APPROXIMATION OF SOLUTIONS TO RETARDED DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO POPULATION DYNAMICS

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We consider a retarded differential equation with applications to population dynamics. We establish the convergence of a finite-dimensional approximations of a unique solution, the existence and uniqueness of which are also proved in the process.

1. Introduction

Consider the following partial differential equation with delay:

$$\begin{aligned}\frac{\partial u}{\partial t}(t, x) &= \frac{\partial^2 u}{\partial x^2}(t, x) + f(t, x, u(t, x), u(t-r, x)), \quad t > 0, x \in [0, 1], \\ \frac{\partial u}{\partial x}(t, 0) &= 0 = \frac{\partial u}{\partial x}(t, 1), \quad t \geq 0, \\ u(s, x) &= h(s, x), \quad s \in [-r, 0], x \in [0, 1],\end{aligned}\tag{1.1}$$

where $f : [0, \infty) \times [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $h : [-r, 0] \times [0, 1] \rightarrow \mathbb{R}$ is a given function. The above problem for $f(t, x, u, v) = -d(x)u + b(x)v$ models a linear growth of a population in $[0, 1]$, where $u(t, \cdot)$ is the population density at time t , and the term $\partial^2 u / \partial x^2$ represents the internal migration. The continuous functions $d, b : [0, 1] \rightarrow [0, \infty)$ represent space-dependent death and birth rates, respectively, and r is the delay due to pregnancy (cf. Engel and Nagel [10, page 434]).

We formulate (1.1) as the following retarded differential equation:

$$\begin{aligned}u'(t) + Au(t) &= f(t, u(t), u(a(t))), \quad 0 < t \leq T < \infty, \\ u(t) &= h(t), \quad t \in [-r, 0],\end{aligned}\tag{1.2}$$

in a Hilbert space H , where $-A$ is the infinitesimal generator of a C_0 semigroup $\{S(t) : t \geq 0\}$ of bounded linear operators in H , $h \in \mathcal{C}_0 := C([-r, 0]; H)$ is a given function and the function a is defined from $[0, T]$ into $[-r, T]$ with the delay property $a(t) \leq t$ for $t \in [0, T]$. For (1.1), we may take $X = L^2[0, 1]$ and $D(A) = \{u \in H^2[0, 1] : u'(0) = u'(1) = 0\}$ with $Au = -d^2u/dx^2$ for $u \in D(A)$. It is known that the semigroup $S(t)$ generated by $-A$ is analytic in H (cf. Engel and Nagel [10, page 454]).

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For $t \in [0, T]$, we will use the notation $\mathcal{C}_t := C([-r, t]; H)$ for the Banach space of all continuous functions from $[-r, t]$ into H endowed with the supremum norm

$$\|\psi\|_t := \sup_{-r \leq \eta \leq t} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_t. \quad (1.3)$$

The linear case of (1.2) in which $f(t, \psi) = L\psi$, with a bounded linear operator $L : \mathcal{C}_T \rightarrow X$ is recently considered by Bátkai et al. [7] using the theory of perturbed Hille-Yosida operators. A particular semilinear case of (1.2) is considered by Alaoui [1].

For the earlier works on existence, uniqueness, and stability of various types of solutions of differential and functional differential equations, we refer to Bahuguna [2, 3], Balachandran and Chandrasekaran [6], Lin and Liu [13], and the references therein. The related results for the approximation of solutions may be found in [4, 5].

Initial studies concerning existence, uniqueness, and finite-time blowup of solutions for the equation

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \end{aligned} \quad (1.4)$$

have been considered by Segal [17], Murakami [15], and Heinz and von Wahl [12]. Bazley [8, 9] has considered the semilinear wave equation

$$\begin{aligned} u''(t) + Au(t) &= g(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \quad u'(0) = \psi, \end{aligned} \quad (1.5)$$

and has established the uniform convergence of approximations of solutions to (1.5) using the existence results of Heinz and von Wahl [12]. Göthel [11] has proved the convergence of approximations of solutions to (1.4), but assumed g to be defined on the whole of H . Based on the ideas of Bazley [8, 9], Miletta [14] has proved the convergence of approximations to solutions of (1.4). The existence, uniqueness, and continuation of classical solutions to (1.2) are considered by Bahuguna [3]. In the present work, we use the ideas of Miletta [14] and Bahuguna [2, 3] to establish the convergence of finite-dimensional approximations of the solutions to (1.2).

