MIXED PROBLEM WITH A WEIGHTED INTEGRAL CONDITION FOR A PARABOLIC EQUATION WITH THE BESSEL OPERATOR

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In this paper, we prove the existence, uniqueness and continuous dependence on the data of a solution of a mixed problem with a weighted integral condition for a parabolic equation with the Bessel operator. The proof uses a functional analysis method based on an a priori estimate and on the density of the range of the operator generated by the considered problem.

Key words: Singular Parabolic Equations, Weighted Integral Condition, A Priori Estimate.

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1. Introduction

We consider the problem of finding a solution $v = v(r, t)$ of the problem:

$$\mathcal{L}v: = v_t - \frac{1}{r}(rv_r)_r = f(r, t), \quad r \in (0, R), \quad t \in (0, T). \quad (1.1)$$

where R, T are given positive constants.

We adjoin to equation (1.1) the initial condition

$$\ell v: = v(r, 0) = \Phi(r), \quad r \in (0, R), \quad (1.2)$$

the Neumann condition

$$v_r(R, t) = \mu(t), \quad t \in (0, T), \quad (1.3)$$

and the weighted integral condition

$$\int_0^R r v(r, t) dr = m(t), \quad (1.4)$$

where f, Φ, μ and m are given functions.

Problem (1.1)-(1.4) can be viewed as a nonlocal problem for parabolic equation with the Bessel operator. The analogous problem which combines a homogeneous Dirichlet condition and the linear constraint

$$\int_0^R v(r, t) dr = 0,$$

has been studied by Benouar-Yurchuk [1]. In the case when in Equation (1.1) instead of the Bessel operator, we have the operator $\frac{\partial}{\partial r} \left(a(r, t) \frac{\partial}{\partial r} \right)$, with Neumann and the linear constraint defined above, we refer the reader to Cannon [5], Cannon-Esteva-van der Hoek [6], Cannon-van der Hoek [7], Muravey-Filinovskii [8], Shi [9], and Bouziani [2]. For other problems with integral conditions, see Bouziani [3, 4] and references therein.

2. Preliminaries

We first transform problem (1.1)-(1.4) with inhomogeneous boundary conditions (1.3) and (1.4) to an equivalent problem with homogeneous conditions by introducing a new unknown function u defined as follows

$$u(r, t) = v(r, t) - U(r, t),$$

where

$$U(r, t) := \mu(t) \cdot r + \frac{12(r-R)^2}{R^4} \left(m(t) - \frac{R^3}{3} \mu(t) \right).$$

Then problem (1.1)-(1.4) can be formulated as follows:

$$\mathcal{L}u = f(r, t) - \mathcal{L}U = f(r, t), \quad (2.1)$$

$$\ell u = u(r, 0) = \Phi(r) - \ell U = \varphi(r), \quad (2.2)$$

$$u_r(R, t) = 0, \quad (2.3)$$

$$\int_0^R r u(r, t) dr = 0. \quad (2.4)$$

Instead of searching for the function v , we search for the function u . So the solution of problem (1.1)-(1.4) will be given by $v(r, t) = u(r, t) + U(r, t)$.

For the investigation of the problem (2.1)-(2.4), we use some function spaces. Let

$L_r^2(Q)$ and $L_{\sqrt{r}}^2(Q)$ be weighted spaces of square integrable functions with finite norms

$$\| u \|_{L_r^2(Q)}^2 = \| ru \|_{L^2(Q)}^2,$$

$$\| u \|_{L_{\sqrt{r}}^2(Q)}^2 = \| \sqrt{r}u \|_{L^2(Q)}^2,$$

respectively, and let $V_r^{1,0}(Q)$ be the subspace of $L_r^2(Q)$ with the finite norm

$$\|u\|_{V_r^{1,0}(Q)}^2 = \|u\|_{L_r^2(Q)}^2 + \|u_r\|_{L_r^2(Q)}^2.$$

We also use weighted spaces on the interval $(0, R)$ such as $L_r^2(0, R)$, $L_{\sqrt{r}}^2(0, R)$ and $V_r^1(0, R)$, whose definitions are analogous to those for functions defined on Q . For example, $V_r^1(0, R)$ is the subspace of $L_r^2(0, R)$ with the finite norm

