STRONG RESULT FOR REAL ZEROS OF RANDOM ALGEBRAIC POLYNOMIALS

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An estimate is given for the lower bound of real zeros of random algebraic polynomials whose coefficients are non-identically distributed dependent Gaussian random variables. Moreover, our estimated measure of the exceptional set, which is independent of the degree of the polynomials, tends to zero as the degree of the polynomial tends to infinity.

Key words: Random Polynomial, Dependent Normal Distribution, Real Roots.

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1. Introduction

Let \( N_n(R, w) \) be the number of real roots of the random algebraic equation

\[
F_n(x, w) = \sum_{\nu=0}^{n} a_\nu(w)x^\nu = 0,
\]

where the \( a_\nu(w), \nu = 0, 1, \ldots, n \) are random variables defined on a fixed probability space \((\Omega, \mathcal{A}, \Pr)\) assuming real values only.

The problem of estimating the lower bound of \( N_n(R, w) \) was initiated by Littlewood and Offord [4]. They considered the case when the coefficients are normally distributed, uniformly distributed in \([-1, 1]\) or assume only the values +1 and −1 with equal probabilities, and proved that there exists an integer \( n_0 \) such that for \( n > n_0 \), \( N_n(R, w) > \frac{C \log n}{\log \log \log n} \) except for a set of measure at most \( C' \frac{\log \log n_0}{\log n_0} \), where \( C \) and \( C' \) are constants.

The lower bound has been studied especially in 1960's and early 1970's (cf. Bharucha-Reid and Sambandham [1] and Farahmand [3]). Taking the coefficients as normal random variables, Evans [2] proved that there exists an integer \( n_0 \) such that for each \( n > n_0 \), \( N_n(R, w) > \frac{C \log n}{\log \log \log n} \) except for a set of measure at most \( C' \frac{\log \log n_0}{\log n_0} \).

The above result of Evans is called the "strong" result for the lower bound in the following sense. The result of Littlewood and Offord is of the form,
In this case, the exceptional set depends on the degree $n$ of the equation. While the “strong” result of Evans is of the form,

$$\Pr \left( \inf_{n > n_0} \frac{N_n(R, w)}{\log \log n} > C \right) \geq 1 - \frac{C'}{\log n_0}.$$  

In such case, the exceptional set is independent of the degree $n$.

Since Evans’ paper appeared, there has been a stream of papers on the lower bound by many workers, like Samal and Mishra [7, 8], although they mainly worked with independent and identically distributed coefficients.

For non-identically distributed coefficients, Samal and Mishra [9] considered the following type of the random algebraic equation:

$$f_n(x, w) = \sum_{v=0}^{n} a_v(w) b_v x^v = 0,$$  \hspace{1cm} (1.2)

where the $a_v(w)$'s have a symmetric stable distribution and the $b_v$'s are non-zero real numbers, and estimated the lower bound and the “strong” result for it. In this case, the coefficients $a_v(w) b_v$'s are non-identically distributed.

For dependent coefficients, Renganathan and Sambandham [6] and Nayak and Mohanty [5] took up several cases. Both of them defined the random variable $\eta_m$ in their proofs and treated the $\eta'_m$'s as independent random variables. Uno [10] pointed out that the $\eta'_m$'s were dependent and that their required results had not been completed, and obtained the lower bound in the case of the type of (1.2), where the $a_v(w)$'s are normally distributed with mean zero and joint density function

$$|M|^{1/2}(2\pi)^-(n+1)/2 \exp(-1/2)a'Ma),$$  \hspace{1cm} (1.3)

where $M^{-1}$ is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i = j) \\ \rho |i - j| & (1 \leq |i - j| \leq m) \\ 0 & (|i - j| > m) \end{cases}$$  \hspace{1cm} (1.4)

for a positive integer $m$, where $0 \leq \rho_j < 1$, $j = 1, 2, \ldots, m$ and $a'$ is the transpose of the column vector $a$, and the $b_v$'s are positive numbers. However, the result of Uno is not the “strong” result for the lower bound.

The object of this paper is to show the “strong” result for the lower bound when the coefficients are non-identically distributed and dependent normal, that is, to obtain a “strong” result of Uno. We assume the same conditions of the $a_v(w)$'s and the $b_v$'s as those of Uno. We remark that this assumption of the $a_v(w)$'s is called stationary $m$-dependent Gaussian and equivalent to the following two statements for a stationary Gaussian sequence:

1. $\{a_v\}$ is $\ast$-mixing,
2. $\{a_v\}$ is $\phi$-mixing,

according to Yoshihara [11].
Throughout the paper, we suppose $n$ is sufficiently large. We shall follow the line of proof of Samal and Mishra [8] and Uno [10].