2. Preliminaries and assumptions

Existence of a solution to (1.2) is closely associated with the existence of a function $u \in \mathcal{C}_{\tilde{T}}$, $0 < \tilde{T} \leq T$ satisfying

$$u(t) = \begin{cases} h(t), & t \in [-r, 0], \\ S(t)h(0) + \int_0^t S(t-s)f(s, \psi(s), \psi(a(s)))ds, & t \in [0, \tilde{T}], \end{cases} \quad (2.1)$$

and such a function u is called a *mild solution* of (1.2) on $[-r, \tilde{T}]$. A function $u \in \mathcal{C}_{\tilde{T}}$ is called a *classical solution* of (1.2) on $[-r, \tilde{T}]$ if $u \in C^1((0, \tilde{T}); H)$ and u satisfies (1.2) on $[-r, \tilde{T}]$.

We assume that in (1.2), the linear operator A satisfies the following hypothesis.

(H1) A is a closed, positive definite, selfadjoint linear operator from the domain $D(A) \subset H$ into H such that $D(A)$ is dense in H , A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots \quad (2.2)$$

and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, that is,

$$Au_i = \lambda_i u_i, \quad (u_i, u_j) = \delta_{ij}, \quad (2.3)$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied, then the semigroup $S(t)$ generated by $-A$ is analytic in H . It follows that the fractional powers A^α of A for $0 \leq \alpha \leq 1$ are well defined from $D(A^\alpha) \subseteq H$ into H (cf. Pazy [16, pages 69–75]). $D(A^\alpha)$ is a Banach space endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha). \quad (2.4)$$

For $t \in [0, T]$, we denote $\mathcal{C}_t^\alpha := C([-r, t]; D(A^\alpha))$ endowed with the norm

$$\|\psi\|_{t,\alpha} := \sup_{-r \leq \eta \leq t} \|\psi(\eta)\|_\alpha. \quad (2.5)$$

The nonlinear function f is assumed to satisfy the following hypotheses.

(H2) The function $h \in \mathcal{C}_0^\alpha$.

(H3) The map f is defined from $[0, \infty) \times D(A^\alpha) \times D(A^\alpha)$ into $D(A^\beta)$ for $0 < \beta \leq \alpha < 1$ and there exists a nondecreasing function L_R from $[0, \infty)$ into $[0, \infty)$ depending on $R > 0$ such that

$$\|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)\| \leq L_R(t)[|t - s|^\gamma + \|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_\alpha], \quad (2.6)$$

for all $(t_i, u_i, v_i) \in [0, \infty) \times B_R(D(A^\alpha), h(0)) \times B_R(D(A^\alpha), h(a(0)))$, for $i = 1, 2$, where $0 < \gamma < 1$, $B_R(Z, z_0) = \{z \in Z : \|z - z_0\|_Z \leq R\}$ is the ball of radius R centered at z_0 in a Banach space Z with its norm $\|\cdot\|_Z$.

(H4) The function $a : [0, T] \rightarrow [-r, T]$ is continuous and satisfies the delay property $a(t) \leq t$ for $t \in [0, T]$.

