ON SOME STOCHASTIC PARABOLIC DIFFERENTIAL EQUATIONS IN A HILBERT SPACE

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We consider some stochastic difference partial differential equations of the form $du(x, t, c) = L(x, t, D)u(x, t, c)dt + M(x, t, D)u(x, t - a, c)dw(t)$, where $L(x, t, D)$ is a linear uniformly elliptic partial differential operator of the second order, $M(x, t, D)$ is a linear partial differential operator of the first order, and $w(t)$ is a Weiner process. The existence and uniqueness of the solution of suitable mixed problems are studied for the considered equation. Some properties are also studied. A more general stochastic problem is considered in a Hilbert space and the results concerning stochastic partial differential equations are obtained as applications.

1. Introduction

Consider the stochastic linear system

$$du(t, c) = Au(t, c)dt + \sum_{i=1}^n \sum_{j=1}^k b_{ij}(B_i u(t - c_j, c))dw_{ij}(t), \quad (1.1)$$

where A is a linear closed operator generating the strongly continuous semigroup $Q(t)$ on a separable Hilbert space H , and w_{ij} are mutually independent Wiener processes on a separable Hilbert space K with covariance operators W_{ij} , positive nuclear operators in the space $L(K, K)$ of continuous linear mapping of K into itself.

It is assumed that A is defined on $S_1 \subset H$ into H and S_1 is dense in H (see [4]).

It is assumed also that $B_1, \dots, B_n$ are linear closed operators defined on $S_2 \supset S_1$, $S_2 \subset H$, and with values in H .

$b_{ij}(\cdot)$ are elements of $L(H, L(K, H))$, (see [1, 2, 4]). We will study the existence and uniqueness of mild solutions, in other words, the existence and uniqueness of a solution of the equation

$$u(t, c) = Q(t)u_0 + \sum_{i=1}^n \sum_{j=1}^k \int_0^t Q(t - \theta)b_{ij}(B_i u(\theta - c_j, c))dw_{ij}(\theta). \quad (1.2)$$

We write $\|u\|$ for the Hilbert space norm of u , and $\|b_{ij}(u)\|$ for the norms of $b_{ij}(u)$ in $L(H, L(K, H))$. We write $\text{tr } W_{ij}$ for the trace of W_{ij} . The processes $w_{ij}(t)$ are defined on a probability space (Ω, F, P) . We denote by $E[u]$ the expectation of u . We suppose that the initial condition u_0 is independent of

$$w_{ij}(t) - w_{ij}(s), \quad t \geq s > 0, \quad (1.3)$$

for all $i = 1, \dots, n, j = 1, \dots, k$.

We suppose also that

$$E[\|u_0\|^2] < \infty, \quad (1.4)$$

and that there is a number $\gamma \in (0, 1)$ such that

$$\|Q(t)b_{ij}(B_i f)\| \leq \frac{\alpha}{t^{\gamma/2}} \|f\|, \quad (1.5)$$

where α is a positive constant and $f \in S_2$.

For $f \in H$, we suppose that

$$\|BQ(t)b_{ij}(f)\| \leq \frac{\alpha}{t^{\gamma/2}} \|f\|. \quad (1.6)$$

In Section 2, we will study the uniqueness and existence of w_{ij} adapted solution $u(t, c)$ of (1.2) in the space $C(0, T; L_2(\Omega, H))$, where $C([0, T], \Lambda)$ denotes the space of continuous functions mapping $[0, T]$ into $\Lambda \subset K$.

In Section 3, we study a mixed problem (initial and boundary value problem) of some stochastic difference partial differential equations.

2. Uniqueness and existence of mild solutions

Let $u(t, c)$ satisfy the condition

$$u(t, c) = F(t), \quad -T_0 < t < 0, \quad (2.1)$$

where F is a given function in the space $C([-T_0, 0], L_2(\Omega, H) \cap S_2)$.

We assume that

$$F(0) = u_0. \quad (2.2)$$

We prove now the following theorem.

