Journal of Applied Mathematics and Stochastic Analysis, **14**:4 (2001), 351-359.

STRONG RESULT FOR REAL ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS

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(Received February, 2001, Revised July, 2001)

An estimate is given for the lower bound of real zeros of random algebraic polynomials whose coefficients are non-identically distributed dependent Gaussian random variables. Moreover, our estimated measure of the exceptional set, which is independent of the degree of the polynomials, tends to zero as the degree of the polynomial tends to infinity.

Key words: Random Polynomial, Dependent Normal Distribution, Real Roots.

AMS subject classifications: 60XX, 60F99.

1. Introduction

Let $N_n(\boldsymbol{R}, w)$ be the number of real roots of the random algebraic equation

$$F_n(x, w) = \sum_{\nu=0}^{n} a_v(w) x^v = 0, \tag{1.1}$$

where the $a_v(w)$, $v = 0, 1, \ldots, n$ are random variables defined on a fixed probability space $(\Omega, \mathcal{A}, \Pr)$ assuming real values only.

The problem of estimating the lower bound of $N_n(\boldsymbol{R}, w)$ was initiated by Littlewood and Offord [4]. They considered the case when the coefficients are normally distributed, uniformly distributed in $[-1, 1]$ or assume only the values $+1$ and -1 with equal probabilities, and proved that there exists an integer n_0 such that for $n > n_0$, $N_n(\boldsymbol{R}, w) > \frac{C \log n}{\log \log \log n}$ except for a set of measure at most $\frac{C'}{\log n}$, where C and C' are constants.

The lower bound has been studied especially in 1960's and early 1970's (cf. Bharucha-Reid and Sambandham [1] and Farahmand [3]). Taking the coefficients as normal random variables, Evans [2] proved that there exists an integer n_0 such that for each $n > n_0$, $N_n(\boldsymbol{R}, w) > \frac{C \log n}{\log \log n}$ except for a set of measure at most $\frac{C' \log \log n_0}{\log n_0}$. The above result of Evans is called the "strong" result for the lower bound in the following sense. The result of Littlewood and Offord is of the form,

$$\Pr\left(\frac{N_n(\boldsymbol{R},w)}{\frac{\log n}{\log\log\log n}} > C\right) \geq 1 - \frac{C'}{\log n}.$$

In this case, the exceptional set depends on the degree n of the equation. While the "strong" result of Evans is of the form,

$$\Pr\left(\inf_{n>n_0} \frac{N_n(\boldsymbol{R},w)}{\frac{\log n}{\log\log n}} > C\right) \geq 1 - \frac{C'\log\log n_0}{\log n_0}.$$

In such case, the exceptional set is independent of the degree n.

Since Evans' paper appeared, there has been a stream of papers on the lower bound by many workers, like Samal and Mishra [7, 8], although they mainly worked with independent and identically distributed coefficients.

For non-identically distributed coefficients, Samal and Mishra [9] considered the following type of the random algebraic equation:

$$f_n(x,w) = \sum_{v=0}^{n} a_v(w) b_v x^v = 0, \tag{1.2}$$

where the $a_v(w)$'s have a symmetric stable distribution and the b_v's are non-zero real numbers, and estimated the lower bound and the "strong" result for it. In this case, the coefficients $a_v(w)b_v$'s are non-identically distributed.

For dependent coefficients, Renganathan and Sambandham [6] and Nayak and Mohanty [5] took up several cases. Both of them defined the random variable η_m in their proofs and treated the η'_ms as independent random variables. Uno [10] pointed out that the η'_ms were dependent and that their required results had not been completed, and obtained the lower bound in the case of the type of (1.2), where the $a_v(w)$'s are normally distributed with mean zero and joint density function

$$|M|^{1/2}(2\pi)^{-(n+1)/2}\exp(-1/2)\boldsymbol{a}'M\boldsymbol{a}), \tag{1.3}$$

where M^{-1} is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i=j) \\ \rho_{|i-j|} & (1 \leq |i-j| \leq m) \\ 0 & (|i-j| > m) \qquad i,j = 0,1,\ldots,n, \end{cases} \tag{1.4}$$

for a positive integer m, where $0 \leq \rho_j < 1$, $j = 1,2,\ldots,m$ and $\boldsymbol{a}'$ is the transpose of the column vector $\boldsymbol{a}$, and the b_v's are positive numbers. However, the result of Uno is not the "strong" result for the lower bound.