3. Approximate solutions and convergence

Let H_n denote the finite-dimensional subspace of H spanned by $\{u_0, u_1, \dots, u_n\}$ and let $P^n : H \rightarrow H_n$ be the corresponding projection operator for $n = 0, 1, 2, \dots$. Let $0 < t < \tilde{T} \leq T$ be such that

$$\begin{aligned} \|(S(t) - I)A^\alpha h(0)\| &\leq \frac{R}{3}, \\ \|A^\alpha(h_n(0) - h(0))\| &\leq \frac{R}{3}. \end{aligned} \quad (3.1)$$

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Let $\bar{h}$ be the extension of h by the constant value $h(0)$ on $[0, T]$. We set

$$T_0 = \min \left\{ \tilde{T}, \left(\frac{(1-\alpha)R}{3M_0C_\alpha} \right)^{1/(1-\alpha)}, \left(\frac{3(1-\alpha)}{8L_R(T_0)C_\alpha} \right)^{1/(1-\alpha)} \right\}, \quad (3.2)$$

where $M_0 = [L_R(T_0)(2R + T^\gamma + 4\|\bar{h}\|_{T_0, \alpha}) + \|f(0, h(0), h(a(0)))\|]$ and C_α is a positive constant such that $\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}$ for $t > 0$.

We define

$$\begin{aligned} f_n &: [0, T_0] \times H \times H \longrightarrow D(A), \\ f_n(t, u, v) &= P^n f(t, P^n u, P^n v), \quad (t, u, v) \in [0, T_0] \times H \times H, \\ h_n &: [-r, 0] \longrightarrow D(A), \quad h_n(t) = P^n h(t), \quad t \in [-r, 0]. \end{aligned} \quad (3.3)$$

Let $A^\alpha : \mathcal{C}_t^\alpha \rightarrow \mathcal{C}_t$ be given by $(A^\alpha \psi)(s) = A^\alpha(\psi(s))$, $s \in [-r, t]$, $t \in [0, T_0]$. We define a map $F_n : B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h}) \rightarrow \mathcal{C}_{T_0}$ as follows:

$$(F_n \psi)(t) = \begin{cases} A^\alpha h_n(t), & t \in [-r, 0], \\ S(t)A^\alpha h_n(0) \\ + \int_0^t A^\alpha S(t-s) f_n(s, A^{-\alpha} \psi(s), A^{-\alpha} \psi(a(s))) ds, & t \in [0, T_0], \end{cases} \quad (3.4)$$

for $\psi \in B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$.

PROPOSITION 3.1. *For each $n \geq n_0$, where n_0 is large enough and $n, n_0 \in \mathbb{N}$, there exists a unique $w_n \in B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$ such that $F_n w_n = w_n$ on $[-r, T_0]$.*

Proof. First, we show that for any $\psi \in B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$, $F_n \psi \in B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$. For $t \in [-r, 0]$,

$$(F_n \psi)(t) - A^\alpha \bar{h}(t) = A^\alpha (P^n - I)h(t) = A^\alpha (P^n - I)h(t) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (3.5)$$

Thus, for $n \geq n_0$, n_0 large enough, for $t \in [-r, 0]$, we have

$$\|(F_n \psi)(t) - A^\alpha \bar{h}(t)\| \leq R. \quad (3.6)$$

Now, for $t \in (0, T_0]$, we have

$$\begin{aligned} \|(F_n \psi)(t) - A^\alpha \bar{h}(t)\| &\leq \|(S(t) - I)A^\alpha h(0)\| + \|A^\alpha (h_n(0) - h(0))\| \\ &+ \int_0^t \|A^\alpha S(t-s)\| \|f_n(s, A^{-\alpha} \psi(s), A^{-\alpha} \psi(a(s)))\| ds. \end{aligned} \quad (3.7)$$

