A NOTE ON CONTROLLABILITY OF SEMILINEAR INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

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Sufficient conditions for controllability of semilinear integrodifferential systems in a Banach space are established. The results are obtained by using the Schaefer fixed-point theorem.

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1. Introduction

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. Several authors have extended the concept to infinite-dimensional systems in Banach spaces with bounded operators. Naito [8, 9] studied the controllability of semilinear systems whereas Yamamoto and Park [13] considered the same problem for parabolic equation with uniformly bounded nonlinear term. Lasiecka and Triggiani [5] studied exact controllability of abstract semilinear equations. Chukwu and Lenhart [3] discussed the controllability of nonlinear systems in abstract spaces and Naito [10] established the controllability for nonlinear Volterra integrodifferential systems. Do [4] and Zhou [14] investigated the approximate controllability for a class of semilinear abstract equations. Recently Balachandran et al. [1, 2] established sufficient conditions for the controllability of nonlinear integrodifferential systems in Banach spaces by using Schauder's fixed-point theorem. The purpose of this paper is to study the controllability of semilinear integrodifferential systems in Banach spaces by suitably applying the Schaefer fixed-point theorem.

2. Preliminaries

Consider the semilinear integrodifferential system

$$\dot{x}(t) = A[x(t) + \int_0^t F(t-s)x(s)ds] + (Bu)(t) + f(t, x(t), \int_0^t g(t, s, x(s))ds), \quad t \in J = [0, b],$$

$$x(0) = x_0, \quad (1)$$

where the state $x(\cdot)$ takes values in a Banach space X and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. Here A is the generator of a strongly continuous semigroup, B is a bounded linear operator from U into X , and $g: J \times J \times X \rightarrow X$ and $f: J \times X \times X \rightarrow X$ are given functions. $F(t): Y \rightarrow Y$ and for $x(\cdot)$ continuous in Y , $AF(\cdot)x(\cdot) \in L^1([0, b], X)$. $F(t) \in B(X)$, $t \in J$ and for some $x \in X$, $F'(t)x$ is continuous in $t \in [0, b]$, where $B(X)$ is the space of all bounded linear operators on X , and Y is the Banach space formed from $D(A)$, the domain of A endowed with the graph norm.

We need the following fixed point theorem due to Schaefer [12].

Schaefer Theorem: Let S be a convex subset of a normed linear space E and $0 \in S$. Let $F: S \rightarrow S$ be a completely continuous operator and let

$$\zeta(F) = \{x \in S; x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

The system (1) has a mild solution of the following form [11]:

$$x(t) = R(t)x_0 + \int_0^t R(t-s) \left[(Bu)(s) + f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) \right] ds, \quad (2)$$

where $R(t)$ is a resolvent operator [6].

In order to study the controllability problem of (1), we consider the following system as in [7]:

$$\begin{aligned} \dot{x}(t) = & \lambda A \left[x(t) + \int_0^t F(t-s)x(s) ds \right] + \lambda (Bu)(t) \\ & + \lambda f \left(t, x(t), \int_0^t g(t, s, x(s)) ds \right), \quad \lambda \in (0, 1), \quad t \in J, \end{aligned} \quad (3)$$

$$x(0) = x_0.$$

Then for system (3), there exists a mild solution of the form

$$x(t) = \lambda R(t)x_0 + \lambda \int_0^t R(t-s) \left[(Bu)(s) + f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) \right] ds.$$

Definition: System (1) is said to be *controllable on the interval J* if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = x_1$.

We assume the following hypotheses:

(i) The resolvent operator $R(t)$ is compact such that

$$\max_{t > 0} \|R(t)\| \leq M_1,$$

where $M_1 > 0$.

- (ii) The linear operator W from $L^2(J, U)$ into X , defined by

$$Wu = \int_0^b R(b-s)Bu(s)ds$$

has an invertible operator W^{-1} defined on $L^2(J, U)/\ker W$ and there exist positive constants M_2, M_3 such that $\|B\| \leq M_2$ and $\|W^{-1}\| \leq M_3$.

- (iii) For each $t \in J$, the function $g(t, s, \cdot): X \rightarrow X$ is continuous and for each $x \in X$, the function $g(\cdot, \cdot, x): J \times J \rightarrow X$ is strongly measurable.
- (iv) For each $t \in J$, the function $f(t, \cdot, \cdot): X \times X \rightarrow X$ is continuous and for each $x, y \in X$, the function $f(\cdot, x, y): J \rightarrow X$ is strongly measurable.
- (v) For every positive integer k , there exists $h_k \in L^1(0, b)$ such that for a.a. $t \in J$,

$$\sup_{\|x\| \leq k} \left\| f(t, x(t), \int_0^t g(t, s, x(s))ds) \right\| \leq h_k(t).$$

- (vi) There exists a continuous function $m: J \times J \rightarrow [0, \infty)$ such that

$$\|g(t, s, x)\| \leq m(t, s)\Omega(\|x\|), \quad t, s \in J, \quad x \in X,$$

where $\Omega: [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

- (vii) There exists a continuous function $p: J \rightarrow [0, \infty)$ such that

$$\|f(t, x, y)\| \leq p(t)\Omega_0(\|x\| + \|y\|), \quad t \in J, \quad x, y \in X,$$

where $\Omega_0: [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function.