$$\|u\|_{V_r^1(0,R)}^2 = \|u\|_{L_r^2(0,R)}^2 + \|u_r\|_{L_r^2(0,R)}^2.$$

Problem (2.1)-(2.4) is equivalent to the operator equation

$$Lu = \mathcal{F},$$

where $L = (\mathcal{L}, \ell)$ and $\mathcal{F} = (f, \varphi)$. The domain $D(L)$ of operator L is the set of all functions $u \in L_r^2(Q)$ satisfying conditions (2.3) and (2.4) for which $u_t, u_r, u_{tr}, u_{rr} \in L_r^2(Q)$. The operator L is viewed as being from E to F , where E is the Banach space consisting of functions $u \in L_r^2(Q)$, satisfying (2.3) and (2.4), with finite norm

$$\|u\|_E^2 = \|u_t\|_{L_r^2(Q)}^2 + \|(ru_r)_r\|_{L^2(Q)}^2 + \sup_{0 \leq \tau \leq T} \|u(r, \tau)\|_{V_r^1(0,R)}^2,$$

and F is the Hilbert space of vector valued functions $\mathcal{F} = (f, \varphi) \in L_r^2(Q) \times V_r^1(0, R)$ with finite norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L_r^2(Q)}^2 + \|\varphi\|_{V_r^1(0,R)}^2.$$

We assume that φ satisfies the compatibility conditions of the form (2.3) and (2.4):

$$\varphi_r(R) = 0 \text{ and } \int_0^R r\varphi(r)dr = 0.$$

3. Two-Sided A Priori Estimates

Theorem 1: For any function $u \in D(L)$, we have a priori estimate

$$\|Lu\|_F \leq 2 \|u\|_E. \quad (3.1)$$

Proof: From equation (2.1), it follows that

$$\|f\|_{L_r^2(Q)}^2 \leq 2 \|u_t\|_{L_r^2(Q)}^2 + \|(ru_r)_r\|_{L^2(Q)}^2. \quad (3.2)$$

The initial condition (2.2) implies that

$$\|\varphi\|_{V_r^1(0,R)}^2 \leq \sup_{0 \leq \tau \leq T} \|u(r, \tau)\|_{V_r^1(0,R)}^2. \quad (3.3)$$

Addition of (3.2) and (3.3) yields the desired inequality (3.1), and this proves Theorem 1.

Theorem 2: For any function $u \in D(L)$, we have

$$\| u \|_E \leq c \| Lu \|_F, \quad (3.4)$$

where c is a positive constant independent of the solution u .

Proof: Taking the scalar product in $L^2(Q^\tau)$ of equation (2.1) and the operator

$$Mu = r\mathcal{F}_r(\rho u_t) + r^2 u_t - ru_r,$$

where

$$\mathcal{F}_r(\rho u) = \int_0^r \rho u(\rho, t) d\rho,$$

and

$$Q^\tau = (0, R) \times (0, \tau),$$

with $0 \leq \tau \leq T$, we get

$$\begin{aligned} & \int_{Q^\tau} ru_t \mathcal{F}_r(\rho u_t) dr dt + \int_{Q^\tau} r^2 u_t^2 dr dt \\ & - \int_{Q^\tau} ru_r u_t dr dt - \int_{Q^\tau} (ru_r)_r \mathcal{F}_r(\rho u_t) dr dt \end{aligned} \quad (3.5)$$

$$- \int_{Q^\tau} ru_t (ru_r)_r dr dt + \int_{Q^\tau} u_r (ru_r)_r dr dt = (f, Mu)_{L_t^2(Q^\tau)}.$$

Using condition (2.2)-(2.4) and integrating by parts, we obtain the following relations:

$$\int_{Q^\tau} ru_t \mathcal{F}_r(\rho u_t) dr dt = \frac{1}{2} \int_{Q^\tau} (\mathcal{F}_r(\rho u_t))^2 \Big|_{r=0}^{r=R} dt = 0, \quad (3.6)$$

$$\begin{aligned} - \int_{Q^\tau} (ru_r)_r \mathcal{F}_r(\rho u_t) dr dt &= - \int_0^\tau \left(ru_r \mathcal{F}_r(\rho u_t) \right) \Big|_{r=0}^{r=R} dt + \int_{Q^\tau} r^2 u_r u_t dr dt \\ &= \int_{Q^\tau} r^2 u_r u_t dr dt, \end{aligned} \quad (3.7)$$

$$\begin{aligned}
-\int_{Q^\tau} ru_t(ru_r)_r dr dt &= -\int_0^\tau r^2 u_r u_t \Big|_{r=0}^{r=R} dt + \int_{Q^\tau} r^2 u_r u_{tr} dr dt + \int_{Q^\tau} ru_r u_t dr dt \\
&= \frac{1}{2} \int_0^R r^2 (u_r(r, \tau))^2 dr - \frac{1}{2} \int_0^R r^2 \varphi_r^2 dr + \int_{Q^\tau} ru_r u_t dr dt,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\int_{Q^\tau} u_r(ru_r)_r dr dt &= \frac{1}{2} \int_0^\tau ru_r^2 \Big|_{r=0}^{r=R} dt + \frac{1}{2} \int_{Q^\tau} u_r^2 dr dt \\
&= \frac{1}{2} \int_{Q^\tau} u_r^2 dr dt.
\end{aligned} \tag{3.9}$$