For $s \in [0, T_0]$,

$$\begin{aligned}
& \|f_n(s, A^{-\alpha}\psi(s), A^{-\alpha}\psi(a(s)))\| \\
& \leq \|f(s, P^n A^{-\alpha}\psi(s), P^n A^{-\alpha}\psi(a(s)))\| \\
& \leq \|f(s, P^n A^{-\alpha}\psi(s), P^n A^{-\alpha}\psi(a(s))) - f(s, P^n \bar{h}(s), P^n \bar{h}(a(s)))\| \\
& \quad + \|f(s, P^n \bar{h}(s), P^n \bar{h}(a(s))) - f(0, h(0), h(a(0)))\| \\
& \quad + \|f(0, h(0), h(a(0)))\| \\
& \leq L_R(T_0)(2R + T^\gamma + 4\|\bar{h}\|_{T_0, \alpha}) + \|f(0, h(0), h(a(0)))\|.
\end{aligned} \tag{3.8}$$

It follows from the choice of T_0 that $F_n : B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h}) \rightarrow B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$ for n large enough. Now, we show that F_n is a strict contraction. For $\psi_1, \psi_2 \in B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$, $(F_n \psi_1)(t) - (F_n \psi_2)(t) = 0$ on $[-r, 0]$ and for $t \in [0, T_0]$, we have

$$\|(F_n \psi_1)(t) - (F_n \psi_2)(t)\| \leq 2L_R(T_0)C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha} \|\psi_1 - \psi_2\|_{T_0} \leq \frac{3}{4} \|\psi_1 - \psi_2\|_{T_0}. \tag{3.9}$$

Taking the supremum over $[-r, T_0]$, it follows that F_n is a strict contraction on $B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$ and hence there exists a unique $w_n \in B_R(\mathcal{C}_{T_0}, A^\alpha \bar{h})$ with $w_n = F_n w_n$ on $[-r, T_0]$. This completes the proof of the proposition. $\square$

Let $u_n = A^{-\alpha} w_n$. Then, $u_n \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$ and satisfies

$$u_n(t) = \begin{cases} h_n(t), & t \in [-r, 0], \\ S(t)h_n(0) + \int_0^t S(t-s)f_n(s, u_n(s), u_n(a(s)))ds, & t \in [0, T_0]. \end{cases} \tag{3.10}$$

Remarks 3.2. The above solution $u_n(t)$ is known as the Faedo-Galerkin approximate solution of (1.2).

COLLORARY 3.3. If $h(t) \in D(A)$ for all $t \in [-\tau, 0]$, then $w_n(t) \in D(A^\beta)$ for all $t \in [-\tau, T_0]$, where $0 \leq \beta < 1$, $0 \leq \alpha + \beta < 1$, and $w_n(t)$ is the solution of the integral equation (3.4).

Proof. For any $g \in D(A^\beta)$ and $t \in [-\tau, 0]$, we have

$$\|(A^\beta g, w_n(t))\| \leq \|g\| \|A^{\beta+\alpha} h_n(t)\|. \tag{3.11}$$

Now, for any $t \in (0, T_0]$, we have

$$\begin{aligned}
(A^\beta g, w_n(t)) &= (g, A^{\beta+\alpha} S(t)h_n(0)) \\
&\quad + \int_0^t (g, A^{\beta+\alpha} S(t-s)f_n(s, A^{-\alpha} w_n(s), A^{-\alpha} w_n(a(s))))ds.
\end{aligned} \tag{3.12}$$

The first term is bounded for $t \in (0, T]$ as

$$\|(A^\beta g, S(t)h_n(0))\| \leq \|g\| M \|A^{\beta+\alpha} h(0)\|. \tag{3.13}$$

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The second term is treated as follows:

$$\left\| \int_0^t (g, A^{\beta+\alpha} S(t-s), f_n(s, u_n(s), u_n(a(s)))) ds \right\| \leq M_0 \|g\| C_{\beta+\alpha} \frac{T_0^{1-(\beta+\alpha)}}{1-(\beta+\alpha)}. \quad (3.14)$$

Hence the corollary is proved. $\square$

COLLORARY 3.4. *If $h(t) \in D(A)$ for all $t \in [-\tau, 0]$, then for any $t \in [-\tau, T_0]$, there exists a constant M_1 , independent of n , such that*

$$\|A^\beta w_n(t)\| \leq M_0 \quad (3.15)$$

for all $-\tau \leq t \leq T_0$ and $0 \leq \beta < 1$.