**Theorem:** Let

$$f_n(x, w) = \sum_{v=0}^{n} a_v(w)b_vx^v = 0$$

be random algebraic equation of degree $n$, where the $a_v(w)$'s are dependent normally distributed with mean zero, joint density function (1.3) and the moment matrix given by (1.4) and the $b_v$, $v = 0, 1, \ldots, n$ are positive numbers such that $\log(\frac{k_n}{t_n}) = o(\log n)$, where $k_n = \max_{0 \leq v \leq n} b_n$ and $t_n = \min_{0 \leq v \leq n} b_v$.

Then there exists an integer $n_0$ such that for each $n > n_0$, the number of real roots of most of the equations $f_n(x, w) = 0$ is at least $\frac{C \log n}{\log(\frac{k_n}{t_n} \log n)}$ except for a set of measure at most $\frac{C'}{\log(\frac{k_n}{t_n} \log n)}$, where $C$ and $C'$ are positive constants.

2. Proof of the Theorem

Let

$$\lambda_l = \sqrt{l \log l}$$

and $M_l$, $l = 1, 2, \ldots$ be a sequence of integers defined by

$$M_l = \left[ \frac{k_n}{t_n} \frac{\lambda_l^2}{l} \right] + 1$$

where $\alpha$ is a positive constant and $[x]$, as usual, denotes the greatest integer not exceeding $x$. Let $k$ be the integer determined by

$$(2k)!M_n^{2k} \leq n < (2k + 2)!M_n^{2k+2}.\quad (2.3)$$

It follows from (2.1), (2.2) and (2.3) that

$$\frac{C_1 \log n}{\log(\frac{k_n}{t_n} \log n)} < k$$

for a constant $C_1$. Hence $k$ is large when $n$ is large.

We shall consider $f_n(x, w)$ at the points

$$x_l = \left(1 - \frac{1}{(2l)!M_l^{2l}}\right)^{\frac{1}{2}},$$

for $l = \left[\frac{k}{2}\right] + 1, \left[\frac{k}{2}\right] + 2, \ldots, k$.

We write
\[
 f_n(x, w) = U_t(w) + R_t(w) \\
 = \sum_1^n a_v(w)b_vx_i^v + \left( \sum_2^3 + \sum_3^3 \right) a_v(w)b_vx_i^v,
\]

where \( v \) ranges from \((2l - 1)!M_{l-1}^2 + 1\) to \((2l + 1)!M_{l+1}^2 + 1\) in \( \sum_1 \), from \( 0 \) to \((2l - 1)!M_{l-1}^2 - 1\) in \( \sum_2 \) and from \((2l + 1)!M_{l+1}^2 + 1\) to \( n \) in \( \sum_3 \). The following lemmas are necessary for the proof of the theorem.

**Lemma 2.1:** For \( \alpha_1 > 0 \) and

\[
\sigma_l = \frac{(2l + 1)!M_{l+1}^2}{(2l)!M_l^2 - 1 + 1} \sum_{i = (2l - 1)!M_{l-1}^2 + 1}^{(2l)!M_{l+1}^2 - 1 + 1} b_i^2x_i^{2i} + 2 \sum_{i = (2l - 1)!M_{l-1}^2 + 1}^{(2l)!M_{l+1}^2 + 1} b_i^2x_i^{2i} + j \rho_j - i,
\]

we have

\[
\sigma_l > \alpha_1 t_n \sqrt{(2l)!M_l^2}.
\]

**Proof:** First for \( t_n = \min_{0 < v < nb_v} \), we have

\[
\sum_{i = (2l - 1)!M_{l-1}^2 + 1}^{(2l)!M_{l+1}^2 - 1 + 1} b_i^2x_i^{2i} > \frac{2t_n^2}{(2l)!M_l^2} \sum_{i = (2l - 1)!M_{l-1}^2 + 1}^{(2l)!M_{l+1}^2 - 1 + 1} x_i^{2i} > \left( \frac{B}{Ae} \right) t_n^2(2l)!M_l^2
\]

where \( A \) and \( B \) are positive constants such that \( A > 1 \) and \( 0 < B < 1 \). Next, for \( m \) given in (1.4), we get

\[
\sum_{i = (2l - 1)!M_{l-1}^2 + 1}^{(2l)!M_{l+1}^2 - 1 + 1} b_i^2x_i^{2i} + j \rho_j - i \]