THEOREM 2.1. *Let $u \in C([0, T], L_2(\Omega, H)) \cap S_2$ be the solution of (1.2). If $F(t) = 0$ on $[-T_0, 0]$, then $u(t) = 0$ for all $t \geq 0$.*

Proof. The solution of the above equation can be written in the form

$$u(t, c) = \sum_{i=1}^n \sum_{j=1}^k \int_{\gamma_j(t)}^t Q(t-\theta) b_{ij}(B_i u(\theta - c_j, c)) dw_{ij}(\theta), \quad (2.3)$$

where

$$\gamma_j(t) = \begin{cases} t, & t \leq c_j, \\ c_j, & t > c_j. \end{cases} \quad (2.4)$$

Thus,

$$E[||u(t, c)||^2] \leq \sum_{i=1}^n \sum_{j=1}^k \text{tr } W_{ij} \int_{\gamma_j(t)}^t \frac{\alpha^2}{(t-\theta)^\gamma} E[||u(\theta - c_j, c)||^2] d\theta. \quad (2.5)$$

So, there is a positive constant λ such that

$$E[||u(t, c)||^2] \leq \frac{\lambda M t^{1-\gamma}}{1-\gamma}, \quad (2.6)$$

where

$$M = \sup_{\theta, c} E[||u(\theta, c)||^2]. \quad (2.7)$$

It is easy to see that

$$E[||u(t, c)||^2] \leq \frac{1}{1-\gamma} t^{2(1-\gamma)} \lambda^2 M \beta(1-\gamma, 2-\gamma), \quad (2.8)$$

where $\beta(m, n)$ is the β function. Now for every $r = 1, 2, \dots$, we can prove that

$$E[||u(t, c)||^2] \leq \frac{\lambda^r M t^{r(1-\gamma)} (\Gamma(1-\gamma))^r}{\Gamma(r(1-\gamma)) + 1}, \quad (2.9)$$

where Γ is the gamma function.

Taking the limit as $r \rightarrow \infty$, we get the required result.

Now to prove the existence of solutions, we suppose that

$$B_1 = B_2 = \dots = B_k = B \quad (2.10)$$

and $u_0 \in S_2$. □

THEOREM 2.2. *There exists a unique mild solution $u \in C([0, T], L_2(\Omega, H)) \cap S_2$ of (1.2).*

Proof. We apply the method of successive approximation. To do this, we set

$$\begin{aligned} u_{r+1}(t, c) = & Q(t)u_0 + \sum_{i=1}^n \sum_{j=1}^k \int_0^{\gamma_j(t)} Q(t-\theta) b_{ij}(BF(\theta - c_j)) dw_{ij}(\theta) \\ & + \sum_{i=1}^n \sum_{j=1}^k \int_{\gamma_j(t)}^t Q(t-\theta) b_{ij}(Bu_r(\theta - c_j, c)) dw_{ij}(\theta). \end{aligned} \quad (2.11)$$

Thus,

$$\begin{aligned} v_{r+1}(t, c) = & BQ(t)u_0 + \sum_i \sum_j \int_0^{\gamma_j(t)} BQ(t-\theta)b_{ij}(BF(\theta - c_j))dw_{ij}(\theta) \\ & + \sum_i \sum_j \int_{\gamma_j(t)}^t BQ(t-\theta)b_{ij}(v_r(\theta - c_j, c))dw_{ij}(\theta), \end{aligned} \quad (2.12)$$

where

$$v_r(t, c) = Bu_r(t, c). \quad (2.13)$$

The zero approximation is taken to be zero.

