A CLASS OF NONLINEAR VARIATIONAL INEQUALITIES INVOLVING PSEUDOMONOTONE OPERATORS

RAM U. VERMA
International Publications
12046 Coed Drive
Orlando, FL 32826 USA

(Received June, 1998; Revised January, 1999)

We present the solvability of a class of nonlinear variational inequalities involving pseudomonotone operators in a locally convex Hausdorff topological vector spaces setting. The obtained result generalizes similar variational inequality problems on monotone operators.

Key words: Nonlinear Variational Inequality, Pseudomonotone Operator, Locally Convex Space, Monotone Operator.

AMS subject classifications: 49J40.

1. Introduction

Recently, the author [4] studied a class of nonlinear variational inequalities in a locally convex Hausdorff topological vector space setting and extended a class of nonlinear variational inequalities involving monotone operators by utilizing a fixed point theorem [5]. The aim of this note is to present a class of generalized nonlinear variational inequalities involving pseudomonotone operators in a locally convex Hausdorff topological vector space setting. The obtained result generalizes the variational inequality results involving monotone operators in a similar setting, especially [4, 5]. For more details on variational and hemivariational inequalities in a Hilbert space setting, we refer to [3].

Let K be a subset of a real locally convex Hausdorff topological vector space X with its dual X' and let $T:K \rightarrow X'$ be a pseudomonotone nonlinear mapping. Let $\langle w, x \rangle$ denote the duality pairing between the elements w in X' and elements x in X . We consider the nonlinear variational inequality (NVI) problem: Find an element u_0 in K such that

$$\langle Tu_0, v - u_0 \rangle \geq 0 \text{ for all } v \text{ in } K. \quad (1)$$

A mapping $T:K \rightarrow X'$ is said to be:

(i) pseudomonotone [1] if

$$\langle Tu, v - u \rangle \geq 0 \Rightarrow \langle Tv, v - u \rangle \geq 0 \text{ for all } u, v \text{ in } K;$$

(ii) quasimonotone [1] if

$$\langle Tu, v - u \rangle > 0 \Rightarrow \langle Tv, v - u \rangle \geq 0 \text{ for all } u, v \text{ in } K;$$

(iii) monotone if for all u, v in K , we have

$$\langle Tv - Tu, v - u \rangle \geq 0.$$

We note that the following implication holds:

$$\text{monotone} \Rightarrow \text{pseudomonotone} \Rightarrow \text{quasimonotone}.$$

Before we consider our main result, we need to recall an auxiliary result (a variant of [5, Proposition 9.9] and similar to [2, Theorem 1] crucial to the NVI problem (1) at hand.

Lemma 1.1: *Let K be a nonempty, compact and convex subset of a locally convex space X and let $S: K \rightarrow P(K)$ be a multivalued mapping such that the following assumptions hold:*

- (i) *The set $S(x)$ is nonempty and convex for all $x \in K$.*
- (ii) *The preimages $S^{-1}(y)$ contain relatively open subsets U_y with respect to K for all y in K .*
- (iii) *$K = \bigcup \{U_y : y \in K\}$.*

Then there exists an element x_0 in K such that $x_0 \in S(x_0)$.

2. The Main Result

This section deals with the main result on the solvability of the NVI problem (1).

Theorem 2.1: *Let K be a nonempty, compact and convex subset of a locally convex Hausdorff topological vector space X and let $T: K \rightarrow X'$ be a pseudomonotone mapping. Then the NVI problem (1) has a solution u_0 in K .*

Proof: Assume the NVI problem (1) has no solution. Then for each $u \in K$, the set $\{v \text{ in } K : \langle Tu, v - u \rangle < 0\}$ is nonempty. As a result, if we define a multivalued mapping $F: K \rightarrow P(K)$ by

$$F(u) = \{v \text{ in } K : \langle Tu, v - u \rangle < 0\},$$

then clearly $F(u)$ is nonempty and convex for each u in K . It follows that

$$F^{-1}(v) = \{u \in K : v \in F(u)\} = \{u \in K : \langle Tu, v - u \rangle < 0\}.$$

It suffices to show that $F^{-1}(v)$ is relatively open with respect to K . For each v in K , the complement of $F^{-1}(v)$ in K ,

$$c_K(F^{-1}(v)) = \{u \in K : \langle Tu, v - u \rangle \geq 0\}$$

$$\subset \{u \in K : \langle Tv, v - u \rangle \geq 0\},$$

by the pseudomonotonicity of T , denoted by $G(v)$. We need to show that $G(v)$ is convex and closed in K . To show $G(v)$ is convex, if $u_1, u_2 \in G(v)$, $0 < t < 1$ and $u_t = tu_1 + (1-t)u_2$, then $u_t \in K$ and, $\langle Tv, v - u_1 \rangle \geq 0$ and $\langle Tv, v - u_2 \rangle \geq 0$. As a result of this, we have

$$\begin{aligned} \langle Tv, v - u_t \rangle &= \langle Tv, v - tu_1 - (1-t)u_2 \rangle = \langle Tv, t(v - u_1) + (1-t)(v - u_2) \rangle \\ &= t\langle Tv, v - u_1 \rangle + (1-t)\langle Tv, v - u_2 \rangle \geq 0. \end{aligned}$$

This implies $u_t \in G(v)$. Next, to show $G(v)$ is relatively closed with respect to K , let $\{u_a\}$ be a Moore-Smith sequence in $G(v)$. Then $\langle Tv, v - u_a \rangle \geq 0$. Assume $u_a \rightarrow u$ in K . Now all we need is to show $u \in G(v)$. We can now express

$$\begin{aligned} \langle Tv, v - u \rangle &= \langle Tv, v - u_a \rangle + \langle Tv, u_a - u \rangle \\ &\geq 0 + \langle Tv, u_a - u \rangle = 0. \end{aligned}$$

Thus, $u \in G(v)$. Since $G(v)$ is a closed convex subset of K , it implies $c_K(G(v)) = \{u \in K : \langle Tv, v - u \rangle < 0\}$ is open in K . This results in

$$c_K(G(v)) \subset F^{-1}(v) \text{ since any } u \notin F^{-1}(v) \text{ implies } u \notin c_K(G(v)).$$

Hence, for each v in K , $F^{-1}(v)$ contains an open subset $c_K(G(v))$. Thus for all u in K , there exists an element v in K such that $\langle Tv, v - u \rangle < 0$. Therefore, $K = \bigcup \{c_K(G(v)) : v \in K\}$. Now by Lemma 1.1, there exists an element x_0 in K such that $x_0 \in F(x_0)$, that means $0 > \langle Tx_0, x_0 - x_0 \rangle = 0$, a contradiction. This completes the proof.

Corollary 2.1: *Let X be a locally convex Hausdorff topological vector space and K a nonempty, compact and convex subset of X . Suppose that $T: K \rightarrow X'$ (dual of X) is a monotone mapping. Then the NVI problem (1) has a solution.*

References

- [1] Karamardian, S., Complementarity problems over cones with monotone and pseudomonotone maps, *J. Optimiz. Theor. Appl.* **18** (1970), 445-454.
- [2] Tarafdar, E., On some nonlinear variational inequalities, *Proc. AMS* **67** (1977), 95-98.
- [3] Verma, R.U., Nonlinear variational and constrained hemivariational inequalities involving relaxed operators, *ZAMM* **77**:5 (1997), 387-391.
- [4] Verma, R.U., Strongly monotone nonlinear variational inequalities, *Math. Sci. Res. Hot-Line* **2**:2 (1998), 21-24.
- [5] Zeidler, E., *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, New York 1986.