Journal of Applied Mathematics and Stochastic Analysis, **14**:2 (2001), 189-194.

EXISTENCE OF PERIODIC SOLUTION FOR FIRST ORDER NONLINEAR NEUTRAL DELAY EQUATIONS

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(Received: March, 1999; Revised October, 1999)

In this paper by using the coincidence degree theory, sufficient conditions are given for the existence of periodic solutions of the first order nonlinear neutral delay differential equation.

Key words: Nonlinear, Neutral Delay Equation, Periodic Solution.
AMS subject classifications: 34K15, 34C25.

1. Introduction

In [3], Kuang and Feldstein proposed to study the existence of a periodic solution for the first order periodic neutral delay equation. In particular, Gopalsamy, He and Wen [2] studied the existence of periodic solutions of the first order neutral delay logistic equation. In this paper, we discuss the following nonlinear neutral delay equation:

$$[x(t) + cx(t-\tau)]' + g(t, x(t-\sigma)) = p(t), \tag{1}$$

where τ, σ and c are constants, and $\tau \geq 0$, $\sigma \geq 0$, $|c| < 1$; $g \in C(R^2, R)$, $g(t,x)$ is a function with period $T(>0)$ for t, and $g(t,x)$ is nondecreasing for x in $[0, +\infty)$; $p \in C(R,R)$, $p(t,T) = p(t)$ for $t \in R$ and $\int_0^T p(t)dt = 0$. Using coincidence degree theory developed by Mawhin [1], we establish a theorem of the existence of periodic solutions with period T of Equation (1).

2. Main Result

The following result provides sufficient conditions for the existence of periodic solution of Equation (1).

Theorem: *Assume that there exist constants $D > 0$ and $M > 0$ such that*

$$xg(t,x) > 0 \text{ for } t \in R \text{ and } |x| \geq D, \tag{2}$$

$$g(t,x) \geq -M \text{ for } t \in R \text{ and } x \leq -D, \tag{3}$$

and

$$|g(t,x)| \leq g(t, |x|) \text{ for } (t,x) \in R^2. \tag{4}$$

Then there exists a periodic solution with period T of Equation (1).

In order to prove the above theorem, we introduce the following preliminaries.

Let X and Z be two Banach spaces. Consider an operator equation,

$$Lx = \lambda Nx,$$

where $L: \mathrm{Dom}L \cap X \to Z$ is a linear operator and $\lambda \in [0,1]$ is a parameter. Let P and Q denote two projectors,

$$P: \mathrm{Dom}L \cap X \to \mathrm{Ker}L \text{ and } Q: Z \to Z/\mathrm{Im}L.$$

We will use the following result of Mawhin [1].

Lemma 1: *Let X and Y be two Banach spaces and L be a Fredholm mapping with index null. Assume that Ω is open bounded in X and $N: \bar{\Omega} \to Z$ is L-compact on $\bar{\Omega}$. Furthermore, suppose that*

(*a*) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \in \mathrm{Dom}L$,*

$$Lx \neq \lambda Nx;$$

(*b*) *for each $x \in \partial\Omega \cap \mathrm{Ker}L$,*

$$QNx \neq 0,$$

and

$$\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0,$$

then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{Dom}L$.

To prove Lemma 2, we make the following preparations. Set

$$X: = \{x \in C^1(R,R) \mid x(t+T) = x(t)\}$$

and define the norm on X as $\|x\| = \max_{t \in [0,T]} \{|x(t)|, |x'(t)|\}$. Similarly, set

$$Z: = \{z \in C(R,R) \mid z(t+T) = z(t)\}$$

and define the norm on Z as $\|z\|_0 = \max_{t \in [0,T]} |z(t)|$. Then both $(X, \|\cdot\|)$ and $(Z, \|\cdot\|_0)$ are Banach spaces. Define respectively the operators L and N as

$$L: X \to Z, \quad x(t) \mapsto x'(t), \tag{5}$$

and

$$N: X \to Z,\ x(t) \mapsto -cx'(t-\tau) - g(t, x(t-\sigma)) + p(t). \tag{6}$$

We know that $\mathrm{Ker}L = R$. Define, respectively, the projective operators P and Q as

$$P: X \to \mathrm{Ker}L,\ \ x \mapsto Px = \frac{1}{T}\int_0^T x(t)dt, \tag{7}$$

and

$$Q: Z \to Z/\mathrm{Im}L,\ \ z \mapsto Qz = \frac{1}{T}\int_0^T z(t)dt. \tag{8}$$

Hence, we have $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q$. Consider the equation

$$x'(t) + \lambda c x'(t-\tau) + \lambda g(t, x(t-\sigma)) = \lambda p(t), \tag{9}$$

where $\lambda \in (0,1)$ is a parameter.

