G-H-KKM SELECTIONS WITH APPLICATIONS TO MINIMAX THEOREMS

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Based on the G-H-KKM selections, some nonempty intersection theorems and their applications to minimax inequalities are presented.

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1. Introduction

Minimax inequalities have numerous applications to variational inequalities, while variational inequalities turn out to be a powerful tool to the solvability of problems in elasticity and plasticity theory, heat conduction, diffusion theory, optimization theory, mathematical economics, and others.

Chang and Zhang [1] introduced the notion of the generalized quasiconcavity and obtained some nonempty intersection theorems and their applications to minimax inequalities in a linear topological space setting. Recently, Tan [4] extended this notion to the case of a G-convex space with applications to minimax theorems and saddle points. Our aim here is to present some G-H-KKM selection theorems and related applications to minimax inequalities in a G-H-space setting.

Let X be a topological space, $P(X)$ denote the power set of X , and $\langle X \rangle$, a family of all nonempty finite subsets of X . Let Δ^n denote a standard $(n-1)$ simplex $\{e_1, e_2, \dots, e_n\}$ of R^n .

Definition 1.1: A triple $(X, H, \{p\})$ is called a *G-H-space* [6] if X is a topological space and $H: \langle X \rangle \rightarrow P(X) \setminus \{\emptyset\}$ is a mapping such that:

- (i) For each $F, G \in \langle X \rangle$, there exists $F_1 \subset F$ such that $F_1 \subset G$ implies $H(F_1) \subset H(G)$.
- (ii) For $F = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$, there is a continuous mapping $p: \Delta^n \rightarrow H(F)$ such that for $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, we have $p(\{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}) \subset H(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})$, where $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subset F$.

A subset D of X is called *finitely G-H-closed in X* if for each $A \in \langle X \rangle$, there exists $\tilde{A} \subset A$ such that $D(\tilde{})H(\tilde{A})$ is closed in $H(\tilde{A})$.

A subset K of X is said to be *compactly closed in X* if $K(\tilde{})L$ is closed in L for all compact subsets L of X .

Definition 1.2: Let $(X, H, \{p\})$ be a G-H-space and $T: X \rightarrow P(X)$ a multivalued mapping. T is called a *G-H-KKM mapping* if for each $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$, there exists $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subset \{x_1, x_2, \dots, x_n\}$ such that

$$H(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}) \subset \left(\bigcap_{j=1}^k\right) T(x_{i_j}) \text{ for } \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}.$$

Definition 1.3: Let $(X, H, \{p\})$ be a G-H-space and let $M_1, M_2, \dots, M_n$ be subsets of X . A subset $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ is said to be a *G-H-KKM selection for $M_1, M_2, \dots, M_n$* if for any $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subset \{x_1, x_2, \dots, x_n\}$, we have

$$H(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}) \subset \left(\bigcap_{j=1}^k\right) M_{i_j} \text{ for } \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}.$$

This generalizes the notion of a KKM selection in a pseudoconvex space by Joo' and Kassay [2].

Definition 1.4: Let X be nonempty set and $(Y, H, \{p\})$ a G-H-space. Let $f: X \times Y \rightarrow R$, $e: Y \rightarrow R$ and $h: X \rightarrow R$ be functions. The function f is said to be *0-generalized G-H-quasiconcave* (resp. *0-generalized G-H-quasiconvex*) in its first variable x if for each $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$, there exists $\{v_1, v_2, \dots, v_n\} \in \langle Y \rangle$ such that for each $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \subset \{v_1, v_2, \dots, v_n\}$ and for any $y_0 \in H(\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\})$, we have

$$\begin{aligned} & \min_{1 \leq j \leq k} [f(x_{i_j}, y_0) + e(y_0) - h(x_{i_j})] \leq 0 \\ & \text{(resp. } \max_{1 \leq j \leq k} [f(x_{i_j}, y_0) + e(y_0) - h(x_{i_j})] \geq 0), \end{aligned}$$

where $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$.

This generalizes the notion of a 0-generalized quasiconcavity (0-generalized quasiconvexity) by Chang and Zhang [1].

2. G-H-KKM Theorems and Applications

In this section we first recall and obtain some auxiliary results and then establish some minimax theorems.

Lemma 2.1: [7] Let $(X, H, \{p\})$ be a G-H-space and $M_1, M_2, \dots, M_n$ be finitely G-H-closed subsets of X . Suppose that $M_1, M_2, \dots, M_n$ have a G-KKM selection. Then $(\bigcap_{i=1}^n M_i) \neq \emptyset$.

Proposition 2.1: Let X be a nonempty set and $(Y, H, \{p\})$ a G-H-space. Let $f: X \times Y \rightarrow R$, $e: Y \rightarrow R$ and $h: X \rightarrow R$ be functions. Then the following statements are equivalent:

(a) A mapping $T: X \rightarrow P(Y)$ defined by

$$T(x) = \{y \in Y: f(x, y) + e(y) - h(x) \leq 0\}$$

$$(resp. T(x) = \{y \in Y: f(x, y) + e(y) - h(x) \geq 0\}),$$

is a G-H-KKM mapping.

- (b) f is 0-generalized G-H-quasiconcave (resp. 0-generalized G-H-quasiconvex) in its first variable x .