- (viii)

$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)},$$

where $c = M_1(\|x_0\|) + M_1Nb$, $\hat{m}(t) = \max\{M_1p(t), Lm(t, t)\}$,

$$N = M_2M_3 \left(\|x_1\| + M_1\|x_0\| + M_1 \int_0^b p(s)\Omega_0(\|x\| \right. \\ \left. + L \int_0^s m(s, \tau)\Omega(\|x\|)d\tau)ds \right).$$

3. Main Result

Theorem: If the hypotheses (i)-(viii) are satisfied, then system (1) is controllable on J .

Proof: Using hypothesis (ii) for an arbitrary function $x(\cdot)$, define the control

$$u(t) = W^{-1} \left[x_1 - R(b)x_0 - \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \right](t).$$

We shall now show that when using this control the operator defined by

$$(Fx)(t) = R(t)x_0 + \int_0^t R(t-s)[(Bu)(s) + f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)]ds, t \in J,$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly, $(Fx)(b) = x_1$, which means that the control u steers the semilinear integrodifferential system from the initial state x_0 to x_1 in time b , provided we can obtain a fixed point of the nonlinear operator F .

First, we obtain a priori bounds for the following equation:

$$\begin{aligned} x(t) = & \lambda R(t)x_0 + \lambda \int_0^t R(t-\eta)BW^{-1}[x_1 - R(b)x_0 \\ & - \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta \\ & + \lambda \int_0^t R(t-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds. \end{aligned}$$

We have, from the assumptions,

$$\begin{aligned} \|x(t)\| \leq & M_1 \|x_0\| + \int_0^t \|R(t-\eta)\| M_2 M_3 [\|x_1\| + M_1 \|x_0\| \\ & + M_1 \int_0^b p(s)\Omega_0(\|x(s)\| + \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds]d\eta \\ & + M_1 \int_0^t p(s)\Omega_0(\|x(s)\| + \int_0^s m(s, \tau)\Omega(\|x(\tau)\|)d\tau)ds \\ & \leq M_1 \|x_0\| + \int_0^t M_1 N ds \end{aligned}$$

$$\begin{aligned}
 & + M_1 \int_0^t p(s) \Omega_0(\|x(s)\|) + \int_0^s m(s, \tau) \Omega(\|x(\tau)\|) d\tau ds \\
 & \leq M_1 \|x_0\| + M_1 N b + M_1 \int_0^t p(s) \Omega_0(\|x(s)\| \\
 & \quad + \int_0^s m(s, \tau) \Omega(\|x(\tau)\|) d\tau ds.
 \end{aligned}$$

Denoting by $v(t)$ the right-hand side of the above inequality, we have $v(0) = M_1 \|x_0\| + M_1 N b$, $\|x(t)\| \leq v(t)$ and

$$\begin{aligned}
 v'(t) & = M_1 p(t) \Omega_0(\|x(t)\|) + \int_0^t m(t, \tau) \Omega(\|x(\tau)\|) d\tau \\
 & \leq M_1 p(t) \Omega_0(v(t)) + \int_0^t m(t, \tau) \Omega(v(\tau)) d\tau.
 \end{aligned}$$

Let

$$w(t) = v(t) + \int_0^t m(t, \tau) \Omega(v(\tau)) d\tau.$$

Then $w(0) = v(0) = c$, $v(t) \leq w(t)$, and

$$\begin{aligned}
 w'(t) & = v'(t) + m(t, t) \Omega(v(t)) \leq M_1 p(t) \Omega_0(w(t)) + m(t, t) \Omega(w(t)) \\
 & \leq \hat{m}(t) [\Omega_0(w(t)) + \Omega(w(t))].
 \end{aligned}$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leq \int_0^b \hat{m}(s) ds < \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \quad t \in J,$$

which in turn implies that there is a constant K such that $w(t) \leq K$, $t \in J$, and hence $\|x(t)\| \leq K$, $t \in J$, where K depends only on b and on the functions m , Ω_0 , and Ω .

Second, we must prove that the operator $F: C = C(J, X) \rightarrow C$ defined by

$$\begin{aligned}
 (Fx)(t) & = R(t)x_0 + \int_0^t R(t-\eta) B W^{-1} [x_1 - R(b)x_0 \\
 & \quad - \int_0^b R(b-s) f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau ds](\eta) d\eta
 \end{aligned}$$

$$+ \int_0^t R(t-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds$$

is a completely continuous operator.