Lemma 2: *Suppose that conditions* (2)-(4) *are satisfied. If $x(t)$ is any periodic solution with period T of Equation* (9), *then there exist positive constants $D_j (j = 0, 1)$ independent of λ and such that*

$$|\, x(t) \,| \le D_0 \text{ and } |\, x'(t) \,| \le D_1,\ \ t \in [0, T]. \tag{10}$$

Proof: Suppose that $x(t)$ is a periodic solution with period T of Equation (9). By integrating (9) from 0 to T, we find

$$\int_0^T g(t, x(t-\sigma))dt = 0. \tag{11}$$

Set

$$E_1 = \{t \in [0,T] \,|\, x(t-\sigma) > D\},\ \ E_2 = [0,T] \backslash E_1.$$

Since $g \in C(R^2, R)$ and $g(t,x)$ is a function with period T for t, we know that

$$\sup_{(t,x) \in R \times [-D,D]} |\, g(t,x) \,| = \max_{(t,x) \in [0,T] \times [-D,D]} |\, g(t,x) \,| < \infty.$$

From (2) and (3) we see that

$$\int_{E_2} |\, g(t, x(t-\sigma)) \,|\, dt \le T \max\{M, \sup_{(t,x) \in R \times [-D,D]} |\, g(t,x) \,|\}. \tag{12}$$

Using (2) and (11), we have

$$\int_{E_1} |\, g(t, x(t-\sigma)) \,|\, dt = \int_{E_1} g(t, x(t-\sigma))dt$$

$$= -\int_{E_2} g(t, x(t-\sigma))dt \tag{13}$$

$$\leq \int_{E_2} |\, g(t, x(t-\sigma))\,|\, dt.$$

By (12) and (13), we have

$$\int_0^T |\, g(t, x(t-\sigma))\,|\, dt \leq 2T \max\{M, \sup_{(t,x) \in R \times [-D, D]} |\, g(t,x)\,|\,\}.$$

Thus

$$\int_0^T |\, g(t, x(t-\sigma))\,|\, dt \leq K_0, \tag{14}$$

where K_0 is a positive constant. Since $x'(t)$ is a periodic function with period T, it follows from (9) that

$$\begin{aligned} \int_0^T |\, x'(t)\,|\, dt &\leq \lambda\,|\,c\,|\int_0^T |\, x'(t-\sigma)\,|\, dt + \lambda \int_0^T |\, g(t, x(t-\sigma))\,|\, dt + \lambda \int_0^T |\, p(t)\,|\, dt \\ &\leq |\,c\,| \int_0^T |\, x'(t)\,|\, dt + \int_0^T |\, g(t, x(t-\sigma))\,|\, dt + T \max_{t \in [0,T]} |\, p(t)\,|\,. \end{aligned} \tag{15}$$

From (14) and (15) we see that

$$\int_0^T |\, x'(t)\,|\, dt \leq |\,c\,| \int_0^T |\, x'(t)\,|\, dt + K_1, \tag{16}$$

where $K_1 = K_0 + T \max_{t \in [0,T]} |\, p(t)\,|$. It follows from (16) that

$$\int_0^T |\, x'(t)\,|\, dt \leq K_2, \tag{17}$$

where $K_2 = K_1/(1 - |\,c\,|)$. By (2) and (11), there exists $t_1 \in [0, T]$ such that $|\, x(t_1 - \sigma)\,| < D$. Taking $t_1 - \sigma = nT + t_2$, where n is an integer and $t_2 \in [0, T]$, we have

$$|\, x(t_2)\,| < D. \tag{18}$$

Then, by (17) and (18), we conclude that for any $t \in [0, T]$,

$$|x(t)| = |x(t_2) + \int_{t_2}^{t} x'(s)ds| \leq |x(t_2)| + \int_0^T |x'(t)|dt$$