Proof: (a) $\Rightarrow$ (b) Since T is G-H-KKM, it implies for each $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ and corresponding $\{v_1, v_2, \dots, v_n\} \in \langle Y \rangle$ that there exist $\{x_{i1}, x_{i2}, \dots, x_{ik}\} \subset \{x_1, x_2, \dots, x_n\}$, $\{v_{i1}, v_{i2}, \dots, v_{ik}\} \subset \{v_1, v_2, \dots, v_k\}$ and any $y_0 \in H(\{v_{i1}, v_{i2}, \dots, v_{ik}\})$ such that

$$H(\{v_{i1}, v_{i2}, \dots, v_{ik}\}) \subset \left(\frac{k}{j=1}\right)T(x_{ij}).$$

This implies $y_0 \in \left(\frac{k}{j=1}\right)T(x_{ij})$, and as a result, there exists some index m ($i \leq m \leq k$) such that $y_0 \in T(x_{im})$. Hence, $f(x_{im}, y_0) + e(y_0) - h(x_{im}) \leq 0$ (resp. $f(x_{im}, y_0) + e(y_0) - h(x_{im}) \geq 0$). It follows that

$$\begin{aligned} \min_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] &\leq 0 \\ (resp. \max_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] &\geq 0). \end{aligned}$$

(b) $\Rightarrow$ (a) Since f is 0-generalized G-H-quasiconcave (resp. 0-generalized G-H-quasiconvex) in x , it implies for any $\{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ and $\{v_1, v_2, \dots, v_n\} \in \langle Y \rangle$, there exist $\{v_{i1}, v_{i2}, \dots, v_{ik}\} \subset \{v_1, v_2, \dots, v_n\}$, and any $y_0 \in H(\{v_{i1}, v_{i2}, \dots, v_{ik}\})$ such that

$$\begin{aligned} \min_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] &\leq 0 \\ (resp. \max_{1 \leq j \leq k} [f(x_{ij}, y_0) + e(y_0) - h(x_{ij})] &\geq 0). \end{aligned}$$

It follows that there exists some index m ($1 \leq m \leq k$) such that $y_0 \in T(x_m) \subset \left(\frac{k}{j=1}\right)T(x_{ij})$. This completes the proof.

Theorem 2.1: Let X be a nonempty set and $(Y, H, \{p\})$ a G-H-space such that $H(F)$ is compact for all $F \in \langle Y \rangle$. Suppose that $f: X \times Y \rightarrow R$, $e: Y \rightarrow R$ and $h: X \rightarrow R$ are functions satisfying the following assumptions:

- (i) f is lower semicontinuous in y on compact subsets of Y .
- (ii) e is lower semicontinuous on compact subsets of Y .
- (iii) f is 0-generalized G-H-quasiconcave in x .
- (iv) There exists an element $x_0 \in X$ such that the set

$$\{y \in Y: f(x_0, y) + e(y) - h(x_0) \leq 0\}$$

is a compact subset of Y .

Then there exists an element $\tilde{y} \in Y$ such that

$$f(x, \tilde{y}) + e(\tilde{y}) - h(x) \leq 0 \text{ for all } x \in X.$$

Proof: Let us define a mapping $T: X \rightarrow P(Y)$ by

$$T(x) = \{y \in Y: f(x, y) + e(y) - h(x) \leq 0\} \text{ for all } x \in X.$$

Since f is 0-generalized G-H-quasiconcave, it implies that $T(x)$ is nonempty. It follows from Proposition 2.1 that T is a G-H-KKM mapping. By (i) and (ii), each $T(x)$ is finitely G-H-closed, that is, for each $\mathcal{A} \subset A \in \langle Y \rangle$, we have

$$\begin{aligned} T(x)(-)H(\mathcal{A}) &= \{y \in H(\mathcal{A}): f(x, y) + e(y) - h(x) \leq 0\} \\ &= \{y \in H(\mathcal{A}): f(x, y) + e(y) \leq h(x)\}, \end{aligned}$$

is closed in $H(\mathcal{A})$ by the lower semicontinuity of f and e , so the family $\{T(x): x \in X\}$ has the finite intersection property by Lemma 2.1. Now applying (iv), we find that $\{T(x)(-)T(x_0): x \in X\}$ is a family of compact subsets of Y . Hence, $(-)_{x \in X} T(x) \neq \emptyset$. That means, there exists an element $\tilde{y} \in Y$ such that

$$f(x, \tilde{y}) + e(\tilde{y}) - h(x) \leq 0 \text{ for all } x \in X.$$

This completes the proof.

For Y compact, Theorem 2.1 reduces to:

Theorem 2.2: Let X be a nonempty set and $(Y, H, \{p\})$ a compact G-H-space with $H(F)$ compact for all $F \in \langle Y \rangle$. Suppose that $f: X \times Y \rightarrow R$, $e: Y \rightarrow R$ and $h: X \rightarrow R$ are functions such that:

- (i) f is lower semicontinuous in second variable y .
- (ii) e is lower semicontinuous.
- (iii) f is 0-generalized G-H-quasiconcave in first variable x .

Then there is an element $\tilde{y} \in Y$ such that

$$f(x, \tilde{y}) + e(\tilde{y}) - h(x) \leq 0 \text{ for all } x \in X.$$

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