Corollary 3.4 is a consequence of Corollary 3.3.

PROPOSITION 3.5. *The sequence $\{u_n\} \subset \mathcal{C}_{T_0}$ is a Cauchy sequence and therefore converges to a function $u \in \mathcal{C}_{T_0}$ if the assumptions (H1)–(H4) hold.*

Proof. From Proposition 3.1 we have (3.10). With $u_n = A^{-\alpha} w_n$, (3.10) becomes

$$w_n(t) = \begin{cases} A^\alpha h_n(t), & t \in [-r, 0], \\ S(t)A^\alpha h_n(0) \\ + \int_0^t A^\alpha S(t-s) f_n(s, A^{-\alpha} u_n(s), A^{-\alpha} u_n(a(s))) ds, & t \in [0, T_0]. \end{cases} \quad (3.16)$$

For $n \geq m \geq n_0$, where n_0 is large enough, $n, m, n_0 \in \mathbb{N}$, $t \in [-r, 0]$, we have

$$\begin{aligned} \|w_n(t) - w_m(t)\| &\leq \|h_n(t) - h_m(t)\|_\alpha \\ &\leq \|(P^n - P^m)h(t)\|_\alpha \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned} \quad (3.17)$$

For $t \in (0, T_0]$ and n, m , and n_0 as above, we have

$$\begin{aligned} \|w_n(t) - w_m(t)\| &\leq \|(P^n - P^m)S(t)A^\alpha h(0)\| \\ &\quad + \int_0^t \|A^\alpha S(t-s)[f_n(s, A^{-\alpha} w_n(s), A^{-\alpha} w_n(a(s))) \\ &\quad - f_m(s, A^{-\alpha} w_m(s), A^{-\alpha} w_m(a(s)))]\| ds. \end{aligned} \quad (3.18)$$

Now, using Corollaries 3.3 and 3.4, we have

$$\begin{aligned} &\|f_n(s, A^{-\alpha} w_n(s), A^{-\alpha} w_n(a(s))) - f_m(s, A^{-\alpha} w_m(s), A^{-\alpha} w_m(a(s)))\| \\ &\leq \|(P^n - P^m)f(s, P^m A^{-\alpha} w_m(s), P^m A^{-\alpha} w_m(a(s)))\| \\ &\quad + L_R(T_0)[\|(P^n - P^m)w_m(s)\| + \|(P^m - P^m)w_m(a(s))\|] \\ &\quad + 2L_R(T_0)\|w_n - w_m\|_s \\ &\leq C_1 + C_2 \frac{1}{\lambda_m^\beta} + C_2 \|w_n - w_m\|_s \end{aligned} \quad (3.19)$$

for some positive constants C_1 and C_2 , where $C_1 = \|(P^n - P^m)\| [L_R(T_0)(2R + T^\gamma + 4\|\bar{h}\|_{T_0, \alpha}) + \|f(0, h(0), h(a(0)))\|]$ and $C_2 = 2L_R(T_0)$. Thus, we have the following estimate:

$$\begin{aligned} \|w_n(t) - w_m(t)\| &\leq C_0 \|(P^n - P^m)A^\alpha h(0)\| + \frac{C_1 C_\alpha T_0^{1-\alpha}}{(1-\alpha)} + \frac{C_2 C_\alpha T_0^{1-\alpha}}{(1-\alpha)\lambda_m^\beta} \\ &\quad + C_2 C_\alpha \int_0^t (t-s)^{-\alpha} \|w_n - w_m\|_s ds, \end{aligned} \quad (3.20)$$

where $C_0 = Me^{\omega T}$. Since $\|w_n - w_m\| = \|h_n - h_m\|_\alpha$ on $[-r, 0]$, we have

$$\begin{aligned} \|w_n - w_m\|_t &\leq \|h_n - h_m\|_\alpha + C_0 \|(P^n - P^m)A^\alpha h(0)\| + \frac{C_1 C_\alpha T_0^{1-\alpha}}{(1-\alpha)} \\ &\quad + \frac{C_2 C_\alpha T_0^{1-\alpha}}{(1-\alpha)\lambda_m^\beta} + C_2 C_\alpha \int_0^t (t-s)^{-\alpha} \|w_n - w_m\|_s ds. \end{aligned} \quad (3.21)$$