The object of this paper is to show the "strong" result for the lower bound when the coefficients are non-identically distributed and dependent normal, that is, to obtain a "strong" result of Uno. We assume the same conditions of the $a_v(w)$'s and the b_v's as those of Uno. We remark that this assumption of the $a_v(w)$'s is called stationary m-dependent Gaussian and equivalent to the following two statements for a stationary Gaussian sequence:

1. $\{a_v\}$ is $*$-mixing,
2. $\{a_v\}$ is ϕ-mixing,

according to Yoshihara [11].

Throughout the paper, we suppose n is sufficiently large. We shall follow the line of proof of Samal and Mishra [8] and Uno [10].

Theorem: *Let*

$$f_n(x,w) = \sum_{v=0}^{n} a_v(w) b_v x^v = 0$$

be random algebraic equation of degree n, where the $a_v(w)$'s are dependent normally distributed with mean zero, joint density function (1.3) *and the moment matrix given by* (1.4) *and the b_v, $v = 0,1,\ldots,n$ are positive numbers such that* $\log(\frac{k_n}{t_n}) = o(\log n)$, *where $k_n = \max_{0 \le v \le n} b_n$ and $t_n = \min_{0 \le v \le n} b_v$.*

Then there exists an integer n_0 such that for each $n > n_0$, the number of real roots of most of the equations $f_n(x,w) = 0$ is at least $\dfrac{C \log n}{\log\left(\frac{k_n}{t_n} \log n\right)}$ *except for a set of measure at most* $\dfrac{C'}{\log\left\{\dfrac{\log n_0}{\log\left(\frac{k_{n_0}}{t_{n_0}} \log n_0\right)}\right\}}$, *where C and C' are positive constants.*

2. Proof of the Theorem

Let

$$\lambda_l = \sqrt{l} \log l \tag{2.1}$$

and M_l, $l = 1,2\ldots$ be a sequence of integers defined by

$$M_l = \left[\frac{\alpha\left(\frac{k_n}{t_n}\right)^2 \lambda_l^2}{l}\right] + 1 \tag{2.2}$$

where α is a positive constant and $[x]$, as usual, denotes the greatest integer not exceeding x. Let k be the integer determined by

$$(2k)! M_n^{2k} \le n < (2k+2)! M_n^{2k+2}. \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that

$$\frac{C_1 \log n}{\log\left(\frac{k_n}{t_n} \log n\right)} < k \tag{2.4}$$

for a constant C_1. Hence k is large when n is large.

We shall consider $f_n(x,w)$ at the points

$$x_l = \left(1 - \frac{1}{(2l)! M_l^{2l}}\right)^{\frac{1}{2}},$$

for $l = [\frac{k}{2}] + 1, [\frac{k}{2}] + 2, \ldots, k$.

We write

$$f_n(x_l, w) = U_l(w) + R_l(w)$$
$$= \sum_1 a_v(w) b_v x_l^v + \left(\sum_2 + \sum_3 \right) a_v(w) b_v x_l^v,$$

where v ranges from $(2l-1)!M_l^{2l-1}+1$ to $(2l+1)!M_l^{2l+1}$ in $\sum_1$, from 0 to $(2l-1)!M_l^{2l-1}$ in $\sum_2$ and from $(2l+1)!M_l^{2l+1}+1$ to n in $\sum_3$.

The following lemmas are necessary for the proof of the theorem.

Lemma 2.1: *For $\alpha_1 > 0$ and*

$$\sigma_l^2 = \sum_{i=(2l-1)!M_l^{2l-1}+1}^{(2l+1)!M_l^{2l+1}} b_i^2 x_l^{2i} + 2 \sum_{i=(2l-1)!M_l^{2l-1}+1}^{(2l+1)!M_l^{2l+1}-1} \sum_{j=i+1}^{(2l+1)!M_l^{2l+1}} b_i b_j x_l^{i+j} \rho_{j-i},$$

we have

$$\sigma_l > \alpha_1 t_n \sqrt{(2l)!} M_l^l.$$

Proof: First for $t_n = \min_{0 \le v \le n} b_v$, we have

$$\sum_{i=(2l-1)!M_l^{2l-1}+1}^{(2l+1)!M_l^{2l+1}} b_i^2 x_l^{2i} > t_n^2 \sum_{i=(2l-1)!M_l^{2l-1}+1}^{(2l)!M_l^{2l}} x_l^{2i} > \left(\frac{B}{Ae}\right) t_n^2 (2l)! M_l^{2l}$$