Substitution of relations (3.6)-(3.9) into (3.5) gives

$$\begin{aligned}
&\int_{Q^\tau} r^2 u_t^2 dr dt + \frac{1}{2} \int_0^R r^2 (u_r(r, \tau))^2 dr + \frac{1}{2} \int_{Q^\tau} u_r^2 dr dt \\
&= \int_{Q^\tau} rf \mathcal{F}_r(\rho u_t) dr dt + \int_{Q^\tau} r^2 f u_t dr dt - \int_{Q^\tau} rf u_r dr dt \\
&\quad - \int_{Q^\tau} r^2 u_r u_t dr dt + \frac{1}{2} \int_0^R r^2 \varphi_r^2 dr.
\end{aligned} \tag{3.10}$$

We estimate the first four terms on the right-hand side of (3.10). By applying Cauchy's inequality we have

$$\int_{Q^\tau} rf \mathcal{F}_r(\rho u_t) dr dt \leq \frac{1}{8} \|u_t\|_{L_r^2(Q^\tau)}^2 + R^2 \|f\|_{L_r^2(Q^\tau)}^2, \tag{3.11}$$

$$\int_{Q^\tau} r^2 u_t \cdot f dr dt \leq \frac{1}{8} \|u_t\|_{L_r^2(Q^\tau)}^2 + 2 \|f\|_{L_r^2(Q^\tau)}^2, \tag{3.12}$$

$$-\int_{Q^r} r u_r \cdot f dr dt \leq \frac{1}{2} \|u_r\|_{L^2(Q^r)}^2 + \frac{1}{2} \|f\|_{L_r^2(Q^r)}^2, \quad (3.13)$$

$$= \int_{Q^r} r^2 u_r \cdot u_t dr dt \leq \frac{1}{4} \|u_t\|_{L_r^2(Q^r)}^2 + \|u_r\|_{L_r^2(Q^r)}^2. \quad (3.14)$$

Therefore, from equality (3.10), by virtue of inequalities (3.11)-(3.14), we obtain

$$\begin{aligned} & \frac{1}{2} \|u_t\|_{L_r^2(Q^r)}^2 + \frac{1}{2} \|u_r(r, \tau)\|_{L_r^2(0, R)}^2 \\ & \leq (R^2 + \frac{5}{2}) \|f\|_{L_r^2(Q^r)}^2 + \frac{1}{2} \|\varphi_r\|_{L_r^2(0, R)}^2 + \|u\|_r \|u\|_{L_r^2(Q^r)}^2. \end{aligned} \quad (3.15)$$

From equation (2.1) we have

$$\frac{1}{16} \|(ru_r)_r\|_{L^2(Q^r)}^2 \leq \frac{1}{8} \|f\|_{L_r^2(Q^r)}^2 + \frac{1}{8} \|u_t\|_{L_r^2(Q^r)}^2. \quad (3.16)$$

We have the elementary inequality

$$\frac{3}{16} \|u(r, \tau)\|_{L_r^2(0, R)}^2 \leq \frac{3}{16} \|\varphi\|_{L_r^2(0, R)}^2 + \frac{3}{16} \|u\|_{L_r^2(Q^r)}^2 + \frac{3}{16} \|u_t\|_{L_r^2(Q^r)}^2. \quad (3.17)$$

Adding (3.15)-(3.17) yields the inequality

$$\begin{aligned} & \|u_t\|_{L_r^2(Q^r)}^2 + \|(ru_r)_r\|_{L^2(Q^r)}^2 + \|u(r, \tau)\|_{V_r^2(0, R)}^2 \\ & \leq (42 + 16R^2) \cdot (\|f\|_{L_r^2(Q^r)}^2 + \|\varphi\|_{V_r^1(0, R)}^2) \\ & \quad + (42 + 16R^2) \|u\|_{V_r^1(Q^r)}^2. \end{aligned} \quad (3.18)$$

Applying Lemma 3.1 from [2] to (3.18), we get

$$\begin{aligned} & \|u_t\|_{L_r^2(Q^r)}^2 + \|(ru_r)_r\|_{L^2(Q^r)}^2 + \|u(r, \tau)\|_{V_r^1(0, R)}^2 \\ & \leq k_1 e^{k_1 T} (\|f\|_{L_r^2(Q)}^2 + \|\varphi\|_{V_r^1(0, R)}^2), \end{aligned} \quad (3.19)$$

where $k_1 = 42 + 16R^2$.

