \left( e^{-c_t z} (\rho_{\epsilon}^{-1})^z \right) dx dt = \int_{0}^{\tau} e^{-c_t (\rho_{\epsilon}^{-1})^z} z dt \tag{4.25}
\]

in relation (4.24), where $c_\tau$ is a constant such that $c_\tau c_0 - c_2 - c_2^2 / 2 c_0 \geq 0$, and integrating by parts with respect to $t$ on each term of the right-hand side of the obtained equality, we obtain, by taking into account (2.4) and due to $u \in D_0(L)$ that

\[
\int \int_{Q} (l-x)a(x,t) \frac{\partial u}{\partial x} \frac{\partial^2 (\rho_{\epsilon}^{-1})^z}{\partial x \partial t} dx dt = - \int \int_{Q} (l-x)e^{-c_t z} a(x,t) \left( \frac{\partial (\rho_{\epsilon}^{-1})^z}{\partial x} \right)^2 dx dt \tag{4.26}
\]

\[
- \int \int_{Q} (l-x) \frac{\partial a(x,t)}{\partial t} \frac{\partial u}{\partial x} \frac{\partial (\rho_{\epsilon}^{-1})^z}{\partial x} dx dt,
\]

\[
\int \int_{Q} (l-x)a(x,t) \frac{\partial u}{\partial x} \frac{\partial (\rho_{\epsilon}^{-1})^z}{\partial x} dx dt
\]

\[
= - \int \int_{Q} (l-x)e^{-c_t z} a(x,t) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} dx dt \tag{4.27}
\]

\[
= - \frac{1}{2} \int_{0}^{l} (l-x)e^{-c_t} a(x,T) \left( \frac{\partial u(x,T)}{\partial x} \right)^2 dx
\]
Elementary calculations, starting from (4.26) and (4.27), yield the inequalities

\[
\frac{\epsilon}{2} \int \int_Q (l - x)e^{-c_T t} \left(c_T a(x, t) - \frac{\partial a(x, t)}{\partial t} \right)(\frac{\partial u}{\partial x})^2 \text{d}x \text{d}t
\]

and

\[
- \frac{c_T e^2}{4c_0} \int \int_Q (l - x)e^{-c_T t} \left(\frac{\partial u}{\partial x}\right)^2 \text{d}x \text{d}t
\]

Substituting (4.28) and (4.29) into (4.24), we get

\[
\epsilon \int \int_Q (l - x)a(x, t)\frac{\partial u}{\partial x} \cdot \frac{\partial^2 (\rho_{\epsilon}^{-1})^*}{\partial x \partial t} \text{d}x \text{d}t
\]

Hence, for sufficiently small \( \epsilon \leq 1 \), we have

\[
\frac{1}{2} \int \int_Q (l - x)\mathcal{F}_x z - 2\mathcal{F}_x^2 z \text{d}x \text{d}t
\]

Passing to the limit in the above inequality and integrating by parts with respect to \( x \), we obtain, by Lemma 4, that

\[
\frac{1}{2} \int \int_Q (l - x)\mathcal{F}_x z - 2\mathcal{F}_x^2 z \text{d}x \text{d}t \leq 0.
\]

Hence, for sufficiently small \( \epsilon \leq 1 \), we have

\[
\int \int_Q e^{c_T t}(\frac{\partial (\rho_{\epsilon}^{-1})^*}{\partial x} z((l - x)\mathcal{F}_x z - 2\mathcal{F}_x^2 z) \text{d}x \text{d}t \leq 0.
\]

Passing to the limit in the above inequality and integrating by parts with respect to \( x \), we obtain, by Lemma 4, that

\[
\int \int_Q e^{c_T t}(\mathcal{F}_x z)^2 \text{d}x \text{d}t \leq 0
\]

and thus \( z = 0 \). Hence, \( h = 0 \), which completes the proof.

Now, we return to the proof of Theorem 2. Since \( F \) is a Hilbert space, we have that \( R(L) = \mathcal{F}_x \) is equivalent to the orthogonality of vector \((h, h_0) \in F\) to the set \( R(L) \), i.e., if and only if, the relation

\[
(\mathcal{F}_x h_0, Q + \left(\frac{\partial \mathcal{F}_x h_0}{\partial x}, \frac{\partial h_0}{\partial x}\right)^2_{L^2_\sigma(0, t)} = 0,
\]
where \( u \) runs over \( B \) and \((h, h_0) \in F\), implies that \( h = 0 \) and \( h_0 = 0 \).

Putting \( u \in D_0(L) \) in (4.31), we obtain

\[
(Lu, h)_{0, Q} = 0.
\]

Hence Proposition 2 implies that \( h = 0 \). Thus, (4.31) takes the form

\[
\left( \frac{\partial^2 u}{\partial x^2}, \frac{\partial h_0}{\partial x} \right)_{L^2_0(0,1)} = 0, \quad u \in D(L).
\]

Since the range of the trace operator \( \ell \) is dense in the Hilbert space with the norm \( \| \frac{\partial h_0}{\partial x} \|_{L^2_0(0,1)} \), from the last equality, it follows that \( h_0 = 0 \) (we recall that \( h_0 \) satisfies the compatibility conditions (1.11)). Hence, \( R(L) \) is dense in \( F \). \( \square \)

**References**


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