Let $B_k = \{x \in C: \|x\| \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq b$,

$$\begin{aligned} & \| (Fx)(t_1) - (Fx)(t_2) \| \leq \| R(t_1) - R(t_2) \| \| x_0 \| \\ & + \| \int_0^{t_1} [R(t_1 - \eta) - R(t_2 - \eta)]BW^{-1}[x_1 - R(b)x_0 \\ & - \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta \| \\ & + \| \int_{t_1}^{t_2} R(t_2 - \eta)BW^{-1}[x_1 - R(b)x_0 \\ & - \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta \| \\ & + \| \int_0^{t_1} [R(t_1 - s) - R(t_2 - s)]f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \| \\ & + \| \int_{t_1}^{t_2} R(t_2 - s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \| \\ & \leq \| R(t_1) - R(t_2) \| \| x_0 \| \\ & + \int_0^{t_1} \| R(t_1 - \eta) - R(t_2 - \eta) \| M_2 M_3 [\| x_1 \| + M_1 e^{\omega b} \| x_0 \| \\ & + M_1 \int_0^b e^{\omega(b-s)} h_k(s) ds] d\eta \\ & + \int_{t_1}^{t_2} \| R(t_2 - \eta) \| M_2 M_3 [\| x_1 \| + M_1 e^{\omega b} \| x_0 \| \\ & + M_1 \int_0^b e^{\omega(b-s)} h_k(s) ds] d\eta \end{aligned}$$

$$+ \int_0^{t_1} \| [R(t_1 - s) - R(t_2 - s)] \| h_k(s) ds$$

$$+ \int_{t_1}^{t_2} \| R(t_2 - s) \| h_k(s) ds.$$

The right-hand side tends to zero as $t_2 - t_1 \rightarrow 0$, since the compactness of $R(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

Thus, F maps B_k into an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Next, we show that $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem, to show that F maps B_k into a precompact set in X .

Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$(F_\epsilon x)(t) = R(t)x_0 + \int_0^{t-\epsilon} R(t-\eta)BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta$$

$$+ \int_0^{t-\epsilon} R(t-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds$$

$$= R(t)x_0 + R(\epsilon) \int_0^{t-\epsilon} R^{-1}(\epsilon)R(t-\eta)BW^{-1}[x_1 - R(b)x_0$$

$$- \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta)d\eta$$

$$+ R(\epsilon) \int_0^{t-\epsilon} R^{-1}(\epsilon)R(t-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds.$$

Since $R(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t): x \in B_k\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have

$$\| (Fx)(t) - (F_\epsilon x)(t) \| \leq \int_{t-\epsilon}^t \| R(t-\eta)BW^{-1}[x_1 - R(b)x_0$$

$$\begin{aligned}
& - \int_0^b R(b-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds](\eta) \parallel d\eta \\
& + \int_{t-\epsilon}^t \parallel R(t-s)f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \parallel ds \\
& \leq \int_{t-\epsilon}^t \parallel R(t-\eta) \parallel M_2 M_3 [\parallel x_1 \parallel + M_1 e^{\omega b} \parallel x_0 \parallel \\
& \quad + M_1 \int_0^b e^{\omega(b-s)} h_k(s)ds]d\eta \\
& \quad + \int_{t-\epsilon}^t \parallel R(t-s) \parallel h_k(s)ds.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Fx)(t): x \in B_k\}$. Hence the set $\{(Fx)(t): x \in B_k\}$ is precompact in X .

It remains to show that $F: C \rightarrow C$ is continuous. Let $\{x_n\}_0^\infty \subseteq C$ with $x_n \rightarrow x$ in C . Then there is an integer r such that $\parallel x_n(t) \parallel \leq r$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$.

By (iv),

$$f(t, x_n(t), \int_0^t g(t, s, x_n(s))ds) \rightarrow f(t, x(t), \int_0^t g(t, s, x(s))ds)$$

for each $t \in J$ and since

$$\parallel f(t, x_n(t), \int_0^t g(t, s, x_n(s))ds) - f(t, x(t), \int_0^t g(t, s, x(s))ds) \parallel \leq 2h_r(t),$$

we have by dominated convergence theorem,

$$\begin{aligned}
\parallel Fx_n - Fx \parallel &= \sup_{t \in J} \parallel \int_0^t R(t-\eta)BW^{-1} [\int_0^b R(b-s) \\
& [f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau) - f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)]ds](\eta)d\eta \\
& + \int_0^t R(t-s)[f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau)
\end{aligned}$$

$$\begin{aligned}
 & -f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau)ds \| \\
 & \leq \int_0^b \| R(t-\eta) \| M_2 M_3 [M_1 \int_0^b e^{\omega(b-s)} \| f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau) \\
 & \quad - f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \| ds] d\eta \\
 & \quad + \int_0^b \| R(t-s) \| \| f(s, x_n(s), \int_0^s g(s, \tau, x_n(\tau))d\tau) \\
 & \quad - f(s, x(s), \int_0^s g(s, \tau, x(\tau))d\tau) \| ds \rightarrow 0, \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally, the set $\zeta(F) = \{x \in C: x = \lambda Fx, \lambda \in (0, 1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem, the operator F has a fixed point in C . This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus system (1) is controllable on J .

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