HARMONIC CLOSE-TO-CONVEX MAPPINGS

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Sufficient coefficient conditions for complex functions to be close-to-convex harmonic or convex harmonic are given. Construction of close-to-convex harmonic functions is also studied by looking at transforms of convex analytic functions. Finally, a convolution property for harmonic functions is discussed.

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1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied by differential geometers such as Choquet [2], Kneser [7], Lewy [8], and Rado [9]. Recent interest in harmonic complex functions has been triggered by geometric function theorists Clunie and Sheil-Small [3].

A continuous function $f = u + iv$ is a complex-valued harmonic functions in a domain $D \subset C$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain, we can write

$$f = h + \overline{g},$$

where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient conditions (see [3] or [8]) for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$.

Denote by $S_H$ the class of functions $f$ of the form (1) that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z: |z| < 1\}$ for which $f(0) = f_{\alpha}(0) - 1 = 0$. Thus we may write

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1Dedicated to KSU Professor Richard S. Varga on his seventieth birthday.
h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \quad (2)

Note that $S_H$ reduces to $S$, the class of normalized univalent analytic functions if the coanalytic part of $f$ is zero. Since $h'(0) = 1 > |g'(0)| = |b_1|$ for $f \in S_H$, the function \((f - b_1 f) / (1 - b_1^2)\) is also in $S_H$. Therefore, we may sometimes restrict ourselves to $S_H^u$, the subclass of $S_H$ for which $b_1 = f_2(0) = 0$. In [3], it was shown that $S_H$ is normal and $S_H^u$ is compact with respect to the topology of locally uniform convergence. Some coefficient bounds for convex and starlike harmonic functions have recently been obtained by Avci and Zlotkiewicz [1], Jahangiri [5, 6], and Silverman [14].

In this paper, we give sufficient conditions for functions in $S_H$ to be close-to-convex harmonic or convex harmonic. We also construct close-to-convex harmonic functions by looking at transforms of convex analytic functions. Finally, we discuss a convolution property for harmonic functions.

In the sequel, unless otherwise stated, we will assume that $f$ is of the form (1) with $h$ and $g$ of the form (2).

2. Convex and Close-to-Convex Mappings

Let $K$, $K_H$, and $K_H^u$ denote the respective subclasses of $S$, $S_H$ and $S_H^u$ where the images of $f(\Delta)$ are convex. Similarly, $C$, $C_H$, and $C_H^u$ denote the subclass of $S$, $S_H$ and $S_H^u$ where the images of $f(\Delta)$ are close-to-convex. Recall that a domain $D$ is convex if the linear segment joining any two points of $D$ lies entirely in $D$. A domain $D$ is called close-to-convex if the complement of $D$ can be written as a union of non-crossing half-lines. For other equivalent criteria, see [4].

Clunie and Sheil-Small [3] proved the following results.

**Theorem A:** If $h$, $g$ are analytic in $\Delta$ with $|h'(0)| > |g'(0)|$ and $h + eg$ is close-to-convex for each $\epsilon$, $|\epsilon| = 1$, then $f = h + \overline{g}$ is harmonic close-to-convex.

**Theorem B:** If $f = h + \overline{g}$ is locally univalent in $\Delta$ and $h + eg$ is convex for some $\epsilon$, $|\epsilon| \leq 1$, then $f$ is univalent close-to-convex.

A domain $D$ is called convex in the direction $\phi$ ($0 \leq \phi < \pi$) if every line parallel to the line through 0 and $e^{i\phi}$ has a connected intersection with $D$. Such a domain is close-to-convex. The convex domains are those convex in every direction. We will also make use of the following result, which may be found in [3].

**Theorem C:** A function $f = h + \overline{g}$ is harmonic convex if and only if the analytic functions $h(z) - e^{i\phi}g(z)$, $0 \leq \phi < 2\pi$, are convex in the direction $\phi/2$ and $f$ is suitably normalized.

The harmonic Koebe function $k_0 = h + \overline{g} \in S_H^u$ is defined by $h(z) - g(z) = z/(1 - z)^2$, $g(z) = zh(z)$, which leads to

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{3}z^3}{(1 - z)^2}, \quad g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{3}z^3}{(1 - z)^2}.$$}

The function $k_0$ maps $\Delta$ onto the complex plane minus the real slit from $-1/6$ to $-\infty$. The coefficients of $k_0$ are $a_n = (2n + 1)(n + 1)/6$ and $b_n = (2n - 1)(n - 1)/6$. These coefficient bounds are known to be extremal for the subclass of $S_H^u$ consisting of typically real functions (e.g., see [3]) and functions that are either starlike or convex in one direction (e.g., see [12]). It is not known if the coefficients of $k_0$ are extremal for all of $S_H^u$.

Necessary coefficient conditions were found in [3] for functions to be in $C_H$ and $K_H$. We now give some sufficient condition for functions to be in these classes. But first we need the following results. See, for example, [13].
Lemma 1: If \( q(z) = z + \sum_{n=2}^{\infty} c_n z^n \) is analytic in \( \Delta \), then \( q \) maps onto a starlike domain if \( \sum_{n=2}^{\infty} |c_n| \leq 1 \) and onto a convex domains if \( \sum_{n=2}^{\infty} n^2 |c_n| \leq 1 \).