Application of Gronwall's inequality gives the required result. This completes the proof of the proposition. $\square$

With the help of Propositions 3.1 and 3.5, we may state the following existence, uniqueness, and convergence result.

THEOREM 3.6. *Suppose that (H1)–(H4) hold. Then, there exist unique functions $u_n \in C([-r, T_0]; H_n)$ and $u \in C([-r, T_0]; H)$ satisfying (3.10) and*

$$u(t) = \begin{cases} h(t), & t \in [-r, 0], \\ S(t)h(0) \\ \quad + \int_0^t S(t-s)f(s, u(s), u(a(s)))ds, & t \in [0, T_0], \end{cases} \quad (3.22)$$

such that $u_n \rightarrow u$ in $C([-r, T_0]; H)$ as $n \rightarrow \infty$, where $h_n(t) = P^n h(t)$ and $f_n(t, u, v) = P^n f(t, P^n u, P^n v)$.

4. Regularity

The functions u_n and u in Theorem 3.6 satisfying (3.10) and (3.22) may be called approximate mild solution and mild solution of (1.2) on $[-r, T_0]$, respectively. In this section, we establish the regularity of the mild solution u of (1.2) under an additional assumption of Hölder continuity of the function a on $[0, T]$. We note that if a is Lipschitz continuous on $[0, T]$, then it is also Hölder continuous on $[0, T]$. We establish the following regularity result.

THEOREM 4.1. *Suppose that (H1)–(H4) hold and, in addition, suppose that $a : [0, T] \rightarrow [-r, T]$ is Hölder continuous, that is, there exist constants $0 < \delta < 1$ and $L_a \geq 0$ such that*

$$|a(t) - a(s)| \leq L_a |t - s|^\delta. \quad (4.1)$$

Then, the mild solution u given by (3.10) of (1.2) is a unique classical solution of (1.2) on $[-r, T_0]$.

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We prove that u is in fact a unique classical solution. For this, we first prove that the mild solution

$$u(t) = \begin{cases} h(t), & t \in [-r, 0], \\ S(t)h(0) + \int_0^t S(t-s)f(s, u(s), u(a(s)))ds, & t \in [0, T_0], \end{cases} \quad (4.2)$$

is locally Hölder continuous on $(0, \tilde{T}]$. Let $v(t) = A^\alpha u(t)$. Then,

$$v(t) = \begin{cases} A^\alpha h(t), & t \in [-r, 0], \\ S(t)A^\alpha h(0) + \int_0^t S(t-s)A^\alpha f(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s)))ds, & t \in [0, T_0]. \end{cases} \quad (4.3)$$

Let

$$N = \sup_{t \in [0, \tilde{T}]} \|f(t, A^{-\alpha}v(t), A^{-\alpha}v(a(t)))\|. \quad (4.4)$$

It is known that (cf. [16, page 197]) for every β with $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have

$$\|(S(h) - I)A^\alpha S(t-s)\| \leq C_\beta h^\beta \|A^{\alpha+\beta} S(t-s)\| \leq Ch^\beta (t-s)^{-\alpha+\beta}, \quad 0 < s < t. \quad (4.5)$$

For $0 < t < t+h \leq T_0$, we have

$$\begin{aligned} \|v(t+h) - v(t)\| &\leq \|(S(h) - I)A^\alpha S(t)\chi(0)\| \\ &\quad + \int_0^t \|(S(h) - I)A^\alpha S(t-s)f(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s)))\|ds \\ &\quad + \int_t^{t+h} \|A^\alpha S(t+h-s)f(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s)))\|ds. \end{aligned} \quad (4.6)$$