where A and B are positive constants such that $A > 1$ and $0 < B < 1$. Next, for m given in (1.4), we get

$$\sum_{i=(2l-1)!M_l^{2l-1}+1}^{(2l+1)!M_l^{2l+1}-1} \sum_{j=i+1}^{(2l+1)!M_l^{2l+1}} b_i b_j x_l^{i+j} \rho_{j-i}$$

$$> t_n^2 \sum_{i=(2l-1)!M_l^{2l-1}+1}^{(2l)!M_l^{2l}-1} \sum_{j=i+1}^{(2l)!M_l^{2l}} x_l^{i+j} \rho_{j-1}$$

$$= t_n^2 \frac{x_l^{2\{(2l-1)!M_l^{2l-1}+1\}}}{1-x_l^2} \left\{ \sum_{i=1}^m \rho_i x_l^i - \sum_{i=1}^m \rho_i x_l^{2\{(2l)!M_l^{2l}-(2l-1)!M_l^{2l-1}\}-i} \right\}$$

$$\ge \left(\frac{B'}{A'}\right) \rho_0 t_n^2 (2l)! M_l^{2l}$$

where $\rho_0 = \sum_{j=1}^m \rho_j$ and A' and B' are positive constants satisfying $A' > 1$ and $0 < B' < 1$. So we get

$$\sigma_l^2 \geq \alpha_1^2 t_n^2 (2l)! M_l^{2l},$$

where α_1 is a positive constant, as required.

The following lemmas (Lemmas 2.2 and 2.3), which are required to prove Lemma 2.4, can be proved by Feller's inequality.

Lemma 2.2:

$$\Pr\left(\left\{w: \left|\sum_2 a_v(w) b_v x_l^v\right| > \lambda_l \tilde{\sigma}_l\right\}\right) < \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\lambda_l^2}{2}}}{\lambda_l},$$

where

$$\tilde{\sigma}_l^2 = \sum_{i=0}^{(2l-1)!M_l^{2l-1}} b_i^2 x_l^{2i} + 2 \sum_{i=0}^{(2l-1)!M_l^{2l-1}-1} \sum_{j=i+1}^{(2l-1)!M_l^{2l-1}} b_i b_j x_l^{i+j} \rho_{j-i}.$$

Lemma 2.3:

$$\Pr\left(\left\{w: \left|\sum_3 a_v(w) b_v x_l^v\right| > \lambda_l \tilde{\tilde{\sigma}}_l\right\}\right) < \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\lambda_l^2}{2}}}{\lambda_l},$$

where

$$\tilde{\tilde{\sigma}}_l^2 = \sum_{i=(2l+1)!M_l^{2l+1}+1}^{n} b_i^2 x_l^{2i} + 2 \sum_{i=(2l+1)!M_l^{2l+1}+1}^{n-1} \sum_{j=i+1}^{n} b_i b_j x_l^{i+j} \rho_{j-i}.$$

Lemma 2.4: *For a fixed l,*

$$\Pr(\{w: |R_l(w)| < \sigma_l\}) > 1 - 2\sqrt{\frac{2}{\pi}} \frac{1}{\lambda_l} e^{-\frac{\lambda_l^2}{2}}.$$

Proof: By Lemmas 2.2 and 2.3, we get for a given l,

$$|R_l(w)| < \lambda_l(\tilde{\sigma}_l + \tilde{\tilde{\sigma}}_l),$$

outside a set of measure at most $2\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\lambda_l^2}{2}}}{\lambda_l}$. Again we have

$$\sum_{i=0}^{(2l-1)!M_l^{2l-1}} b_i^2 x_l^{2i} \leq 2k_n^2 (2l-1)! M_l^{2l-1}$$

and

$$\sum_{i=0}^{(2l-1)!M_l^{2l-1}-1} \sum_{j=i+1}^{(2l-1)!M_l^{2l-1}} b_i b_j x_l^{i+j} \rho_{j-i}$$

$$\leq k_n^2 \sum_{i=1}^{m} \rho_i \sum_{j=1}^{(2l-1)!M_l^{2l-1}-(i-1)} x_l^{2j+i-2} \leq \rho_0 k_n^2 (2l-1)! M_l^{2l-1}.$$

Hence, we get for a positive constant α_2,

$$\tilde{\sigma}_l^2 \leq \alpha_2^2 k_n^2 (2l-1)! M_l^{2l-1}.$$

Similarly we have

$$\tilde{\tilde{\sigma}}_l^2 \leq \alpha_3^2 k_n^2 (2l-1)! M_l^{2l-1}$$

for a positive constant α_3. Therefore we obtain outside the exceptional set,

$$| R_l(w) | < \lambda_l(\alpha_2+\alpha_3) k_n M_l^{l-\frac{1}{2}} < \left(\frac{\alpha_2+\alpha_3}{\alpha_1} \frac{k_n}{t_n} \frac{1}{\sqrt{2l}} \lambda_l \sigma_l \right) / M_l^{\frac{1}{2}} < \sigma_l,$$

by Lemma 2.1 and (2.2).