\[
\sum_{i = (2l - 1)!M_{l-1}^2 + 1}^{(2l)!M_{l+1}^2 - 1 + 1} x_i^{2i} + j \rho_j - 1
\]

\[
= t_n^2 \left\{ \sum_{i = 1}^m \rho_i x_i^2 \right\} + \sum_{i = 1}^m \left( (2l)!M_l^2 - (2l)!M_l^2 - 1 \right) \rho_i x_i^2 \]

\[
\geq \left( \frac{B'}{A'} \right) t_n^2(2l)!M_l^2
\]

where \( \rho_0 = \sum_{j = 1}^m \rho_j \) and \( A' \) and \( B' \) are positive constants satisfying \( A' > 1 \) and \( 0 < B' < 1 \). So we get
where $\alpha_1$ is a positive constant, as required.

The following lemmas (Lemmas 2.2 and 2.3), which are required to prove Lemma 2.4, can be proved by Feller’s inequality.

**Lemma 2.2:**

$$\Pr\left(\left\{ w : \sum_{2} a_{ij}(w)b_{ij}x_{1}^{i} > \lambda_{l} \tilde{\sigma}_{l} \right\} \right) < \sqrt{\frac{\lambda_{l}}{\pi}} e^{-\frac{\lambda_{l}^{2}}{2}}$$

where

$$\tilde{\sigma}_{l}^{2} = \sum_{i=0}^{\frac{(2l-1)!M_{l}^{2l-1}}{2}} b_{i}^{2}x_{1}^{2i} + 2 \sum_{i=0}^{\frac{(2l-1)!M_{l}^{2l-1}}{2}} \sum_{j=i+1}^{\frac{(2l-1)!M_{l}^{2l-1}}{2}} b_{i}b_{j}x_{1}^{i + j} \rho_{j-i}.$$

**Lemma 2.3:**

$$\Pr\left(\left\{ w : \sum_{3} a_{ij}(w)b_{ij}x_{1}^{i} > \lambda_{l} \tilde{\sigma}_{l} \right\} \right) < \sqrt{\frac{\lambda_{l}}{\pi}} e^{-\frac{\lambda_{l}^{2}}{2}}$$

where

$$\tilde{\sigma}_{l}^{2} = \sum_{i=\frac{(2l+1)!M_{l}^{2l+1}}{2}}^{n} b_{i}^{2}x_{1}^{2i} + 2 \sum_{i=\frac{(2l+1)!M_{l}^{2l+1}}{2}}^{n-1} \sum_{j=i+1}^{n} b_{i}b_{j}x_{1}^{i + j} \rho_{j-i}.$$

**Lemma 2.4:** For a fixed $l$,

$$\Pr(\{w : |R_{l}(w)| < \sigma_{l}\}) > 1 - 2\sqrt{\frac{\lambda_{l}}{\pi}} e^{-\frac{\lambda_{l}^{2}}{2}}.$$

**Proof:** By Lemmas 2.2 and 2.3, we get for a given $l$,

$$|R_{l}(w)| < \lambda_{l}(\tilde{\sigma}_{l} + \tilde{\sigma}_{l}),$$

outside a set of measure at most $2\sqrt{\frac{\lambda_{l}}{\pi}} e^{-\frac{\lambda_{l}^{2}}{2}}$. Again we have

$$\sum_{i=0}^{\frac{(2l-1)!M_{l}^{2l-1}}{2}} b_{i}^{2}x_{1}^{2i} \leq 2k_{n}^{2}(2l-1)!M_{l}^{2l-1}$$

and

$$\sum_{i=0}^{\frac{(2l-1)!M_{l}^{2l-1}}{2}} \sum_{j=i+1}^{\frac{(2l-1)!M_{l}^{2l-1}}{2}} b_{i}b_{j}x_{1}^{i + j} \rho_{j-i} \leq k_{n}^{2} \sum_{i=1}^{m} \rho_{i} \sum_{j=1}^{(2l-1)!M_{l}^{2l-1} - (i-1)} x_{1}^{2j} + i - 2 \leq \rho_{0}k_{n}^{2}(2l-1)!M_{l}^{2l-1}.$$
Hence, we get for a positive constant $\alpha_2$,

$$\tilde{\sigma}_l^2 \leq \alpha_2^2 k_n^2 (2l-1)! M_l^{2l-1}.$$  

Similarly we have

$$\tilde{\sigma}_l^2 \leq \alpha_3^2 k_n^2 (2l-1)! M_l^{2l-1}$$

for a positive constant $\alpha_3$. Therefore we obtain outside the exceptional set,

$$| R_l(w) | < \lambda_l(\alpha_2 + \alpha_3) k_n^l M_l^{l-\frac{1}{2}} \left( \frac{\alpha_2 + \alpha_3}{\alpha_1} \frac{k_n}{\ell_n} \frac{1}{2\ell_1} \right) / M_l^{\frac{1}{2}} < \sigma_l,$$

by Lemma 2.1 and (2.2).