Using (4.5), we get

$$\|(S(h) - I)A^\alpha S(t)\chi(0)\| \leq Ct^{-(\alpha+\beta)}h^\beta \|\chi(0)\| \leq M_1 h^\beta, \quad (4.7)$$

where M_1 depends on t and $M_1 \rightarrow \infty$ as $t \rightarrow 0$. Now,

$$\begin{aligned} &\int_0^t \|(S(h) - I)A^\alpha S(t-s)f(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s)))\|ds \\ &\leq CNh^\beta \int_0^t (t-s)^{-(\alpha+\beta)}ds \\ &\leq M_2 h^\beta, \end{aligned} \quad (4.8)$$

where M_2 is independent of t . For the last integral in (4.6), we have

$$\begin{aligned}
 & \int_t^{t+h} \|A^\alpha S(t+h-s) f(s, A^{-\alpha} v(s), A^{-\alpha} v(a(s)))\| ds \\
 & \leq NC_\alpha \int_t^{t+h} (t+h-s)^{-\alpha} ds \\
 & \leq \frac{NC_\alpha}{1-\alpha} h^{1-\alpha} \\
 & \leq M_3 h^\beta,
 \end{aligned} \tag{4.9}$$

where M_3 is also independent of t . The above estimates imply that

$$\|v(t) - v(s)\| \leq L_v |t - s|^\beta, \quad |t - s| < 1, \quad 0 < s, t \leq T_0. \tag{4.10}$$

For any $0 < s < t \leq T_0$, with $t - s \geq 1$, we insert $t_1 < t_2 < \dots < t_n$ between s and t such that $1/2 \leq t_{i+1} - t_i < 1$ for $i = 1, 2, \dots, n-1$ and $t - t_n < 1$. Clearly, $n \leq 2T_0 \leq 2T$. Then, for $0 < s < t \leq T_0$, with $t - s \geq 1$, we have

$$\begin{aligned}
 \|v(t) - v(s)\| & \leq \|v(t) - v(t_n)\| + \sum_{i=1}^{n-1} \|v(t_{i+1}) - v(t_i)\| + \|v(t_1) - v(s)\| \\
 & \leq L_v \left[(t - t_n)^\beta + \sum_{i=1}^{n-1} (t_{i+1} - t_i)^\beta + (t_1 - s)^\beta \right] \\
 & \leq (2T + 1)L_v |t - s|^\beta = \tilde{L}_v |t - s|^\beta,
 \end{aligned} \tag{4.11}$$

where $\tilde{L}_v = (2T + 1)L_v$. Now, for $0 < s, t \leq T_0$, we have

$$\begin{aligned}
 & \|f(t, A^{-\alpha} v(t), A^{-\alpha} v(a(t))) - f(s, A^{-\alpha} v(s), A^{-\alpha} v(a(s)))\| \\
 & \leq L_f(R) [|t - s|^\gamma + \|v(t) - v(s)\| + \|v(a(t)) - v(a(s))\|] \\
 & \leq L_f(R) [|t - s|^\gamma + \tilde{L}_v |t - s|^\beta + \tilde{L}_v |a(t) - a(s)|^\beta] \\
 & \leq L_f(R) [|t - s|^\gamma + \tilde{L}_v |t - s|^\beta + \tilde{L}_v L_a^\beta |t - s|^{\delta \cdot \beta}] \\
 & \leq L_f(R) (1 + \tilde{L}_v + \tilde{L}_v L_a^\beta) |t - s|^{\max\{\gamma, \beta, \delta \cdot \beta\}},
 \end{aligned} \tag{4.12}$$

which shows that the function $t \mapsto f(t, A^{-\alpha} v(t), A^{-\alpha} v(a(t)))$ is locally Hölder continuous on $(0, T_0]$.