Since the right-hand side of (3.19) is independent of τ , in the left-hand side, we take the least upper bound with respect to τ from 0 to T . We thus obtain inequality (3.4), with $c = k_1^{1/2} e^{k_1 T/2}$.

4. Solvability of the Problem

It follows from inequality (3.1) that the operator $L: E \rightarrow F$ is continuous. From inequality (3.4), it follows that the range $R(L)$ of L is closed in F . Therefore, there exists a continuous inverse operator L^{-1} yielding the solution. That is, L is a linear homeomorphism from the space E on the closed set $R(L) \subset F$. To prove that there exists a unique solution of problem (2.1)-(2.4), we have to show that $R(L) = F$.

Theorem 3: For any $\Phi = (f, \varphi) \in F$, there is a unique solution $u = L^{-1}\Phi$ of the problem (2.1)-(2.4), satisfying the estimate

$$\|u\|_E \leq c \|\Phi\|_F,$$

where c is a positive constant independent of the solution u .

Proof: To establish the proof of this theorem, we need the following:

Proposition: If, for all u in the set $D_0(L) = \{u \in D(L) : \ell u = 0\}$ and for some function $\omega \in L_r^2(Q)$, we have

$$(\mathcal{L}u, \omega)_{L_r^2(Q)} = 0, \quad (4.1)$$

then $\omega \equiv 0$ almost everywhere on Q .

Proof of Proposition: Using the fact that (4.1) holds for any function $u \in D_0(L)$, we can express it in a particular form. Let us define the function ϑ by the relation

$$\vartheta(r, t) = \int_t^T \omega(r, s) ds,$$

and let u_t be a solution of the equation

$$\frac{1}{r}(u_t - \mathcal{F}_r^2(\rho u_t)) = \vartheta(r, t), \quad (4.2)$$

where

$$\mathcal{F}_r^2(\rho u_t) = \int_0^r \int_0^\rho \eta u_t(\eta, t) d\eta d\rho.$$

Let the function u be given by

$$u = \begin{cases} \int_v^t u_\xi d\xi, & 0 \leq t \leq v, \\ 0, & v \leq t \leq T. \end{cases} \quad (4.3)$$

We now have

$$\omega(r, t) = \frac{1}{4}(\mathcal{F}_r^2(\rho u_t) - u_t)_t. \quad (4.4)$$

We now have

$$\omega(r, t) = \frac{1}{r}(\mathcal{F}_r^2(\rho u_t) - u_t)_t. \quad (4.4)$$

Relations (4.2) and (4.3) imply that u is in $D_0(L)$.

Lemma: The function u defined by (4.2) and (4.3) had derivatives with respect to t up to the second order belonging to the space $L^2(Q_v)$, where $Q_v = (0, R) \times (v, T)$.

Proof: The proof of this lemma has a purely technical character. We shall not reproduce it here, but instead refer the reader to [2].

Now replacing ω in (4.1) by its representation (4.4), we have

$$\int_Q ru_t \mathcal{F}_r^2(\rho u_{tt}) dr dt - \int_Q u_r \mathcal{F}_r^2(\rho u_{tt}) dr dt + \int_Q u_r u_{tt} dr dt \quad (4.5)$$

$$- \int_Q ru_{rr} \mathcal{F}_r^2(\rho u_{tt}) dr dt - \int_Q ru_t u_{tt} dr dt + \int_Q r_{rr} u_{tt} dr dt = 0.$$