Let us define random events E_p, F_p and G_p by

$$E_p = \{w: U_{3p}(w) \geq \sigma_{3p}, U_{3p+1}(w) < -\sigma_{3p+1}\},$$

$$F_p = \{w: U_{3p}(w) < -\sigma_{3p}, U_{3p+1}(w) \geq \sigma_{3p+1}\}$$

and

$$G_p = \{w: | R_{3p}(w) | < \sigma_{3p}, | R_{3p+1}(w) | < \sigma_{3p+1}\}$$

for $(3p, 3p+1)$ such that $[\frac{k}{2}] + 1 \leq 3p < 3p+1 \leq k$. It can be easily seen that

$$\Pr(E_p \cup F_p) > \delta,$$

where $\delta > 0$ is a certain constant. And we define random variables η_p, ζ_p and ξ_p such that

$$\eta_p = \begin{cases} 1 & \text{on } E_p \cup F_p \\ 0 & \text{elsewhere,} \end{cases}$$

$$\zeta_p = \begin{cases} 0 & \text{on } G_p \\ 1 & \text{elsewhere} \end{cases}$$

and

$$\xi_p = \eta_p - \eta_p \zeta_p.$$

If $\xi_p = 1$, there is a root of the polynomial in the interval (x_{3p}, x_{3p+1}).

Let $p_{\min}$ and $p_{\max}$ be the integers such that

$$p_{\min} = \min\left\{ p \in \boldsymbol{N} \,\middle|\, \left[\frac{k}{2}\right] + 1 \leq 3p < 3p+1 \leq k \right\}$$

and

$$p_{\max} = \max\left\{ p \in \boldsymbol{N} \,\middle|\, \left[\frac{k}{2}\right] + 1 \leq 3p < 3p+1 \leq k \right\}.$$

Then the number of roots in the $(x_{[\frac{k}{2}]+1}, x_k)$ must exceed $\sum_{p=p_{\min}}^{p_{\max}} \xi_p$.

We shall need the strong law of large numbers in the following form.

If $\eta_2, \eta_3, \ldots$ are independent random variables with $\operatorname{var}(\eta_i) < 1$ *for all i, then for given any $\epsilon > 0$, we have*

$$\Pr\left\{\sup_{p_{\max}-p_{\min}+1 \geq k_0}\left|\frac{1}{p_{\max}-p_{\min}+1}\sum_{p=p_{\min}}^{p_{\max}}(\eta_p - E(\eta_p))\right| \geq \epsilon\right\} \leq \frac{D}{\epsilon^2 k_0},$$

where D is a positive constant.

Here we get

$$\left|\sum_{p=p_{\min}}^{p_{\max}}\{\xi_p - E(\eta_p)\}\right| \leq \left|\sum_{p=p_{\min}}^{p_{\max}}\{\eta_p - E(\eta_p)\}\right| + \sum_{p=p_{\min}}^{p_{\max}}\zeta_p.$$

Since

$$E(\zeta_p) \leq 4\sqrt{\frac{2}{\pi}}\,\frac{1}{\lambda_{3p}}e^{-\frac{\lambda_{3p}^2}{2}}$$

from Lemma 2.4, we have

$$\sum_{p=p_{\min}}^{p_{\max}}\zeta_p < (p_{\max}-p_{\min}+1)\epsilon_1$$

outside an exceptional set of measure at most

$$4\sqrt{\frac{2}{\pi}}\sum_{p=p_{\min}}^{p_{\max}}\frac{1}{(p_{\max}-p_{\min}+1)\epsilon_1}\,\frac{1}{\lambda_{3p}}e^{-\frac{\lambda_{3p}^2}{2}} < C_2\frac{1}{\lambda_{3p_{\min}}}e^{-\frac{\lambda_{3p_{\min}}^2}{2}},$$

where C_2 is a constant. Thus we obtain

$$\sup_{p_{\max}-p_{\min}+1 \geq k_0}\frac{1}{(p_{\max}-p_{\min}+1)}\sum_{p=p_{\min}}^{p_{\max}}\zeta_p < \epsilon_1,$$

outside an exceptional set of measure at most

$$C_2\sum_{p_{\max}-p_{\min}+1 \geq k_0}\frac{e^{-\frac{\lambda_{3p_{\min}}^2}{2}}}{\lambda_{3p_{\min}}}.$$