Now, consider the initial value problem

$$\frac{dw(t)}{dt} + Aw(t) = f(t, A^{-\alpha} v(t), A^{-\alpha} v(a(t))), \quad w(0) = \chi(0). \tag{4.13}$$

By [16, Corollary 4.3.3], (4.13) has a unique solution $w \in C^1((0, T_0]; H)$ given by

$$w(t) = S(t)\chi(0) + \int_0^t S(t-s)f(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s)))ds. \quad (4.14)$$

For $t > 0$, each term on the right of (4.14) is in $D(A) \subseteq D(A^\alpha)$, we may apply A^α on w to get

$$A^\alpha w(t) = S(t)A^\alpha\chi(0) + \int_0^t A^\alpha S(t-s)f(s, A^{-\alpha}v(s), A^{-\alpha}v(a(s)))ds. \quad (4.15)$$

The right-hand side of (4.15) is equal to $v(t)$ and therefore $w(t) = u(t)$ on $[0, T_0]$. Thus, $u \in C^1((0, T_0]; H)$ and hence u is a classical solution of (1.2). This completes the proof of the theorem.

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References

- [1] L. Alaoui, *Nonlinear homogeneous retarded differential equations and population dynamics via translation semigroups*, Semigroup Forum **63** (2001), no. 3, 330–356.
- [2] D. Bahuguna, *Existence, uniqueness and regularity of solutions to semilinear nonlocal functional differential problems*, Nonlinear Anal. **57** (2004), no. 7-8, 1021–1028.
- [3] ———, *Existence, uniqueness, and regularity of solutions to semilinear retarded differential equations*, J. Appl. Math. Stochastic Anal. **2004** (2004), no. 3, 213–219.
- [4] D. Bahuguna and R. Shukla, *Approximations of solutions to second order semilinear integrodifferential equations*, Numer. Funct. Anal. Optim. **24** (2003), no. 3-4, 365–390.
- [5] D. Bahuguna, S. K. Srivastava, and S. Singh, *Approximations of solutions to semilinear integrodifferential equations*, Numer. Funct. Anal. Optim. **22** (2001), no. 5-6, 487–504.
- [6] K. Balachandran and M. Chandrasekaran, *Existence of solutions of a delay differential equation with nonlocal condition*, Indian J. Pure Appl. Math. **27** (1996), no. 5, 443–449.
- [7] A. Bátkai, L. Maniar, and A. Rhandi, *Regularity properties of perturbed Hille-Yosida operators and retarded differential equations*, Semigroup Forum **64** (2002), no. 1, 55–70.
- [8] N. W. Bazley, *Approximation of wave equations with reproducing nonlinearities*, Nonlinear Anal. **3** (1979), no. 4, 539–546.
- [9] ———, *Global convergence of Faedo-Galerkin approximations to nonlinear wave equations*, Nonlinear Anal. **4** (1980), no. 3, 503–507.
- [10] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.
- [11] R. Göthel, *Faedo-Galerkin approximations in equations of evolution*, Math. Methods Appl. Sci. **6** (1984), no. 1, 41–54.
- [12] E. Heinz and W. von Wahl, *Zu einem Satz von F. E. Browder über nichtlineare Wellengleichungen*, Math. Z. **141** (1975), 33–45 (German).
- [13] Y. P. Lin and J. H. Liu, *Semilinear integrodifferential equations with nonlocal Cauchy problem*, Nonlinear Anal. **26** (1996), no. 5, 1023–1033.
- [14] P. D. Miletta, *Approximation of solutions to evolution equations*, Math. Methods Appl. Sci. **17** (1994), no. 10, 753–763.

- [15] H. Murakami, *On non-linear ordinary and evolution equations*, Funkcial. Ekvac. **9** (1966), 151–162.
- [16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
- [17] I. Segal, *Non-linear semi-groups*, Ann. of Math. (2) **78** (1963), 339–364.

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