It is easy to see that

$$E[|v_1(t, c)|^2] \leq E[|BQ(t)u_0|^2] + \sum_{i=1}^n \sum_{j=1}^k \text{tr } W_{ij} \int_0^{\gamma_j(t)} \frac{\alpha^2}{(t-\theta)^\gamma} \|b_{ij}\|^2 E[|BF(\theta - c_j)|^2] d\theta. \quad (2.14)$$

Using the method of Theorem 2.1, we can prove that there exists a positive number λ such that

$$E[|v_{r+1}(t, c) - v_r(t, c)|^2] \leq \frac{\lambda^r t^{r(1-\gamma)} (\Gamma(1-\gamma))^r}{\Gamma(r(1-\gamma) + 1)}. \quad (2.15)$$

Since v can be written in the form

$$v(t, c) = \sum_{r=0}^{\infty} [v_{r+1}(t, c) - v_r(t, c)], \quad (2.16)$$

it follows that

$$E[|v(t, c)|^2] \leq \sum_r \frac{1}{r^2(1-\gamma)^2} \sum_r r^2(1-\gamma)^2 E[|v_{r+1}(t, c) - v_r(t, c)|^2]. \quad (2.17)$$

So v represents the solution of the equation

$$\begin{aligned} v(t, c) = & BQ(t)u_0 + \sum_i \sum_j \int_0^{\gamma_j(t)} BQ(t-\theta)b_{ij}(BF(\theta - c_j))dw_{ij}(\theta) \\ & + \sum_i \sum_j \int_{\gamma_j(t)}^t BQ(t-\theta)b_{ij}(v(\theta - c_j, c))dw_{ij}(\theta). \end{aligned} \quad (2.18)$$

Using (1.2) and (2.18), we deduce the existence of the solution of (1.2) in the space

$$C(0, T; L_2(\Omega, H)) \cap S_2. \quad (2.19)$$

The uniqueness of this solution follows from Theorem 2.1. $\square$

We give now conditions for the second moment of $u(t, c)$ to decay exponentially. To state the third theorem, we need the following conditions.

C_1 : there are positive numbers α and μ such that

$$\|Q(t)\| \leq \alpha e^{-\mu t}, \quad t > 0. \quad (2.20)$$

This exponential stability of the semigroup is equivalent to the requirement that for all $\lambda > -\mu$,

$$\|(\lambda I + A)^{-1}\| \leq \alpha(\lambda + \mu)^{-1}. \quad (2.21)$$

$$C_2: \|BQ(t)f\| \leq (\alpha/t^{y/2})e^{-\mu t}\|f\|, \quad t > 0.$$

$$C_3: \|Q(t)Bf\| \leq (\alpha/t^{y/2})e^{-\mu t}\|f\|, \quad t > 0.$$

THEOREM 2.3. *Assume conditions C_1 , C_2 , and C_3 then for sufficiently large μ , constants a and b can be found such that*

$$E[\|u(t, c)\|^2] \leq aE[\|u_0\|^2]e^{-bt}, \quad a > 0, b > 0. \quad (2.22)$$

Proof. Using conditions C_1 , C_2 , and C_3 , and (1.2), we get

$$h(t, c) \leq \lambda_1 + \lambda_2 \sum_{j=1}^k \int_0^t \frac{h(\theta - c_j, c)}{(t - \theta)^y} d\theta, \quad (2.23)$$

where $\lambda_1 = \alpha^2 E[\|u_0\|^2]$, $\lambda_2 > \alpha^2 \text{tr } W_{ij}$, λ_2 is a positive constant, and $h(t, c) = e^{2\mu t} E[\|u(t, c)\|^2]$.

Let $\{h_r\}$ be a sequence of functions such that

$$h_{r+1}(t, c) \leq \lambda_1 + \lambda_2 \sum_{j=1}^k \int_0^t \frac{h_r(\theta - c_j, c)}{(t - \theta)^y} d\theta, \quad (2.24)$$

where the zero approximation is taken to be zero. As $r \rightarrow \infty$, we get

$$h(t, c) \leq \lambda_1 \sum_r \frac{\lambda_2^r t^{r(1-y)} (\Gamma(1-y))^r}{\Gamma(r(1-y) + 1)}. \quad (2.25)$$