By using the strong law of large numbers since the η_p's are independent for sufficiently large n, we have

$$\sup_{p_{\max}-p_{\min}+1 \geq k_0}\frac{1}{(p_{\max}-p_{\min}+1)}\left|\sum_{p=p_{\min}}^{p_{\max}}\{\xi_p - E(\eta_p)\}\right| < \epsilon,$$

outside an exceptional set G_{k_0} of measure at most

$$C_2 \sum_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{e^{-\frac{\lambda_{3p_{\min}}^2}{2}}}{\lambda_{3p_{\min}}} + \frac{C_3}{K_0},$$

where C_3 is a constant.

A simple calculation shows that

$$p_{\max} = \left[\frac{k+2}{3}\right] - 1 \text{ and } p_{\min} = \left[\frac{k}{6}\right] + 1.$$

Hence we obtain

$$\frac{1}{p_{\max} - p_{\min} + 1} \sum_{p = p_{\min}}^{p_{\max}} \xi_p > \frac{1}{p_{\max} - p_{\min} + 1} \sum_{p = p_{\min}}^{p_{\max}} E(\eta_p) - \epsilon$$

for all k such that $p_{\max} - p_{\min} + 1 \geq k_0$ outside an exceptional set G_{k_0}.

Applying $E(\eta_p) > \delta$ and using (2.4), we get

$$N_n > \sum_{p = p_{\min}}^{p_{\max}} \xi_p > (p_{\max} - p_{\min} + 1)(\delta - \epsilon) > C_4 k > \frac{C_5 \log n}{\log\left(\frac{k_n}{t_n} \log n\right)}$$

for all k such that $p_{\max} - p_{\min} + 1 \geq k_0$ outside an exceptional set G_{k_0}, where C_4 and C_5 are constants. It can be seen that the set $\{k \in \boldsymbol{N} \mid p_{\max} - p_{\min} + 1 \geq k_0\}$ is contained in the set $\{k \in \mathbb{N} \mid k \geq 6k_0 - 2\}$.

If $n = n_0$ corresponds to $k = 6k_0 - 2$, then all $n > n_0$ will correspond to $k > 6k_0 - 2$. Therefore, we have for all $n > n_0$,

$$N_n > \frac{C \log n}{\log\left(\frac{k_n}{t_n} \log n\right)},$$

where

$$\Pr(G_{k_0}) < C_2 \sum_{k > 6k_0 - 2} \frac{1}{\lambda_{3p_{\min}}} e^{-\frac{\lambda_{3p_{\min}}^2}{2}} + \frac{C_3}{k_0}$$

$$\leq C_2 \left\{ \frac{1}{\lambda_{3k_0}} e^{-\frac{\lambda_{3k_0}^2}{2}} + 6\left(\frac{1}{\lambda_{3(k_0+1)}} e^{-\frac{\lambda_{3(k_0+1)}^2}{2}} + \frac{1}{\lambda_{3(k_0+2)}} e^{-\frac{\lambda_{3(k_0+2)}^2}{2}} + \cdots \right) \right\} + \frac{C_3}{k_0}$$

$$\leq 6C_2 \sum_{q \geq k_0} \frac{1}{\lambda_{3q}} e^{-\frac{\lambda_{3q}^2}{2}} + \frac{C_3}{k_0} = 6C_2 \sum_{q \geq k_0} \frac{1}{\sqrt{3q} \log(3q)} e^{-\frac{3q(\log(3q))^2}{2}} + \frac{C_3}{k_0}$$

$$\leq \frac{4}{\sqrt{3}} C_2 \sum_{q \geq k_0} \frac{1}{q(\log q)^2} + \frac{C_3}{k_0} \leq \frac{C_6}{\log k_0} + \frac{C_3}{k_0} \leq \frac{C'}{\log \left\{ \dfrac{\log n_n}{\log \left(\dfrac{k_{n_0}}{t_{n_0}} \log n_0 \right)} \right\}},$$

where C_6 is a constant. This completes the proof of the theorem.

Acknowledgement

The author wishes to thank the referee for his valuable comments.

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