$$\leq |x(t_2)| + K_2 < D_0, \tag{19}$$

where $D_0 = D + K_2$. Since $x(t)$ is a periodic function with period T, from (19) we see that $|x(t-\sigma)| < D_0$ for $t \in [0,T]$. Note that $g(t,x)$ is nondecreasing for x in $[0, +\infty)$. Hence we have for any $t \in [0,T]$,

$$|g(t, x(t-\sigma))| \leq g(t, |x(t-\sigma)|) \leq g(t, D_0). \tag{20}$$

Note that if $g(t, D_0)$ is a periodic continuous function, then there exists a positive constant K_3, for any $t \in [0,T]$, such that

$$g(t, D_0) \leq |g(t, D_0)| \leq K_3. \tag{21}$$

From (9), (20), (21) and note that $x'(t)$ is a periodic function with period T, we conclude that for any $t \in [0,T]$

$$\begin{aligned} |x'(t)| &\leq \lambda |c| \, |x'(t-\tau)| + \lambda |g(t, x(t-\sigma))| + \lambda |p(t)| \\ &\leq |c| \max_{t \in [0,T]} |x'(t)| + g(t, x(t-\sigma)) + \max_{t \in [0,T]} |p(t)| \\ &\leq |c| \max_{t \in [0,T]} |x'(t)| + g(t, D_0) + \max_{t \in [0,T]} |p(t)| \\ &\leq |c| \max_{t \in [0,T]} |x'(t)| + K_3 + \max_{t \in [0,T]} |p(t)|. \end{aligned} \tag{22}$$

Letting $K_4 = K_3 + \max_{t \in [0,T]} |p(t)|$, for any $t \in [0,T]$ we have

$$|x'(t)| \leq |c| \max_{t \in [0,T]} |x'(t)| + K_4. \tag{23}$$

By (23) we obtain

$$\max_{t \in [0,T]} |x'(t)| \leq |c| \max_{t \in [0,T]} |x'(t)| + K_4.$$

Thus,

$$\max_{t \in [0,T]} |x'(t)| \leq D_1, \tag{24}$$

where $D_1 = K_4/(1 - |c|)$. The proof of Lemma 2 is complete.

Proof of the Theorem: Suppose that $x(t)$ is any periodic solution with period T of Equation (9). By Lemma 2, there exist positive constants D_j $(j = 0, 1)$, which are independent of λ, such that

$$|x(t)| \leq D_0 \text{ and } |x'(t)| \leq D_1, \; t \in [0,T].$$

Let $D_2 = \max\{D_0, D_1, D\} + 1$, and

$$\Omega: = \{x \in X \mid \| x \| < D_2\}.$$

In view of (2), we see that

$$-\frac{1}{T}\int_0^T g(t, -D_2)dt > 0 \text{ and } -\frac{1}{T}\int_0^T g(t, D_2)dt < 0. \tag{25}$$

By (5)-(7) and (8), we know that L is the Fredholm operator with index null and N is L-compact on $\bar{\Omega}$ (see [1]). In terms of evaluation of a bound of periodic solutions in Lemma 2, we know that for any $x \in \partial\Omega \cap \mathrm{Dom}L$ and $\lambda \in (0,1)$, $Lx \neq \lambda Nx$. Since for any $x \in \partial\Omega \cap \mathrm{Ker}L$, $x = D_2(> D)$ or $x = -D_2$, in view of (25) and $\int_0^T p(t)dt = 0$, we have

$$QNx = \frac{1}{T}\int_0^T [-cx'(t-\tau) - g(t, x(t-\sigma)) + p(t)]dt$$

$$= -\frac{1}{T}\int_0^T g(t, \pm D_2)dt \neq 0,$$

which shows that

$$\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0.$$

By Lemma 1, there exists a periodic solution with period T of Equation (1). The proof is complete.

Example: Consider the equation

$$[x(t) - \frac{1}{3}x(t-\pi)]' + e^{\sin t}x(t-\pi)e^{x(t-\pi)} = \frac{4}{3}\cos t - \sin t. \tag{26}$$

It is easy to verify that for Equation 926), all the conditions of the theorem are satisfied with $D > 0$ and $M = 3$. Thus Equation (26) has a periodic solution with period 2π. We see that $x(t) = \sin t$ is such a periodic solution of Equation (26).

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