Let us define random events $E_p$, $F_p$ and $G_p$ by

$$E_p = \{ w: U_{3p}(w) \geq \sigma_{3p}, U_{3p+1}(w) < -\sigma_{3p+1} \},$$

$$F_p = \{ w: U_{3p}(w) < -\sigma_{3p}, U_{3p+1}(w) \geq \sigma_{3p+1} \}$$

and

$$G_p = \{ w: | R_{3p}(w) | < \sigma_{3p}, | R_{3p+1}(w) | < \sigma_{3p+1} \}$$

for $(3p, 3p+1)$ such that $\left[ \frac{k}{2} \right] + 1 \leq 3p < 3p+1 \leq k$. It can be easily seen that

$$\Pr(E_p \cup F_p) > \delta,$$

where $\delta > 0$ is a certain constant. And we define random variables $\eta_p$, $\zeta_p$ and $\xi_p$ such that

$$\eta_p = \begin{cases} 1 & \text{on } E_p \cup F_p \\ 0 & \text{elsewhere} \end{cases},$$

$$\zeta_p = \begin{cases} 0 & \text{on } G_p \\ 1 & \text{elsewhere} \end{cases}$$

and

$$\xi_p = \eta_p - \eta_p \zeta_p.$$  

If $\xi_p = 1$, there is a root of the polynomial in the interval $(x_{3p}, x_{3p+1})$.

Let $p_{\min}$ and $p_{\max}$ be the integers such that

$$p_{\min} = \min \left\{ p \in \mathbb{N} \middle| \left[ \frac{k}{2} \right] + 1 \leq 3p < 3p+1 \leq k \right\}$$

and

$$p_{\max} = \max \left\{ p \in \mathbb{N} \middle| \left[ \frac{k}{2} \right] + 1 \leq 3p < 3p+1 \leq k \right\}.$$
Then the number of roots in the \((x_{\left[\frac{k}{2}\right] + 1}, x_k)\) must exceed \(\sum_{p = p_{\min}}^{p_{\max}} \xi_p\).

We shall need the strong law of large numbers in the following form.

If \(\eta_2, \eta_3, \ldots\) are independent random variables with \(\text{var}(\eta_i) < 1\) for all \(i\), then for given any \(\epsilon > 0\), we have

\[
\Pr \left\{ \sup_{p_{\max} - p_{\min} + 1 \geq k_0} \left| \frac{1}{p_{\max} - p_{\min} + 1} \sum_{p = p_{\min}}^{p_{\max}} (\eta_p - E(\eta_p)) \right| \geq \epsilon \right\} \leq \frac{D}{\epsilon^2 k_0},
\]

where \(D\) is a positive constant.

Here we get

\[
\left| \sum_{p = p_{\min}}^{p_{\max}} \xi_p - E(\eta_p) \right| \leq \left| \sum_{p = p_{\min}}^{p_{\max}} \eta_p - E(\eta_p) \right| + \sum_{p = p_{\min}}^{p_{\max}} \zeta_p.
\]

Since

\[
E(\zeta_p) \leq 4\sqrt{\frac{2}{\pi}} \frac{1}{\lambda_{3p}^2} e^{-\frac{\lambda_{3p}^2}{2}}
\]

from Lemma 2.4, we have

\[
\sum_{p = p_{\min}}^{p_{\max}} \zeta_p < (p_{\max} - p_{\min} + 1)\epsilon_1
\]

outside an exceptional set of measure at most

\[
4\sqrt{\frac{2}{\pi}} \sum_{p = p_{\min}}^{p_{\max}} \frac{1}{p_{\max} - p_{\min} + 1}\epsilon_1 \frac{1}{\lambda_{3p}^2} e^{-\frac{\lambda_{3p}^2}{2}} < C_2 \frac{1}{\lambda_{3p_{\min}}^2} e^{-\frac{\lambda_{3p_{\min}}^2}{2}},
\]

where \(C_2\) is a constant. Thus we obtain

\[
\sup_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{p_{\max} - p_{\min} + 1} \sum_{p = p_{\min}}^{p_{\max}} \zeta_p < \epsilon_1,
\]

outside an exceptional set of measure at most

\[
C_2 \sum_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{\lambda_{3p_{\min}}^2} e^{-\frac{\lambda_{3p_{\min}}^2}{2}}.
\]