3. Main Results

Theorem 1: If \( f = h + \bar{g} \) with
\[
\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq 1,
\] then \( f \in \mathcal{C}_\mathcal{H} \). The result is sharp.

Proof: In view of Theorem A, we need only prove that \( h + \epsilon g, |\epsilon| = 1 \), is close-to-convex. It suffices to show that
\[
t(z) = \frac{h + \epsilon g}{1 - \epsilon \bar{h}_1} = z + \sum_{n=2}^{\infty} \left( \frac{a_n + \epsilon b_n}{1 - \epsilon \bar{h}_1} \right) z^n \in \mathcal{C}.
\]
Since
\[
\sum_{n=2}^{\infty} \left| \frac{a_n + \epsilon b_n}{1 - \epsilon \bar{h}_1} \right| \leq \sum_{n=2}^{\infty} \frac{n(|a_n| + |b_n|)}{1 - |\epsilon h_1|} \leq 1
\]
if and only if (3) holds, \( t(z) \) maps \( \Delta \) onto a starlike domain and consequently \( t(z) \in \mathcal{C} \).

To see that the upper bound in (3) cannot be extended to \( 1 + \delta, \delta > 0 \), we note that the function \( f(z) = z + \frac{1 + \delta}{n} z^n \) is not univalent in \( \Delta \).

Theorem 2: If \( f \) is locally univalent with \( \sum_{n=2}^{\infty} n^2 |a_n| \leq 1 \), then \( f \in \mathcal{C}_\mathcal{H} \).

Proof: Take \( \epsilon = 0 \) in Theorem B and apply Lemma 1.

Corollary: If \( \sum_{n=2}^{\infty} n^2 |a_n| \leq 1 \) and \( |g'(z)| \leq 1/2, z \in \Delta \), then \( f \in \mathcal{C}_\mathcal{H} \).

Proof: The function \( f \) is locally univalent if \( |h'(z)| > |g'(z)| \) for \( z \in \Delta \). Since
\[
2 \sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} n^2 |a_n| \leq 1,
\]
we have \( h'(z) > 1 - \sum_{n=2}^{\infty} n |a_n| \geq 1/2 \).

We next give a sufficient coefficient condition for \( f \) to be convex harmonic.

Theorem 3: If
\[
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1,
\]
then \( f \in \mathcal{K}_\mathcal{H} \). The result is sharp.

Proof: By Theorem C, it suffices to show that \( h - e^{i\theta} g \) is convex in \( \Delta \). Set
\[
s(z) = \frac{h - e^{i\theta} g}{1 - e^{i\theta} b_1} = z + \sum_{n=2}^{\infty} \left( \frac{a_n - e^{i\theta} b_n}{1 - e^{i\theta} b_1} \right) z^n.
\]
Since
\[
\sum_{n=2}^{\infty} n^2 \left| \frac{a_n - e^{i\theta} b_n}{1 - e^{i\theta} b_1} \right| \leq \sum_{n=2}^{\infty} \frac{n^2(|a_n| + |b_n|)}{1 - |\epsilon h_1|} \leq 1
\]
if and only if (4) holds, we see from Lemma 1 that \( s(z) \in \mathcal{K} \) and consequently \( f \in \mathcal{K}_\mathcal{H} \).

The function \( f(z) = z + \frac{1 + \delta}{n} z^n, \delta > 0 \), shows that the upper bound in (4) cannot be improved.
Remark: The coefficient bound given in Theorem 3 can also be found in [5] and [14]. However, our approach in this paper is different from those given in [5] and [14].

Remark: The well-known results for univalent functions that $f$ is convex if and only if $z f'$ is starlike does not carry over to harmonic univalent functions. See [12]. Hence, we cannot conclude from Theorem 3 that (3) is a sufficient condition for $f$ to map $\Delta$ onto a starlike domain. Nevertheless, we believe this to be the case. See [5, 6, 14].

We now introduce a class of harmonic close-to-convex functions that are constructed from convex analytic functions.

Theorem 4: If $h(z) \in \mathcal{K}$ and $w(z)$ is a Schwartz function, then

$$f(z) = h(z) + \int_0^z w(t)h'(t)dt \in \mathcal{C}_H^n.$$ 

Proof: Write $g'(z) = w(z)h'(z)$. Now for each $\epsilon$, $|\epsilon| = 1$, we observe that

$$\Re \left( \frac{h(z) - \epsilon g(z)}{h(z)} \right) = \Re \left( 1 + \epsilon w(z) \right) \geq 1 - |z| > 0, z \in \Delta.$$ 

Consequently, $h + \epsilon g$ is close-to-convex and the result follows from Theorem A.

Remark: If we only require that $w$ in Theorem 4 be analytic with $|w(z)| < 1$, $z \in \Delta$, then we may conclude that $f \in \mathcal{C}_H$.