Using the properties of Mittag-Leffler function, we get

$$h(t, c) \leq C_1 \exp \left[t \lambda_2^{1/(1-y)} (\Gamma(1-y)^{1/(1-y)}) \right] + \frac{C_2}{1 + t^{(1-y)}}, \quad (2.26)$$

where C_1 and C_2 are positive constants. Thus for a sufficiently large μ , we get the required result. $\square$

3. Stochastic parabolic differential equations

Let $C^m(S)$ be the set of all continuous functions in S together with all their m -partial derivatives. Denote by $C_0^m(S)$ the subset of $C^m(S)$ consisting of all functions which have a compact support. Let $W^m(S)$ be a Sobolev space. In other words, $W^m(S)$ is the complete space of $C^m(S)$ with respect to the norm

$$\|f\|_m = \left[\sum_{|\alpha| \leq m} \int_S |D^\alpha f(x)|^2 dx \right], \quad (3.1)$$

where $x = (x_1, \dots, x_n)$,

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_j = \frac{\partial}{\partial x_j}, \quad j = 1, \dots, n, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \quad (3.2)$$

and $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -dimensional multi-index. We denote by $W_0^m(S)$ the complete space of $C_0^m(S)$ with respect to the norm defined by (3.1).

Let r_b be the cylinder; $r_b = (x, t) : x \in S, 0 < t < b, 0 < b < \infty$, and let Γ_b be the lateral boundary

$$\Gamma_b = \{(x, t) : x \in \partial S, 0 < t < b\}. \quad (3.3)$$

We consider the parabolic stochastic partial differential equations

$$\begin{aligned} du(x, t, c) = & \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u(x, t, c)}{\partial x_i \partial x_j} dt \\ & + \sum_{i=1}^n \sum_{r=1}^k \left[b_{ir}(x, t) \frac{\partial}{\partial x_i} + b_{0r}(x, t) \right] u(x, t - c_r, c) dw_{ir}(t), \end{aligned} \quad (3.4)$$

with the initial and boundary conditions

$$u(x, 0, c) = u_0(x), \quad (3.5)$$

$$u(x, t, c)|_{\Gamma_b} = 0. \quad (3.6)$$

It is assumed that

$$\sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \geq \delta \sum_{i=1}^n \xi_i^2, \quad (3.7)$$

where $\delta > 0$, $(x, t) \in \bar{\Omega}_b$ and $\bar{\Omega}_b$ is the closure of Ω_b , and Ω_b is an open bounded domain in the $n+1$ dimensional Euclidean space. It is assumed also that all the coefficients a_{ij} , b_{ir} , and b_{0r} are continuous on $\bar{\Omega}_b$ and satisfy a uniform Hölder condition in $t \in [0, b]$.

The mixed problem (3.4), (3.5), (3.6) can be written in the abstract form

$$du(t, c) = Au(t, c)dt + \sum_{i=1}^n \sum_{r=1}^k b_{ir}(B_i u(t - c_r, c)) dw_{ir}(t) + \sum_r b_{0r} u(t - c_r, c) dw_{0r}(t), \quad (3.8)$$

where A is the operator with domain $G = W^2(S) \cap W_0^1(S)$ given by

$$Au = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u. \quad (3.9)$$

Let $L_2(S)$ be the space of all square integrable functions on S . The space $H = L_2(S)$ is a Hilbert space and G is dense in H .

The operators $B_1, \dots, B_n$ with domains $W^1(S) \cap W_0^1(S)$ are given by

$$B_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad (3.10)$$

and b_{ir}, b_{or} are the continuous functions defined on $\bar{\Omega}_b$.

Since A is uniformly elliptic on $\bar{\Omega}_b$, it follows that the semigroup $Q(t)$ exists with the properties (1.5) and (1.6) (see [3, 5, 6]).

Consequently, Theorems 2.1, 2.2, and 2.3 can be applied for the considered abstract mixed problem.

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