Using conditions (2.3) and (2.4), and the special form of u given by (4.2) and (4.3), integration by parts of the first and last three terms on the right-hand side of (4.5) give

$$\int_Q ru_t \mathcal{F}_r^2(\rho u_{tt}) dr dt = \frac{1}{2} \| \mathcal{F}_r(\rho u_t(r, v)) \|_{L^2(0, R)}^2, \quad (4.6)$$

$$- \int_Q ru_{rr} \mathcal{F}_r^2(\rho u_{tt}) dr dt = - \int_{Q_v} ru_{tr} \mathcal{F}_r(\rho u_t) dr dt + \int_{Q_v} u_r \mathcal{F}_r^2(\rho u_{tt}) dr dt, \quad (4.7)$$

$$- \int_Q ru_t u_{tt} dr dt = \frac{1}{2} \| u(r, v) \|_{L^2_{\sqrt{r}}(0, R)}^2, \quad (4.8)$$

$$\int_Q ru_{rr} u_{tt} dr dt = \| u_{tr} \|_{L^2_{\sqrt{r}}(Q_v)}^2 = \int_{Q_v} u_r u_{tt} dr dt. \quad (4.9)$$

Substituting equalities (4.6)-(4.9) into (4.5), we obtain

$$\begin{aligned} & \| u_{tr} \|_{L^2_{\sqrt{r}}(Q_v)}^2 + \frac{1}{2} \| \mathcal{F}_r(\rho u_t(\cdot, v)) \|_{L^2(0, R)}^2 + \frac{1}{2} \| u_t(\cdot, v) \|_{L^2_{\sqrt{r}}(0, R)}^2 \\ &= \int_{Q_v} ru_{tr} \mathcal{F}_r(\rho u_t) dr dt. \end{aligned} \quad (4.10)$$

Further, by virtue of an elementary inequality, we estimate the right-hand side of (4.10) to get

$$\begin{aligned} & \| u_{tr} \|_{L^2_{\sqrt{r}}(Q_v)}^2 + \| \mathcal{F}_r(\rho u_t(\cdot, v)) \|_{L^2(0, R)}^2 + \| u_t(\cdot, v) \|_{L^2_{\sqrt{r}}(0, R)}^2 \\ &\leq k_2 (\| \mathcal{F}_r(\rho u_t) \|_{L^2(Q_v)}^2 + \| u_t \|_{L^2_{\sqrt{r}}(Q_v)}^2), \end{aligned} \quad (4.11)$$

where $k_2 = \max(R, 1)$. If we put

$$y(v) = \| \mathcal{F}_r(\rho u_t) \|_{L^2(Q_v)}^2 + \| u_t \|_{L^2_{\sqrt{r}}(Q_v)}^2,$$

then from (4.11) we have

$$-\frac{dy}{dv} \leq k_2 y(v),$$

and consequently,

$$-\frac{d}{dv}(y(v)e^{k_2 v}) \leq 0. \quad (4.12)$$

Integrating (4.12) over the interval (v, T) and taking into account that $y(T) = 0$, we obtain

$$y(v)e^{k_2 v} \leq 0. \quad (4.13)$$

It follows from (4.13), that $\omega \equiv 0$ almost everywhere in Q_v . The length v does not depend on the choice of the origin. Proceeding in this way step by step, we prove that $\omega \equiv 0$ almost everywhere in Q . Thus, the proposition is proved.

Now, we return to the proof of Theorem 3. It is sufficient to prove that the set $R(L)$ is dense in F . Let $\Psi = (\omega, \omega_0) \in R(L)^\perp$ such that

$$(\mathcal{L}u, \omega)_{L^2_r(Q)} + (\ell u, \omega_0)_{V^1_r(0, R)} = 0. \quad (4.14)$$

Putting $u \in D_0(L)$ in the equation (4.14), we get

$$(\mathcal{L}u, \omega)_{L^2_r(Q)} = 0, \quad \forall u \in D_0(L).$$

By virtue of the proposition, we deduce that $\omega \equiv 0$. Thus (4.14) becomes

$$(\ell u, \omega_0)_{V^1_r(0, R)} = 0. \quad (4.15)$$

Since the range $R(\ell)$ of the operator ℓ is everywhere dense in $V^1_r(0, R)$, (4.15) implies that $\omega_0 \equiv 0$. Hence $\Psi \equiv 0$. This completes the proof of Theorem 3.

5. Conclusion

In summary, we have established the existence, uniqueness and continuous dependence on given data for solutions to a mixed problem for a parabolic equation with the Bessel operator which combines inhomogeneous Neumann condition and integral conditions. More precisely, we have constructed a sufficiently smooth function satisfying these conditions. Then we are lead to study an equivalent problem with homogeneous boundary conditions. Thus, we have established two sided a priori estimates for the operator $L: E \rightarrow F$ generated by the considered problem. Then we concluded that the operator L realizes a linear homeomorphism of the space E on the closed set $R(L) \subset F$. To prove that the studied problem possesses a unique solution, we proved that $R(L)$ is dense in F .

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