Corollary: If $h \in \mathcal{K}$ and $n$ is a positive integer, then

$$f_n(z) = \int_0^z \left( \frac{h(t)}{t} \right)^2 dt + \int_0^z t^{n-2}h^2(t)dt \in \mathcal{C}_H^n.$$ 

Proof: A result of Sheil-Small [10] shows that $\int_0^z (h(t)/t)^2 dt \in \mathcal{K}$ whenever $h \in \mathcal{K}$. Set $w(z) = z^n$ in Theorem 4, and the result follows.

We now give some examples from Theorem 4.

Example 1: Suffridge [15] showed for the partial sums $p_n(z)$ of $e^{1+z} = \sum_{k=0}^\infty (1 + z)^k/k!$ that

$$C_n(z) = \frac{p_n(z) - p_n(0)}{p_n(0)} = \sum_{k=1}^n \left( \frac{\sum_{i=0}^{k-1} z^i}{\sum_{i=0}^{k-1} i!} \right) \frac{1}{k+1} z^k \in \mathcal{K}.$$ 

Setting $w(z) = z$ in Theorem 4, we see that

$$f(z) = \sum_{k=1}^n \left( \frac{\sum_{i=0}^{k-1} z^i}{\sum_{i=0}^{k-1} i!} \right) \left( \frac{(k+1)z^k + k\bar{z}^{k+1}}{(k+1)!} \right) \in \mathcal{C}_H^n.$$ 

Example 2: Since $h_k(z) = z + z^k/k^2 \in \mathcal{K}$, we get from the Corollary that

$$f_{k,n}(z) = z + \frac{2}{k^2} z^k + \frac{1}{k^2(2k-1)} z^{2k-1} + \frac{2^n + 1}{k^2(n+1)} \frac{z^{2n+1}}{k^2(n+1)!} \in \mathcal{C}_H^n$$ 

for $k = 2, 3, \ldots$, and $n = 1, 2, \ldots$.

Example 3: Set $h(z) = z/(1 - z)$ and $w(z) = z$ in Theorem 4. Then

$$f(z) = \frac{z}{1-z} + \int_0^z \frac{t}{1-t} dt = 2\Re \left( \frac{z}{1-z} + \log(1 - \tau) \right) \in \mathcal{C}_H^n.$$ 

We can actually state a more general result for which Example 3 is a special case.
Theorem 5: If \( b(z) \) is analytic with \( |b(z)| < 1/|1 - z|^2 \), \( z \in \Delta \), then

\[
f(z) = \frac{z}{1 - z} + \int_0^z b(t)dt \in \mathcal{C}_H.
\]

Proof: Set \( h(z) = z/(1 - z) \) and \( g(z) = \int_0^z b(t)dt \). Then \( |h'(z)| = (1/|1 - z|^2) > |g'(z)| = |b(z)| \), so that \( f \) is locally univalent. Set \( \epsilon = 0 \) in Theorem B, and the result follows.

Corollary: If \( b(z) \) is analytic with \( |b(z)| \leq 1/4 \), \( z \in \Delta \), then

\[
\frac{z}{1 - z} + \int_0^z b(t)dt \in \mathcal{C}_H.
\]

4. Convolution Condition

The convolution of two harmonic functions \( f_1(z) = z + \sum_{n=2}^\infty a_n z^n + \sum_{n=1}^\infty b_n \overline{z}^n \) and \( f_2(z) = z + \sum_{n=2}^\infty A_n z^n + \sum_{n=1}^\infty B_n \overline{z}^n \) is defined by

\[
f_1(z) \ast f_2(z) = (f_1 \ast f_2)(z) = z + \sum_{n=2}^\infty a_n A_n z^n + \sum_{n=1}^\infty b_n B_n \overline{z}^n.
\]

In [3], it was shown for \( \phi \in \mathcal{K} \) and \( f \in \mathcal{K}_H \) that \( (\phi + \overline{\phi}) \ast f \in \mathcal{C}_H \) (\( |\epsilon| \leq 1 \)). We given an example to show that \( \mathcal{K} \) cannot be replaced by \( S^*(\alpha) \), \( 0 \leq \alpha < 1 \), the family of functions starlike of order \( \alpha \).

Set

\[
\phi(z) = z + \frac{1-\alpha}{n-\alpha} z^n \in S^*(\alpha), \quad h(z) = \frac{z-\sqrt{2}}{(1-z)^2}, \quad g(z) = \frac{-z^2}{(1-z)^2}.
\]

Then \( f = h + g \in \mathcal{K}_H \), see [3]. Setting \( \epsilon = 0 \) in \( (\phi + \overline{\phi}) \ast f \) we obtain

\[
\phi \ast f = \phi \ast (h + g) = \phi \ast h = \left( z + \frac{1-\alpha}{n-\alpha} z^n \right) \ast \left( z + \sum_{n=2}^{n+1} \frac{n+1}{2} z^n \right)
\]

\[
= z + \frac{(1-\alpha)(n+1)}{2(n-\alpha)} z^n,
\]

which is not even univalent for \( n > 2\alpha/(1-\alpha) \).

References

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