By using the strong law of large numbers since the \(\eta_p\)'s are independent for sufficiently large \(n\), we have

\[
\sup_{p_{\max} - p_{\min} + 1 \geq k_0} \frac{1}{p_{\max} - p_{\min} + 1} \left| \sum_{p = p_{\min}}^{p_{\max}} (\zeta_p - E(\eta_p)) \right| < \epsilon,
\]
outside an exceptional set $G_{k_0}$ of measure at most

$$C_2 \sum_{p_{\text{max}} - p_{\text{min}} + 1 \geq k_0} e^{-\frac{\lambda^2_{3p_{\text{min}}}}{2}} + \frac{C_3}{k_0},$$

where $C_3$ is a constant.

A simple calculation shows that

$$p_{\text{max}} = \left\lceil \frac{k + 2}{3} \right\rceil - 1 \quad \text{and} \quad p_{\text{min}} = \left\lfloor \frac{k}{6} \right\rfloor + 1.$$

Hence we obtain

$$\frac{1}{p_{\text{max}} - p_{\text{min}} + 1} \sum_{p_{\text{max}} - p_{\text{min}} + 1} \xi_p > \frac{1}{p_{\text{max}} - p_{\text{min}} + 1} \sum_{p_{\text{min}}} \xi_p > C_4 k > \frac{C_5 \log n}{\log (\frac{k}{n} \log n)}$$

for all $k$ such that $p_{\text{max}} - p_{\text{min}} + 1 \geq k_0$ outside an exceptional set $G_{k_0}$.

Applying $E(\eta_p) > \delta$ and using (2.4), we get

$$N_n > \frac{\sum_{p_{\text{max}} - p_{\text{min}} + 1} \xi_p > (p_{\text{max}} - p_{\text{min}} + 1)(\delta - \epsilon) > C_4 k > \frac{C_5 \log n}{\log (\frac{k}{n} \log n)}}$$

for all $k$ such that $p_{\text{max}} - p_{\text{min}} + 1 \geq k_0$ outside an exceptional set $G_{k_0}$, where $C_4$ and $C_5$ are constants. It can be seen that the set \{k \in N \mid p_{\text{max}} - p_{\text{min}} + 1 \geq k_0\} is contained in the set \{k \in N \mid k \geq 6k_0 - 2\}.

If $n = n_0$ corresponds to $k = 6k_0 - 2$, then all $n > n_0$ will correspond to $k > 6k_0 - 2$. Therefore, we have for all $n > n_0$,

$$N_n > \frac{C \log n}{\log (\frac{k}{n} \log n)},$$

where

$$\Pr(G_{k_0}) < C_2 \sum_{k > 6k_0 - 2} \frac{1}{\lambda_{3p_{\text{min}}}} e^{-\frac{\lambda^2_{3p_{\text{min}}}}{2}} + \frac{C_3}{k_0}$$

$$\leq C_2 \left\{ \frac{1}{\lambda_{3k_0}} e^{-\frac{\lambda^2_{3k_0}}{2}} + 6 \left( \frac{1}{\lambda_{3(k_0 + 1)}} e^{-\frac{\lambda^2_{3(k_0 + 1)}}{2}} + \frac{1}{\lambda_{3(k_0 + 2)}} e^{-\frac{\lambda^2_{3(k_0 + 2)}}{2}} + \ldots \right) \right\} + \frac{C_3}{k_0}$$

$$\leq 6C_2 \sum_{q \geq k_0} \frac{1}{\lambda_{3q}} e^{-\frac{\lambda^2_{3q}}{2}} + \frac{C_3}{k_0} = 6C_2 \sum_{q \geq k_0} \frac{1}{\sqrt{3q \log (3q)}} e^{-\frac{3q(\log (3q))^2}{2}} + \frac{C_3}{k_0}$$
\[ \leq \frac{4}{\sqrt{3}} C_2 \sum_{q \geq 0} \frac{1}{q(q\log q)^2} + \frac{C_3}{k_0} \leq \frac{C_6}{\log k_0} + \frac{C_3}{k_0} \leq \frac{C'}{\log \left\{ \frac{k_n}{n_0 \log k_0} \right\}}, \]

where \( C_6 \) is a constant. This completes the proof of the theorem.

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